Power Series Solution of Differential Equations and Bessel Function

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1 POWER SERIES SOLUTION AND SINGULARITIES

We begin with a method, called power series method, in finding solutions for differential equations. We will discuss its necessity and its usefulness. We will also deal with the singularities of differential equations.

1.1 Need for a power series solution

Most of the specific functions encountered in elementary analysis belong to a class known as the elementary functions. Recall that an algebraic function is a polynomial, a rational function or more generally any function $y = f(x)$ that satisfies an equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \cdots + P_1(x)y + P_0(x) = 0$$

where each $P_i$ is a polynomial. The elementary functions consist of the algebraic functions; the elementary transcendental (or nonalgebraic) functions such as trigonometric, inverse trigonometric, exponential and logarithmic functions; and all other functions that can be constructed from these by adding, subtracting, multiplying, dividing or forming a function of a function. For example, consider the following which looks complicated but is still an elementary function:

$$y = \tan\left[\frac{xe^{1/x} + \tan^{-1}(1 + x^2)}{\sin x \cos 2x - \sqrt{\ln x}}\right]^{1/3}$$

Beyond the elementary functions lie the higher transcendental functions or special functions. Most of them arise as solutions of second order linear differential equations. Many of these special functions find application in connection with partial differential equations in mathematical physics. The study of special functions has been developed and strengthened by great mathematicians like Euler, Gauss, Abel, Jacobi, Hermite and many more. Let us see how these functions come into picture. We know that the simple ODE $y'' + y = 0$ can be solved to get the familiar functions $y = \cos x$ and $y = \sin x$ as the solutions. Now, consider the equation $xy'' + y' + xy = 0$ which cannot be solved in terms of elementary functions. In other words, we do not know any familiar procedure which will yield solutions for this equation. We have to look for some alternative method to solve such type of equations: solve it in terms of power series and we use these series to define new special functions.

Definition 1.1 An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

(1.1)

is called a power series in $x$ (about the origin).

The series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

(1.2)

is a power series in $(x - x_0)$, which looks more general than the equation (1.1). But equation (1.2) can be reduced to the form (1.1) by writing $x$ for $(x - x_0)$ which is nothing but only a translation of the coordinate system.

Definition 1.2 The series (1.1) is said to converge at a point $x$ if the limit $\lim_{m \to \infty} \sum_{n=0}^{m} a_n x^n$ exists and in this case the sum of the series is the value of this limit.
1.2 Radius of convergence

Obviously, the series (1.1) always converges at the point \( x = 0 \). With respect to the arrangement of their points of convergence, all power series in \( x \) fall into one of the following three main categories of the type

\[
\sum_{n=0}^{\infty} \frac{n!}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]  

Series (1.3) diverges, i.e. it fails to converge for all \( x \neq 0 \).

Series (1.4) converges for all \( x \).

Series (1.5) converges for \( |x| < 1 \) and diverges for \( |x| > 1 \).

Some power series behave like (1.3) and converge only for \( x = 0 \), we are not interested in these. Some others, like (1.4), converge for all \( x \) and they are obviously the easiest ones. All other series are more or less like (1.5).

Thus, we observe that:

To each series of the kind (1.5) there corresponds a positive real number \( R \), called the **radius of convergence**, with the property that the series converges if \( |x| < R \) and diverges if \( |x| > R \). \([ R = 1 \) for the series (1.5)]

It is customary to put \( R \) equal to 0 when the series converges only for \( x = 0 \) and equal to \( \infty \) when it converges for all \( x \). We can cover all the possibilities by stating:

**Definition 1.3** Each power series in \( x \) has a radius of convergence \( R \), where \( 0 \leq R \leq \infty \), with the property that the series converges if \( |x| < R \) and diverges if \( |x| > R \).

Note that if \( R = 0 \), no \( x \) satisfies \( |x| \leq R \), and if \( r = \infty \), then no \( x \) satisfies \( |x| > R \).

Let \( \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots \) be a series of nonzero constants. Recall that if the limit

\[ \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = L \]

exists, then the ratio tests assert that the series converges if \( L < 1 \) and diverges if \( L > 1 \). For our power series (1.1), for each \( a_n \neq 0 \), \( x \neq 0 \), we have

\[ \lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = L, \]

then (1.1) converges if \( L < 1 \) and diverges if \( L > 1 \). We write

\[ R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \text{ if this limit exists } \left( R = \infty \text{ if } \frac{a_n}{a_{n+1}} \to \infty \right) \]

If \( R \) is infinite and nonzero, then it determines an interval of convergence \( -R < x < R \) such that inside the interval the series converges and outside the interval it diverges. A power series may or may not converge at either endpoint of its interval of convergence.

1.3 Differentiation and integration of power series

Suppose that series (1.1) converges for \( |x| < R \) with \( R > 0 \) and denote its sum by \( f(x) \):

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \]

(1.6)
Then \( f(x) \) is automatically continuous and has derivatives for \( |x| < R \). The series can be differentiated term-wise:

\[
\begin{align*}
  f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \\
  f''(x) &= 2a_2 + 6a_3 x + \cdots
\end{align*}
\]

and so on, and each of the resulting series converges for \( |x| < R \). Continuing this way, we can link \( a_n \)'s to \( f(x) \) and its derivatives by

\[
f^{(n)}(0) = \frac{n! a_n}{n!} \Rightarrow a_n = \frac{f^{(n)}(0)}{n!}
\]

so as to express \( f(x) \) as

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots
\]

Also note that the series (1.6) can be integrated term-wise provided the limits of integration lie inside the interval of convergence.

**Definition 1.4** A function with the property that a power series expansion of the form

\[
f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n
\]

is valid in some neighbourhood of the point \( x_0 \) is said to be analytic at \( x_0 \). In this case \( a_n \)'s are necessarily given by

\[
a_n = \frac{f^{(n)}(x_0)}{n!}
\]

The series (1.8) is called the Taylor series of \( f(x) \) about \( x = x_0 \).

### 1.4 Power series solution for first order ODEs

Let us first start with a very simple first order ODE and try to compare the result obtained by elementary method. Take the ODE

\[
y' - y = 0
\]

Assume that (1.9) has a power series solution of the form

\[
y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots
\]

that converges for \( |x| < R \) with \( R > 0 \), i.e. we assume that (1.9) has a solution that is analytic at the origin. By term-by-term differentiation,

\[
y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + (n+1)a_{n+1} x^n + \cdots
\]

Since \( y' = y \), the coefficients of powers of \( x \) must be equal, i.e.,

\[
a_1 = a_0, \ a_1 = 2a_2, \ a_2 = 3a_3, \ldots, \ (n+1)a_{n+1} = a_n, \cdots
\]

i.e.

\[
a_1 = a_0, \ a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \ a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \cdots, \ a_n = \frac{a_0}{n!}
\]

Hence the power series solution for (1.9) is:

\[
y(x) = a_0 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \right)
\]
We can easily observe that the series in (1.12) is nothing but the power series expansion of $e^x$. Hence the solution can be recognized as a familiar elementary function and written as

$$y = a_0 e^x$$  \hspace{1cm} (1.13)

Now consider another first order equation:

$$x y' - (x + 2)y + 2x^2 + 2x = 0$$  \hspace{1cm} (1.14)

By adopting a similar approach, we can arrive at $a_0 = 0$ and $a_1 = 2$. The recurrence formula for the coefficients would be

$$a_n = \frac{a_{n-1}}{n - 2}$$

which will give us

$$a_3 = a_2, \ a_4 = \frac{a_3}{2} = \frac{a_2}{2}, \ a_5 = \frac{a_4}{3} = \frac{a_2}{6}, \ldots$$

The solution can be written as

$$y(x) = 2x + a_2 \left( x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{6} + \frac{x^6}{24} + \cdots \right)$$

$$= 2x + a_2 x^2 e^x$$  \hspace{1cm} (1.15)

### 1.5 Second order ODEs and singularities

The general homogenous second order linear equation is of the form

$$y'' + P(x)y' + Q(x)y = 0$$  \hspace{1cm} (1.16)

Equation (1.16) can be very easily solved for constant $P$ and $Q$, and also for some simple $P$ and $Q$. For other kinds of $P$ and $Q$, power series solution is the only procedure.

The main fact about (1.16) is that its solutions near a point $x_0$ depends on the behaviour of its coefficient functions $P(x)$ and $Q(x)$ near this point. We restrict ourselves to the case in which $P(x)$ and $Q(x)$ are well-behaved in the sense of being analytic at $x_0$, which means that each has a power series expansion valid in some neighbourhood of this point. In this case $x_0$ is called an ordinary point of equation (1.16) and it turns out that every solution of the equation is also analytic at this point. Any point that is not an ordinary point of (1.16) is called a singular point.

Consider the following problems the solutions of which are already known to us.

1. $y'' + y = 0$
2. $y'' - 16y = 0$

We can approach the problems in a similar manner as in the case of first order equations and get the solutions. Let’s state a theorem which will help us in knowing about the solution of a second order equation.

**Theorem 1.1** Let $x_0$ be an ordinary point of the ODE

$$y'' + P(x)y' + Q(x)y = 0$$  \hspace{1cm} (1.17)

and let $a_0$ and $a_1$ be arbitrary constants. Then there exists a unique function $y(x)$ that is analytic at $x_0$, which is a solution of equation (1.17) in a certain neighbourhood of this point, and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. Furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid on an interval $|x - x_0| < R$ ($R > 0$), then the power series expansion of this solution is also valid on the same interval.
1.6 Classification of singular points

Consider \( x_0 \) to be a singular point of the equation

\[ y'' + P(x)y' + Q(x)y = 0 \]  

Many differential equations have singular points and the choice of physically appropriate solutions is often determined by their behavior near these points. It is interesting to note that we want to avoid the singular points of a differential equation, it is precisely these points that usually demand particular attention. If we consider

\[ y'' + 2xy' - 2x^2y = 0 \]  

then the origin is clearly a singular point. \( y_1 = x \) and \( y_2 = x^{-2} \) are independent solutions for \( x > 0 \), so \( y = c_1x + c_2x^{-2} \) is the general solution on this interval. But if we want the solution to be bounded near the origin, this can be obtained by setting \( c_2 = 0 \).

In most of the applications, the singular points are rather “weak” in the sense that the coefficient functions are only mildly non-analytic and simple modifications will yield satisfying solutions.

Definition 1.5 A singular point \( x_0 \) of the equation (1.18) is called regular if the functions \( (x - x_0)^2P(x) \) and \( (x - x_0)^2Q(x) \) are analytic, and irregular otherwise.

In other words, the singularity in \( P(x) \) cannot be worse than \( 1/(x - x_0)^2 \), and that in \( Q(x) \) cannot be worse than \( 1/(x - x_0)^2 \).

Consider the equations

\[ x^2y'' + xy' + (x^2 - \mu^2)y = 0 \]  

\[ (1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \]

We can easily observe that \( x = 0 \), and \( x = \pm 1 \) are regular singular points for the equations (1.19) and (1.20) respectively.

1.7 Frobenius series

A “quasi power series” of the form

\[ y = x^m \sum_{n=0}^{\infty} a_n x^n = x^m(a_0 + a_1 x + a_2 x^2 + \cdots) \]  

is called a Frobenius series and the procedure for finding solutions of ODEs by using this type of power series is called the method of Frobenius. The exponent \( m \) can be integer, fraction, positive or negative. Frobenius series is used when we want to solve some differential equations which have regular singular point(s). If we want to have a power series solution about an ordinary point of the differential equation, then Frobenius method is not needed. But if we want a power series solution about a regular singular point, use of Frobenius series becomes essential. We see that when we use the power series method as explained in the previous subsection, we do not know how to apply that series expansion about a regular singular point. Frobenius method will always work provided the point of expansion is no worse than a regular singular point. In (1.21) above, the coefficients as well as the exponent \( m \) are undetermined. In the next section, we will discuss in details about this method while considering Bessel’s equation and Legendre equation.

2 Bessel Function

2.0.1 Origin of Bessel equation, solution and different kinds of Bessel functions

The differential equation

\[ x^2y'' + xy' + (x^2 - \mu^2)y = 0 \]  

(2.1)
where \( \mu \) is a constant, is called Bessel’s equation of order \( \mu \) and its solutions are known as Bessel functions of order \( \mu \). Recall that Laplace’s equation in cylindrical coordinates \((r, \theta, z)\) is given by

\[
u_{rr} + \left(1/r\right)u_r + \left(1/r^2\right)u_{\theta\theta} + u_{zz} = 0 \tag{2.2}
\]

In order to solve this equation, we can use the separation of variables method by assuming a solution of (2.2) of the form

\[
u(r, \theta, z) = R(r)T(\theta)Z(z) \tag{2.3}
\]

Using (2.3) in (2.2), we see that the partial differential equation (2.2) is converted to three ordinary differential equations – one each in \( r \) and \( z \) as follows:

\[
\begin{align*}
r^2R'' + rR' + \left(\frac{\lambda^2r^2 - \mu^2}{r}\right) &= 0 \\
T'' + \mu^2T &= 0 \\
Z'' - \lambda^2Z &= 0
\end{align*}
\]

where the constants \( \lambda \) and \( \mu \) are separation constants. The last two equations of the above system give rise to two simple solutions whereas the first one is not known to have some known solutions. This equation is nothing but what we call Bessel’s equation.

We can easily see that \( x = 0 \) is a singular point of the equation (2.1). Moreover, here \( P(x) = 1/x \), \( Q(x) = (x^2 - \mu^2)/x^2 \). Hence \( xP(x) = 1 \) and \( x^2Q(x) = x^2 - \mu^2 \) which show that \( x = 0 \) is a regular singular point. Let’s assume the solution of (2.1) to be of the form (Frobenius series)

\[
y = x^m \sum_{k=0}^{\infty} a_k x^k
\]

The indicial equation is

\[
m^2 - \mu^2 = 0 \Rightarrow m_1 = \mu, \ m_2 = -\mu
\]

Hence we can assume that equation (2.1) has a solution of the form :

\[
y = x^\mu \sum_{k=0}^{\infty} a_k x^{k+\mu} \tag{2.4}
\]

where \( a_0 \neq 0 \) and the power series converges for all \( x \).

By inserting (2.4) in (2.1), we will ultimately have

\[
\sum_{k=0}^{\infty} k(k+2\mu)a_k x^{k+\mu} + \sum_{k=2}^{\infty} a_{k-2} x^{k+\mu} = 0
\]

Equating coefficients of \( x^{k+\mu} \) to zero, we get the recursion formula

\[
a_k = -\frac{a_{k-2}}{k(k+2\mu)} \tag{2.5}
\]

We know that \( a_0 \neq 0 \) and \( a_1 = 0 \) (which is obvious from (2.5) above as \( a_{-1} = 0 \)). Equation (2.5) is the recurrence relation for the coefficients of the power series (2.4). Repeated applications of (2.5) will yield \( a_k = 0 \) for every odd \( k \), i.e.

\[
a_1 = a_3 = a_5 = \cdots = a_{2m-1} = \cdots = 0
\]

All the nonzero constants can be expressed in terms of \( a_0 \).

\[
\begin{align*}
a_2 &= -\frac{a_0}{2(2\mu + 2)} = -\frac{a_0}{2^2(\mu + 2)} \\
a_4 &= -\frac{a_4}{4(2\mu + 4)} = -\frac{a_0}{2^3(\mu + 1)(\mu + 2)} \\
a_6 &= -\frac{a_6}{6(2\mu + 6)} = -\frac{a_0}{2^3(\mu + 1)(\mu + 2)(\mu + 3)}
\end{align*}
\]
Now we can write the solution of (2.1) as
\[ y = a_0 x^\mu \sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{2^k k! (\mu+1)(\mu+2) \cdots (\mu+k)} \] (2.6)

Bessel function of first kind of order \( \mu \), denoted by \( J_\mu(x) \), is defined by putting \( a_0 = 1/(2^\mu \Gamma(\mu+1)) \) so that
\[ J_\mu(x) = \sum_{k=0}^\infty (-1)^k \frac{(x/2)^{2k+\mu}}{k! \Gamma(\mu+k+1)} \] (2.7)

The most useful Bessel functions are the ones of order 0 and 1 which are given by
\[ J_0(x) = \sum_{k=0}^\infty (-1)^k \frac{1}{(2k)!} \frac{x^{2k}}{2^{2k}} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^24^2} - \frac{x^6}{2^24^26^2} + \cdots \] (2.8)
\[ J_1(x) = \sum_{k=0}^\infty (-1)^k \frac{1}{k!(k+1)!} \frac{x^{2k+1}}{2^{2k+1}} = \frac{x}{2} - \frac{1}{1!2!} \left( \frac{x}{2} \right)^3 + \frac{1}{2!3!} \left( \frac{x}{2} \right)^5 + \cdots \] (2.9)

Also, another interesting and important result is
\[ J'_0(x) = -J_1(x) \] (2.10)

Now by taking the other value of \( m \), i.e. \( m = -\mu \), we get
\[ J_{-\mu}(x) = \sum_{k=0}^\infty (-1)^k \frac{1}{2^k k! \Gamma(-\mu+k+1)} x^{-2k} \] (2.11)

which is a second particular solution of Bessel’s equation of order \( \mu \). This function is unbounded at \( x = 0 \) while \( J_\mu(x) \) remains finite. Hence, for non-integral values of \( \mu \), \( J_\mu(x) \) and \( J_{-\mu}(x) \) are two linearly independent solutions. Hence the general solution of (2.1) is given by
\[ y(x) = c_1 J_\mu(x) + c_2 J_{-\mu}(x), \ \mu \text{ not an integer} \] (2.12)

where \( c_1 \) and \( c_2 \) are arbitrary constants.

To construct a second solution, take a linear combination of \( J_\mu(x) \) and \( J_{-\mu}(x) \) as follows:
\[ Y_\mu(x) = \alpha J_\mu(x) + \beta J_{-\mu}(x) \] (2.13)

where \( \alpha \) and \( \beta \) are nonzero arbitrary constants. For standardization, particular values of \( \alpha \) and \( \beta \) are taken to be \( \alpha = \cot \pi \mu \) and \( \beta = -\csc \pi \mu \). Hence, when \( \mu \) is non-integral, the second solution is
\[ Y_\mu(x) = \frac{\cos \pi \mu J_\mu(x) - J_{-\mu}(x)}{\sin \pi \mu} \] (2.14)
This \( Y_m(x) \) is known as Bessel function of second kind of order \( \mu \) with the property that \( Y_m \to -\infty \) as \( x \to 0 \). Accordingly, an equivalent general solution of (2.1) is

\[
y(x) = AJ_\mu(x) + BY_\mu(x)
\]

If we consider \( \mu \) to be an integer, say \( \mu = n \), then we observe that \( J_n(x) \) and \( J_{-n}(x) \) are not linearly independent but rather they are related through a relation

\[
J_{-n}(x) = (-1)^n J_n(x)
\]  

(2.16)

Hence, for integral values of \( \mu \), we cannot write a general solution in the form

\[
y = AJ_n(x) + BJ_{-n}(x)
\]

But analysis shows that, even for integral values of \( \mu \), the equation (2.15) holds as \( Y_m(x) = \lim_{\mu \to m} Y_\mu(x) \) exists. Hence, (2.15) is solution of (2.1) whether or not \( \mu \) is an integer.

In some applied problems, another form of general solution is used:

\[
y(x) = AH^{(1)}_{\mu}(x) + BH^{(2)}_{\mu}(x)
\]

(2.17)

where \( H^{(1)}_{\mu}(x) \) and \( H^{(2)}_{\mu}(x) \) are respectively known as Hankel functions of first and second kind of order \( \mu \), or Bessel function of third kind of order \( \mu \). They are defined by the relations

\[
H^{(1)}_{\mu}(x) = J_{\mu}(x) + iY_{\mu}(x)
\]

(2.18)

\[
H^{(2)}_{\mu}(x) = J_{\mu}(x) - iY_{\mu}(x)
\]

(2.19)

### 2.0.2 Generating Function of Bessel Functions, Properties of Bessel Function

The generating function of the Bessel functions is

\[
\exp \left( \frac{1}{2} x(t - \frac{1}{t}) \right)
\]

This function can be developed into a Laurent series. The coefficient of \( t^n \) in the expansion is the Bessel function of argument \( x \) and order \( n \).

\[
\exp \left( \frac{1}{2} x(t - \frac{1}{t}) \right) = \sum_{n=-\infty}^{\infty} t^n J_n(x)
\]

(2.20)

Using (2.16), we can rewrite (2.20) as

\[
\exp \left( \frac{1}{2} x(t - \frac{1}{t}) \right) = J_0(x) + \sum_{n=1}^{\infty} \left[ t^n + (-1)^n t^{-n} \right] J_n(x)
\]

(2.21)

**Recurrence Formulas:** The following two important recurrence formulas can be obtained either from the expression of \( J_{\mu}(x) \) or from the generating functions.

\[
J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)
\]

(2.22)

\[
J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x)
\]

(2.23)

Another set of important relations is:

\[
\frac{d}{dx} (x^\mu J_{\mu}(x)) = x^\mu J_{\mu-1}(x)
\]

(2.24)

\[
\frac{d}{dx} (x^{-\mu} J_{\mu}(x)) = -x^{-\mu} J_{\mu+1}(x)
\]

(2.25)

The above recurrence relations hold for \( Y_{\mu}(x) \) as well.