MA 102
Mathematics II
Lecture 5

11 March, 2015
Remark: There was a mistake in the last slide of the last lecture.

Example: Recall that we considered the IVP

\[ \frac{dy}{dx} + y = f(x); \quad y(0) = 0, \text{ where } f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}. \]

Observe that this IVP has discontinuous non-homogeneous term and the (continuous) function \( y(x) \) defined by

\[ y(x) = \begin{cases} 1 - e^{-x} & \text{if } 0 \leq x \leq 1 \\ (e - 1)e^{-x} & \text{if } x > 1 \end{cases}. \]

satisfies the above IVP separately on the intervals \([0, 1]\) and \((1, \infty)\).

Exercise: Think why is it technically incorrect to say that the above function \( y(x) \) is a solution to the IVP on the interval \([0, \infty)\).
Differential of a function of 2 variables

**Definition**

**Differential of a function of 2 variables**: If \( f(x, y) \) is a function of two variables with continuous first partial derivatives in a region \( R \) of the \( xy \)-plane, then its differential \( df \) is

\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.
\]

In the special case when \( f(x, y) = c \), where \( c \) is a constant, we have \( \frac{\partial f}{\partial x} = 0 \) and \( \frac{\partial f}{\partial y} = 0 \). Therefore, we have \( df = 0 \), or in other words,

\[
\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0.
\]

So given a one-parameter family of functions \( f(x, y) = c \), we can generate a first order ODE by computing the differential on both sides of the equation \( f(x, y) = c \).
A differential expression \( M(x, y)dx + N(x, y)dy \) is an **exact differential** in a region \( R \) of the \( xy \)-plane if it corresponds to the differential of some function \( f(x, y) \) defined on \( R \). A first order differential equation of the form

\[
M(x, y)dx + N(x, y)dy = 0
\]

is called an **exact equation** if the expression on the left hand side is an exact differential.

**Example:** 1) \( x^2y^3dx + x^3y^2dy = 0 \) is an exact equation since
\[
x^2y^3dx + x^3y^2dy = d\left(\frac{x^3y^3}{3}\right).
\]
2) \( ydx + xdy = 0 \) is an exact equation since \( ydx + xdy = d(xy) \).
3) \( \frac{ydx-xdy}{y^2} = 0 \) is an exact equation since \( \frac{ydx-xdy}{y^2} = d\left(\frac{x}{y}\right) \).
Criterion for an exact differential

**Theorem**

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region $R$ defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition for $M(x, y)dx + N(x, y)dy$ to be an exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

**Example**

Solve the ODE $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$.

This equation can be expressed as $M(x, y)dx + N(x, y)dy = 0$ where $M(x, y) = 3x^2 + 4xy$ and $N(x, y) = 2x^2 + 2y$. It is easy to verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x$. Hence the given ODE is exact.

We have to find a function $f$ such that $\frac{\partial f}{\partial x} = M = 3x^2 + 4xy$ and $\frac{\partial f}{\partial y} = N = 2x^2 + 2y$. Now $\frac{\partial f}{\partial x} = 3x^2 + 4xy \Rightarrow f(x, y) = \int (3x^2 + 4xy)dx = x^3 + 2x^2y + \phi(y)$ for some function $\phi(y)$ of $y$. Again $\frac{\partial f}{\partial y} = 2x^2 + 2y$ and $f(x, y) = x^3 + 2x^2y + \phi(y)$ together imply that $2x^2 + \phi'(y) = 2x^2 + 2y \Rightarrow \phi(y) = y^2 + c_1$ for some constant $c_1$. Hence the solution is $f(x, y) = c$ or $x^3 + 2x^2y + y^2 + c_1 = c$. 
Converting a first order non-exact DE to exact DE

Consider the following example:

Example

The first order DE \( ydx - xdy = 0 \) is clearly not exact. But observe that if we multiply both sides of this DE by \( \frac{1}{y^2} \), the resulting ODE becomes

\[
\frac{dx}{y} - \frac{x}{y^2} dy = 0
\]

which is exact!

Definition

It is sometimes possible that even though the original first order DE \( M(x, y)dx + N(x, y)dy = 0 \) is not exact, but we can multiply both sides of this DE by some function (say, \( \mu(x, y) \)) so that the resulting DE

\[
\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0
\]

becomes exact. Such a function/factor \( \mu(x, y) \) is known as an **integrating factor** for the original DE \( M(x, y)dx + N(x, y)dy = 0 \).
How to find an integrating factor?

If a first order DE has one integrating factor, then it has infinitely many integrating factors. [For a proof, see tutorial problem sheet.]

We will now list down some rules for finding integrating factors, but before that, we need the following definition:

Definition
A function $f(x, y)$ is said to be **homogeneous** of degree $n$ if $f(tx, ty) = t^n f(x, y)$ for all $(x, y)$ and for all $t \in \mathbb{R}$.

Example
1) $f(x, y) = x^2 + y^2$ is homogeneous of degree 2.
2) $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ is homogeneous of degree 0.
3) $f(x, y) = \frac{x(x^2+y^2)}{y^2}$ is homogeneous of degree 1.
4) $f(x, y) = x^2 + xy + 1$ is NOT homogeneous.
How to find an integrating factor? contd...

Definition

A first order DE of the form

\[ M(x, y)dx + N(x, y)dy = 0 \]

is said to be **homogeneous** if both \( M(x, y) \) and \( N(x, y) \) are homogeneous functions of the same degree.

**NOTE:** Here the word “homogeneous” does not mean the same as it did for first order linear equation \( a_1(x)y' + a_0(x)y = g(x) \) when \( g(x) = 0 \).

Some rules for finding an integrating factor: Consider the DE

\[ M(x, y)dx + N(x, y)dy = 0. \]  \((*)\)

**Rule 1:** If \((*)\) is a homogeneous DE with \( M(x, y)x + N(x, y)y \neq 0 \), then

\[ \frac{1}{Mx+Ny} \]

is an integrating factor for \((*)\).
Rule 2: If $M(x, y) = f_1(xy)y$ and $N(x, y) = f_2(xy)x$ and $Mx - Ny \neq 0$, where $f_1$ and $f_2$ are functions of the product $xy$, then \( \frac{1}{Mx - Ny} \) is an integrating factor for (*)

Rule 3: If \( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = f(x) \) (function of $x$-alone), then \( e^{\int f(x) dx} \) is an integrating factor for (*)

Rule 4: If \( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = F(y) \) (function of $y$-alone), then \( e^{-\int F(y) dy} \) is an integrating factor for (*)
Proof of Rule 1

Proof.

Observe that \( Mdx + Ndy = \frac{1}{2}[(Mx + N y)(\frac{dx}{x} + \frac{dy}{y}) - (Mx - Ny)(\frac{dx}{x} - \frac{dy}{y})] \)
\[ \]
= \( \frac{1}{2}[(Mx + Ny)d(log xy) + (Mx - Ny)d(log \frac{x}{y})] \).

Since we have assumed that \( Mx + Ny \neq 0 \), therefore we can divide both sides of the above equation by \( Mx + Ny \) to get
\[ \frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}[d log(xy) + (\frac{Mx - Ny}{Mx + Ny})d log(\frac{x}{y})]. \] Since the DE (*) is homogeneous, therefore \( \frac{Mx - Ny}{Mx + Ny} \) is a function (say, \( f(\frac{x}{y}) \)) of \( \frac{x}{y} \). Hence we have
\[ \frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}[d log(xy) + f(\frac{x}{y})d log(\frac{x}{y})]. \] Since \( \frac{x}{y} = e^{log \frac{x}{y}} \), therefore
\[ \frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}[d log(xy) + F(log(\frac{x}{y}))d(log \frac{x}{y})] \] for some function \( F \). Hence
\[ \frac{Mdx + Ndy}{Mx + Ny} = d[\frac{1}{2}log(xy) + \frac{1}{2} \int F(log \frac{x}{y})d(log \frac{x}{y})], \]
which is an exact differential. This completes the proof.
Proof of Rule 3

Proof.

Let \( f(x) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \). To show: \( \mu(x) := e^{\int f(x)\,dx} \) is an integrating factor. That is, to show \( \frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \).

Since \( \mu \) is a function of \( x \) alone, we have \( \frac{\partial}{\partial y}(\mu M) = \mu \frac{\partial M}{\partial y}. \) Also \( \frac{\partial}{\partial x}(\mu N) = \mu'(x)N + \mu(x) \frac{\partial N}{\partial x}. \) So we must have:

\[
\mu(x)\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \mu'(x)N,
\]
or equivalently we must have,

\[
\frac{\mu'(x)}{\mu(x)} = f(x),
\]

which is anyways true since \( \mu(x) := e^{\int f(x)\,dx}. \) \( \square \)

The proof of Rule 4 is similar. The proof of Rule 2 is an exercise.