

Power Series Solutions to the Bessel Equation

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The Bessel equation

The equation

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0, \quad (1)$$

where α is a nonnegative constant, is called the **Bessel equation**.

The point $x_0 = 0$ is a regular singular point. We shall use the method of Frobenius to solve this equation.

Thus, we seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad x > 0, \quad (2)$$

with $a_0 \neq 0$.

Differentiation of (2) term by term yields

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}.$$

Similarly, we obtain

$$y'' = x^{r-2} \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^n.$$

Substituting these into (1), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} \\ & + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} \alpha^2 a_n x^{n+r} = 0. \end{aligned}$$

This implies

$$x^r \sum_{n=0}^{\infty} [(n+r)^2 - \alpha^2] a_n x^n + x^r \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Now, cancel x^r , and try to determine a_n 's so that the coefficient of each power of x will vanish.

For the constant term, we require $(r^2 - \alpha^2)a_0 = 0$. Since $a_0 \neq 0$, it follows that

$$r^2 - \alpha^2 = 0,$$

which is the **indicial** equation. The only possible values of r are α and $-\alpha$.

Case I. For $r = \alpha$, the equations for determining the coefficients are:

$$\begin{aligned} [(1 + \alpha)^2 - \alpha^2]a_1 &= 0 \quad \text{and,} \\ [(n + \alpha)^2 - \alpha^2]a_n + a_{n-2} &= 0, \quad n \geq 2. \end{aligned}$$

Since $\alpha \geq 0$, we have $a_1 = 0$. The second equation yields

$$a_n = -\frac{a_{n-2}}{(n + \alpha)^2 - \alpha^2} = -\frac{a_{n-2}}{n(n + 2\alpha)}. \quad (3)$$

Since $a_1 = 0$, we immediately obtain

$$a_3 = a_5 = a_7 = \cdots = 0.$$

For the coefficients with even subscripts, we have

$$a_2 = \frac{-a_0}{2(2+2\alpha)} = \frac{-a_0}{2^2(1+\alpha)},$$

$$a_4 = \frac{-a_2}{4(4+2\alpha)} = \frac{(-1)^2 a_0}{2^4 2!(1+\alpha)(2+\alpha)},$$

$$a_6 = \frac{-a_4}{6(6+2\alpha)} = \frac{(-1)^3 a_0}{2^6 3!(1+\alpha)(2+\alpha)(3+\alpha)},$$

and, in general

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n! (1+\alpha)(2+\alpha) \cdots (n+\alpha)}.$$

Therefore, the choice $r = \alpha$ yields the solution

$$y(x) = a_0 x^\alpha \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! (1+\alpha)(2+\alpha) \cdots (n+\alpha)} \right).$$

Note: The ratio test shows that the power series formula converges for all $x \in \mathbb{R}$.

For $x < 0$, we proceed as above with x^r replaced by $(-x)^r$. Again, in this case, we find that r satisfies

$$r^2 - \alpha^2 = 0.$$

Taking $r = \alpha$, we obtain the same solution, with x^α is replaced by $(-x)^\alpha$. Therefore, the function $y_\alpha(x)$ is given by

$$y_\alpha(x) = a_0|x|^\alpha \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! (1+\alpha)(2+\alpha) \cdots (n+\alpha)} \right) \quad (4)$$

is a solution of the Bessel equation valid for all real $x \neq 0$.

Case II. For $r = -\alpha$, determine the coefficients from

$$[(1 - \alpha)^2 - \alpha^2]a_1 = 0 \quad \text{and} \quad [(n - \alpha)^2 - \alpha^2]a_n + a_{n-2} = 0.$$

These equations become

$$(1 - 2\alpha)a_1 = 0 \quad \text{and} \quad n(n - 2\alpha)a_n + a_{n-2} = 0.$$

If 2α is not an integer, these equations give us

$$a_1 = 0 \quad \text{and} \quad a_n = -\frac{a_{n-2}}{n(n - 2\alpha)}, \quad n \geq 2.$$

Note that this formula is same as (3), with α replaced by $-\alpha$. Thus, the solution is given by

$$y_{-\alpha}(x) = a_0|x|^{-\alpha} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! (1 - \alpha)(2 - \alpha) \cdots (n - \alpha)} \right), \quad (5)$$

which is valid for all real $x \neq 0$.

Euler's gamma function and its properties

For $s \in \mathbb{R}$ with $s > 0$, we define $\Gamma(s)$ by

$$\Gamma(s) = \int_{0+}^{\infty} t^{s-1} e^{-t} dt.$$

The integral converges if $s > 0$ and diverges if $s \leq 0$.

Integration by parts yields the functional equation

$$\Gamma(s+1) = s\Gamma(s).$$

In general,

$$\Gamma(s+n) = (s+n-1) \cdots (s+1)s\Gamma(s), \text{ for every } n \in \mathbb{Z}^+.$$

Since $\Gamma(1) = 1$, we find that $\Gamma(n+1) = n!$. Thus, the gamma function is an extension of the factorial function from integers to positive real numbers. Therefore, we write

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \quad s \in \mathbb{R}.$$

Using this gamma function, we shall simplify the form of the solutions of the Bessel equation. With $s = 1 + \alpha$, we note that

$$(1 + \alpha)(2 + \alpha) \cdots (n + \alpha) = \frac{\Gamma(n + 1 + \alpha)}{\Gamma(1 + \alpha)}.$$

Choose $a_0 = \frac{2^{-\alpha}}{\Gamma(1+\alpha)}$ in (4), the solution for $x > 0$ can be written

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + 1 + \alpha)} \left(\frac{x}{2}\right)^{2n}.$$

The function J_α defined above for $x > 0$ and $\alpha \geq 0$ is called the **Bessel function of the first kind** of order α .

When α is a nonnegative integer, say $\alpha = p$, the Bessel function $J_p(x)$ is given by

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p}, \quad (p = 0, 1, 2, \dots).$$

This is a solution of the Bessel equation for $x < 0$.

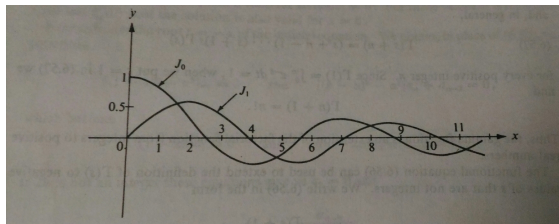


Figure : The Bessel functions J_0 and J_1 .

If $\alpha \notin \mathbb{Z}^+$, define a new function $J_{-\alpha}(x)$ (replacing α by $-\alpha$)

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+1-\alpha)} \left(\frac{x}{2}\right)^{2n}.$$

With $s = 1 - \alpha$, we note that

$$\Gamma(n+1-\alpha) = (1-\alpha)(2-\alpha)\cdots(n-\alpha)\Gamma(1-\alpha).$$

Thus, the series for $J_{\alpha}(x)$ is the same as that for $y_{-\alpha}(x)$ in (5) with $a_0 = \frac{2^{\alpha}}{\Gamma(1-\alpha)}$, $x > 0$. If α is not positive integer, $J_{-\alpha}$ is a solution of the Bessel equation for $x > 0$.

If $\alpha \notin \mathbb{Z}^+$, $J_{\alpha}(x)$ and $J_{-\alpha}(x)$ are linearly independent on $x > 0$. The general solution of the Bessel equation for $x > 0$ is

$$y(x) = c_1 J_{\alpha}(x) + c_2 J_{-\alpha}(x).$$

Useful recurrence relations for J_α

- $\frac{d}{dx}(x^\alpha J_\alpha(x)) = x^\alpha J_{\alpha-1}(x).$

$$\begin{aligned}
 \frac{d}{dx}(x^\alpha J_\alpha(x)) &= \frac{d}{dx} \left\{ x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \alpha + n)} \left(\frac{x}{2}\right)^{2n+\alpha} \right\} \\
 &= \frac{d}{dx} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2\alpha}}{n! \Gamma(1 + \alpha + n) 2^{2n+\alpha}} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 2\alpha) x^{2n+2\alpha-1}}{n! \Gamma(1 + \alpha + n) 2^{2n+\alpha}}.
 \end{aligned}$$

Since $\Gamma(1 + \alpha + n) = (\alpha + n)\Gamma(\alpha + n)$, we have

$$\begin{aligned}
 \frac{d}{dx}(x^\alpha J_\alpha(x)) &= \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+2\alpha-1}}{n! \Gamma(\alpha + n) 2^{2n+\alpha}} \\
 &= x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + (\alpha - 1) + n)} \left(\frac{x}{2}\right)^{2n+\alpha-1} \\
 &= x^\alpha J_{\alpha-1}(x).
 \end{aligned}$$

The other relations involving J_α are:

- $\frac{d}{dx}(x^{-\alpha} J_\alpha(x)) = -x^{-\alpha} J_{\alpha+1}(x).$
- $\frac{\alpha}{x} J_\alpha(x) + J'_\alpha(x) = J_{\alpha-1}(x).$
- $\frac{\alpha}{x} J_\alpha(x) - J'_\alpha(x) = J_{\alpha+1}(x).$
- $J_{\alpha-1}(x) + J_{\alpha+1}(x) = \frac{2\alpha}{x} J_\alpha(x).$
- $J_{\alpha-1}(x) - J_{\alpha+1}(x) = 2J'_\alpha(x).$

Note: Workout these relations.

*** End ***