MA 201: Method of Separation of Variables
Finite Vibrating String Problem
Lecture - 11
IBVP for Vibrating string with no external forces

- We consider the problem in a computational domain
  \[(x, t) \in [0, L] \times [0, \infty)\]

- The IBVP under consideration consists of the following:

- The governing equation:
  \[u_{tt} = c^2 u_{xx}, \ (x, t) \in (0, L) \times (0, \infty).\]  
  \[\tag{1}\]

  - The boundary conditions for all \(t > 0\):
    \[u(0, t) = 0, \quad u(L, t) = 0.\]  
    \[\tag{2}\]

  - The initial conditions for \(0 \leq x \leq L\) are
    \[u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x).\]  
    \[\tag{3}\]
Separation of variables method

- **Method of separation of variables for PDE:**
  a convenient and powerful solution technique of partial differential equations.

- **The main idea of this method is to**
  convert the given partial differential equation into several ordinary differential equations and then obtain the solutions by familiar solution techniques.

- **The success of this method stems from the fact that**
  a function can be expanded in a series in terms of certain elementary/special functions.

- **The basic idea of this method is that**
  the solution is assumed to consist of the product of two or more functions:

- **Each function being the function of one independent variable only.**
Separation of variables method (Contd.)

- The number of functions involved depends on the number of independent variables.

- As for example
  Try solving an equation for \( u = u(x, y) \).

- We will assume a solution in the form \( u(x, y) = X(x)Y(y) \)
  where \( X \) is a function of \( x \) only and \( Y \) is a function of \( y \) only.

- Substituting this solution in the given equation we will have a pair of ODEs.

- Note that
  This method can be used only for bounded domains so that boundary conditions are prescribed.
Boundary and initial conditions

- If the boundary conditions are prescribed in terms of some values of the function on the boundaries, then these conditions are called **Dirichlet conditions**.

- and the corresponding BVPs are called Dirichlet boundary value problems.

- If the boundary conditions are prescribed in terms of some values of the derivatives on the boundaries then these conditions are called **Neumann conditions**.

- and the corresponding BVPs are called Neumann boundary value problems.

- If the boundary conditions for a specific problem contain both types, then these conditions are called **mixed or Robin conditions** and the corresponding BVPs are called **Robin boundary value problem**.
Recall the wave equation

\[ u_{tt} - c^2 u_{xx} = 0. \]  \hspace{1cm} (4)

Assume a solution of the form

\[ u(x, t) = X(x) T(t). \] \hspace{1cm} (5)

Here, \( X(x) \) is a function of \( x \) alone and \( T(t) \) is a function of \( t \) alone.

Substituting (5) in equation (4)

\[ X T'' = c^2 X'' T. \] \hspace{1cm} (6)
IBVP for Vibrating string with no external forces (Contd.)

• Separating the variables

\[
\frac{X''}{X} = \frac{T''}{c^2 T}.
\]

• Here the left side is a function of \(x\) and the right side is a function of \(t\).

• The equality will hold only if both are equal to a constant, say, \(k\).

• We get two differential equations as follows:

\[
\begin{align*}
X'' - kX &= 0, \quad (7a) \\
T'' - c^2 kT &= 0. \quad (7b)
\end{align*}
\]
• Since $k$ is any constant,
  ▶ it can be zero, or
  ▶ it can be positive, or
  ▶ it can be negative.

• Consider all the possibilities and examine what value(s) of $k$ lead to a non-trivial solution.
Case I: $k = 0$.

- In this case the equations (7) reduce to
  \[ X'' = 0, \quad \text{and} \quad T'' = 0 \]

- Giving rise to solutions
  \[ X(x) = Ax + B, \quad T(t) = Ct + D. \]

- But the solution $u(x, t) = X(x)T(t)$ is a trivial solution if it has to satisfy the boundary conditions $u(0, t) = u(L, t) = 0$.

- This case of $k = 0$ is rejected since it gives rise to trivial solutions only.
Case II: \( k > 0 \), let \( k = \lambda^2 \) for some \( \lambda > 0 \).

- In this case the equations (7) reduce to the equations

\[
X'' - \lambda^2 X = 0, \quad \text{and} \quad T'' - c^2 \lambda^2 T = 0
\]

- Giving rise to solutions

\[
X(x) = Ae^{\lambda x} + Be^{-\lambda x}, \\
T(t) = Ce^{c\lambda t} + De^{-c\lambda t}.
\]

- Therefore

\[
u(x, t) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{c\lambda t} + De^{-c\lambda t}).
\]
Using boundary condition \( u(0, t) = 0 \),

\[ A + B = 0, \quad B = -A. \]

Using boundary condition \( u(L, t) = 0 \),

\[
(\text{Ae}^{\lambda L} + \text{Be}^{-\lambda L})(\text{Ce}^{\lambda t} + \text{De}^{-\lambda t}) = 0.
\]

The \( t \) part of the solution cannot be zero as it will lead to a trivial solution.

Then we must have

\[ A(e^{\lambda L} - e^{-\lambda L}) = 0, \]

Which leads to \( A = 0 \) as \( \lambda \neq 0 \).

\( k > 0 \) also gives rise to trivial solution: \( k > 0 \) is also rejected.
Case III: $k < 0$, let $k = -\lambda^2$ for some $\lambda > 0$.

- In this case equations (7) reduce to the equations

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad T'' + c^2 \lambda^2 T = 0$$

- Giving rise to solutions

$$X(x) = A \cos \lambda x + B \sin \lambda x,$$
$$T(t) = C \cos(c \lambda t) + D \sin(c \lambda t).$$

- Hence

$$u(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos(c \lambda t) + D \sin(c \lambda t)).$$
IBVP for Vibrating string with no external forces (Contd.)

• Using boundary condition \( u(0, t) = 0, \ A = 0 \).

• Using boundary condition \( u(L, t) = 0, \)

\[
B \sin \lambda L = 0.
\]

• \( B \neq 0 \) as that will lead to a trivial solution.

• Hence we must have

\[
sin \lambda L = 0
\]

• which gives us

\[
\lambda = \lambda_n = \frac{n\pi}{L}, \ n = 1, 2, 3, \ldots
\]
These $\lambda_n$’s are called eigenvalues and note that corresponding to each $n$ there will be an eigenvalue.

Accordingly, the solution is

$$ u(x, t) = (A \cos \lambda x + B \sin \lambda x)(C \cos(c\lambda t) + D \sin(c\lambda t)) $$

$$ = \sin \lambda_n x (BC \cos(c\lambda_n t) + BD \sin(c\lambda_n t)) $$

$$ = \sin \frac{n\pi x}{L} \left[ A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L} \right] $$

$$ = u_n(x, t). $$

The solution corresponding to each eigenvalue is called an eigenfunction.
IBVP for Vibrating string with no external forces (Contd.)

- Since the wave equation is linear and homogeneous, any linear combination will also be a solution

- Hence, we can expect the solution in the following form:

\[
u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x}{L} \right) \left[ A_n \cos \left( \frac{n\pi ct}{L} \right) + B_n \sin \left( \frac{n\pi ct}{L} \right) \right], \quad (8)
\]

- provided
  1. \( A_n \) and \( B_n \) are determined uniquely and
  2. each of the resulting series for those coefficients converges, and
  3. the limit of the series is twice continuously differentiable with respect to \( x \) and \( t \) so that it satisfies the equation \( u_{tt} - c^2 u_{xx} = 0 \).
What is the idea?

First, we assume that (i) the infinite series converges for some $A_n$ and $B_n$

(ii) term-wise differentiation with respect to $t$ is possible and it converges

Next, we calculate $A_n$ and $B_n$ using the given initial conditions.

Once, both the coefficients $A_n$ and $B_n$ are calculated, we then prove that the series actually holds following properties

i. the resulting series for those coefficients converge, and

ii. the limit of the series is twice continuously differentiable with respect to $x$ and $t$, and it satisfies the partial differential equation.
Using the initial condition \( u(x, 0) = \phi(x) \)

\[
\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}.
\] (9)

This series can be recognized as the half-range sine expansion of a function \( \phi(x) \) defined in the range \((0, L)\). What is the assumption on \( \phi \) for the convergence in \((0, L)\)?

\( A_n \) can be obtained by multiplying equation (9) by \( \sin \frac{n\pi x}{L} \) and integrating with respect to \( x \) from 0 to \( L \).

Therefore

\[
A_n = \frac{2}{L} \int_{0}^{L} \phi(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots
\] (10)

Here, we have used the fact that

\[
\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} \, dx = L.
\]
IBVP for Vibrating string with no external forces (Contd.)

- To use the other initial condition $u_t(x, 0) = \psi(x)$, we need to differentiate (8) w.r.t. $t$ to get

$$u_t(x, t) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x}{L} \right) \left( \frac{n\pi c}{L} \right) \left[ -A_n \sin \left( \frac{n\pi ct}{L} \right) + B_n \cos \left( \frac{n\pi ct}{L} \right) \right].$$

- Then

$$\psi(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \left( \frac{n\pi x}{L} \right).$$

- Similarly

$$B_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin \left( \frac{n\pi x}{L} \right) \, dx, \quad n = 1, 2, 3, \ldots$$ (11)
Now, consider the infinite series

\[
\sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left[ A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right], \tag{12}
\]

with

\[
A_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots \tag{13}
\]

\[
B_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots \tag{14}
\]

Clearly, the series satisfies both initial and boundary conditions.

In fact, we claim that it is the solution of the finite vibrating string problem. How?
IBVP for Vibrating string with no external forces (Contd.)

- Consider following infinite series

\[ w_1(x, t) = \sum_{n=1}^{\infty} A_n \cos \left( \frac{n \pi c t}{L} \right) \sin \left( \frac{n \pi x}{L} \right), \quad (15) \]

\[ w_2(x, t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n \pi c t}{L} \right) \sin \left( \frac{n \pi x}{L} \right). \quad (16) \]

- Using the trigonometric identity

\[ \sin \left( \frac{n \pi x}{L} \right) \cos \left( \frac{n \pi c t}{L} \right) = \frac{1}{2} \sin \left( \frac{n \pi}{L} (x - ct) \right) + \frac{1}{2} \sin \left( \frac{n \pi}{L} (x + ct) \right) \quad (17) \]

- We write

\[
\begin{align*}
    w_1(x, t) &= \frac{1}{2} \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi}{L} (x - ct) \right) + \frac{1}{2} \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi}{L} (x + ct) \right) \\
    &= \frac{1}{2} \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi}{L} \xi \right) + \frac{1}{2} \sum_{n=1}^{\infty} A_n \sin \left( \frac{n \pi}{L} \eta \right),
\end{align*}
\]

\[ \xi = x - ct \in \mathbb{R}, \quad \eta = x + ct \in \mathbb{R}. \]
IBVP for Vibrating string with no external forces (Contd.)

- We know that the Fourier series
\[ \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \text{ with } A_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} \, dx \]

converges to \( \phi(\xi) \) in \( \xi \in (0, L) \). But, \( \xi \in \mathbb{R} \).

- Define the odd periodic extension of \( \phi \) by
\[
\phi_0(x) = \begin{cases} 
\phi(x) & \text{if } 0 < x < L, \\
-\phi(-x) & \text{if } -L < x < 0, \\
\phi(x + 2L) & \text{for rest } x \in \mathbb{R}.
\end{cases}
\]

- From convergence of Fourier series, we conclude that the series
\[ \sum_{n=1}^{\infty} \tilde{A}_n \sin \frac{n\pi x}{L} \text{ with } \tilde{A}_n = \frac{1}{L} \int_{-L}^{L} \phi_0(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_0^L \phi_0(x) \sin \frac{n\pi x}{L} \, dx = A_n \]

converges to \( \phi_0(\xi) \) in \([-L, L]\). What about the convergence in \((-\infty, \infty)\)?
Therefore, the infinite series

\[ \sum_{n=1}^{\infty} A_n \sin \frac{n\pi\xi}{L} \text{ with } A_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} dx \]

converges to \(\phi_0(\xi)\) in \(\xi \in (-\infty, \infty)\).

As a consequence, we have

\[ w_1(x, t) = \frac{1}{2} \phi_0(\xi) + \frac{1}{2} \phi_0(\eta) \]

Assume that \(\phi \in C^2[0, L]\). What about the smoothness of \(\phi_0\) in \((-\infty, \infty)\)?

Then we obtain

\[ \frac{\partial^2 w_1}{\partial t^2} - c^2 \frac{\partial^2 w_1}{\partial x^2} = 0. \]
Similarly, under the assumption $\psi \in C^2[0, L]$, we have

$$\frac{\partial^2 w_2}{\partial t^2} - c^2 \frac{\partial^2 w_2}{\partial x^2} = 0.$$ 

Hence,

$$\frac{\partial^2}{\partial t^2}(w_1 + w_2) - c^2 \frac{\partial^2}{\partial x^2}(w_1 + w_2) = 0.$$

Recall

$$w_1(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi ct}{L} \sin \frac{n\pi x}{L},$$

$$w_2(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi ct}{L} \sin \frac{n\pi x}{L}.$$
Formal Solution of the Finite Vibrating String Problem

- The solution is given by

\[ u(x, t) = w_1(x, t) + w_2(x, t) \]

\[ = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi ct}{L} \sin \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi ct}{L} \sin \frac{n\pi x}{L} \]

- with

\[ A_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} \, dx, \; n = 1, 2, 3, \ldots \]

\[ B_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin \frac{n\pi x}{L} \, dx, \; n = 1, 2, 3, \ldots \]

- Is the solution unique?