Power Series Solutions to the Legendre Equation

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The Legendre equation

\[(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,\]  \hspace{1cm} (1)

where $\alpha$ is any real constant, is called Legendre’s equation. When $\alpha \in \mathbb{Z}^+$, the equation has polynomial solutions called Legendre polynomials. In fact, these are the same polynomial that encountered earlier in connection with the Gram-Schmidt process.

The Eqn. (1) can be rewritten as

\[[(x^2 - 1)y']' = \alpha(\alpha + 1)y,\]

which has the form $T(y) = \lambda y$, where $T(f) = (pf')'$, with $p(x) = x^2 - 1$ and $\lambda = \alpha(\alpha + 1)$. 
Note that the nonzero solutions of (1) are eigenfunctions of $T$ corresponding to the eigenvalue $\alpha(\alpha + 1)$.

Since $p(1) = p(-1) = 0$, $T$ is symmetric with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx.$$ 

Thus, eigenfunctions belonging to distinct eigenvalues are orthogonal.
Power series solution for the Legendre equation

The Legendre equation can be put in the form

\[ y'' + p(x)y' + q(x)y = 0, \]

where

\[ p(x) = -\frac{2x}{1-x^2} \quad \text{and} \quad q(x) = \frac{\alpha(\alpha + 1)}{1-x^2}, \quad \text{if} \ x^2 \neq 1. \]

Since \( \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \) for \( |x| < 1 \), both \( p(x) \) and \( q(x) \) have power series expansions in the open interval \((-1, 1)\).

Thus, seek a power series solution of the form

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in (-1, 1). \]
Differentiating term by term, we obtain

\[ y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}. \]

Thus,

\[ 2xy' = \sum_{n=1}^{\infty} 2na_n x^n = \sum_{n=0}^{\infty} 2na_n x^n, \]

and

\[
(1 - x^2)y'' = \sum_{n=2}^{\infty} (n(n - 1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n - 1)a_n x^n
\]

\[ = \sum_{n=0}^{\infty} [(n + 2)(n + 1)a_{n+2} - n(n - 1)a_n] x^n. \]
Substituting in (1), we obtain

\[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \alpha(\alpha + 1)a_n = 0, \quad n \geq 0,\]

which leads to a recurrence relation

\[a_{n+2} = -\frac{(\alpha - n)(\alpha + n + 1)}{(n + 1)(n + 2)}a_n.\]

Thus, we obtain

\[
\begin{align*}
a_2 &= -\frac{\alpha(\alpha + 1)}{1 \cdot 2} a_0, \\
a_4 &= -\frac{(\alpha - 2)(\alpha + 3)}{3 \cdot 4} a_2 = (-1)^2 \frac{\alpha(\alpha - 2)(\alpha + 1)(\alpha + 3)}{4!} a_0, \\
\vdots \\
a_{2n} &= (-1)^n \frac{\alpha(\alpha - 2) \cdots (\alpha - 2n + 2) \cdot (\alpha + 1)(\alpha + 3) \cdots (\alpha + 2n - 1)}{(2n)!} a_0.
\end{align*}
\]
Similarly, we can compute $a_3, a_5, a_7, \ldots$, in terms of $a_1$ and obtain

\[
    a_3 = -\frac{(\alpha - 1)(\alpha + 2)}{2 \cdot 3} a_1
\]

\[
    a_5 = -\frac{(\alpha - 3)(\alpha + 4)}{4 \cdot 5} a_3 = (-1)^2 \frac{(\alpha - 1)(\alpha - 3)(\alpha + 2)(\alpha + 4)}{5!} a_1
\]

\[
    \vdots
\]

\[
    a_{2n+1} = (-1)^n \frac{(\alpha - 1)(\alpha - 3) \cdots (\alpha - 2n + 1)(\alpha + 2)(\alpha + 4) \cdots (\alpha + 2n)}{(2n + 1)!} a_1
\]

Therefore, the series for $y(x)$ can be written as

\[
    y(x) = a_0 y_1(x) + a_1 y_2(x), \quad \text{where}
\]

\[
    y_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\alpha(\alpha-2)\cdots(\alpha-2n+2)(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)}{(2n)!} x^{2n}, \quad \text{and}
\]

\[
    y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{\alpha(\alpha-3)\cdots(\alpha-2n+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2n)}{(2n+1)!} x^{2n+1}.
\]
Note: The ratio test shows that \( y_1(x) \) and \( y_2(x) \) converges for \( |x| < 1 \). These solutions \( y_1(x) \) and \( y_2(x) \) satisfy the initial conditions

\[
y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1.
\]

Since \( y_1(x) \) and \( y_2(x) \) are independent, the general solution of the Legendre equation over \((-1, 1)\) is

\[
y(x) = a_0 y_1(x) + a_1 y_2(x)
\]

with arbitrary constants \( a_0 \) and \( a_1 \).
Observations

Case I. When $\alpha = 0$ or $\alpha = 2m$, we note that

$$\alpha(\alpha - 2) \cdots (\alpha - 2n + 2) = 2m(2m - 2) \cdots (2m - 2n + 2) = \frac{2^n m!}{(m - n)!}$$

and

$$(\alpha + 1)(\alpha + 3) \cdots (\alpha + 2n - 1) = \frac{(2m + 1)(2m + 3) \cdots (2m + 2n - 1)}{(2m + 2n)!}(2m)! = \frac{(2m + 2n)!m!}{2^n(2m)!(m + n)!}.$$ 

Then, in this case, $y_1(x)$ becomes

$$y_1(x) = 1 + \frac{(m!)^2}{(2m)!} \sum_{k=1}^{m} (-1)^k \frac{(2m + 2k)!}{(m - k)!(m + k)!(2k)!} x^{2k},$$

which is a polynomial of degree $2m$. In particular, for $\alpha = 0, 2, 4 (m = 0, 1, 2)$, the corresponding polynomials are

$$y_1(x) = 1, \quad 1 - 3x^2, \quad 1 - 10x^2 + \frac{35}{3}x^4.$$
Note that the series $y_2(x)$ is not a polynomial when $\alpha$ is even because the coefficients of $x^{2n+1}$ is never zero.

**Case II.** When $\alpha = 2m + 1$, $y_2(x)$ becomes a polynomial and $y_1(x)$ is not a polynomial.

In this case,

$$y_2(x) = x + \frac{(m!)^2}{(2m + 1)!} \sum_{k=1}^{m} (-1)^k \frac{(2m + 2k + 1)!}{(m - k)!(m + k)!(2k + 1)!} x^{2k+1}.$$  

For example, when $\alpha = 1, 3, 5$ ($m = 0, 1, 2$), the corresponding polynomials are

$$y_2(x) = x, \quad x - \frac{5}{3}x^3, \quad x - \frac{14}{3}x^3 + \frac{21}{5}x^5.$$
The Legendre polynomial

Let

\[ P_n(x) = \frac{1}{2^n} \sum_{r=0}^{[n/2]} \frac{(-1)^r (2n - 2r)!}{r!(n - r)!(n - 2r)!} x^{n-2r}, \]

where \([n/2]\) denotes the greatest integer \(\leq n/2\).

- When \(n\) is even, it is a constant multiple of the polynomial \(y_1(x)\).
- When \(n\) is odd, it is a constant multiple of the polynomial \(y_2(x)\).

The first five Legendre polynomials are

\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1) \]

\[ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x). \]
Figure: Legendre polynomial over the interval $[-1, 1]$
Rodrigues’s formula for the Legendre polynomials

Note that
\[
\frac{(2n - 2r)!}{(n - 2r)!} x^{n-2r} = \frac{d^n}{dx^n} x^{2n-2r} \quad \text{and} \quad \frac{1}{r!(n - r)!} = \frac{1}{n!} \binom{n}{r}.
\]

Thus, \( P_n(x) \) in (2) can be expressed as
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} x^{2n-2r}.
\]

When \([n/2] < r \leq n\), the term \(x^{2n-2r}\) has degree less than \(n\), so its \(n\)th derivative is zero. This gives
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^{2n-2r} = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,
\]
which is known as Rodrigues’ formula.
Properties of the Legendre polynomials $P_n(x)$

- For each $n \geq 0$, $P_n(1) = 1$. Moreover, $P_n(x)$ is the only polynomial which satisfies the Legendre equation

$$ (1 - x^2)y'' - 2xy' + n(n + 1)y = 0 $$

and $P_n(1) = 1$.

- For each $n \geq 0$, $P_n(-x) = (-1)^n P_n(x)$.

- $$ \int_{-1}^{1} P_n(x) P_m(x) \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases} $$
• If $f(x)$ is a polynomial of degree $n$, we have

$$f(x) = \sum_{k=0}^{n} c_k P_k(x), \text{ where}$$

$$c_k = \frac{2k + 1}{2} \int_{-1}^{1} f(x)P_k(x)dx.$$  

• It follows from the orthogonality relation that

$$\int_{-1}^{1} g(x)P_n(x)dx = 0$$

for every polynomial $g(x)$ with $\text{deg}(g(x)) < n$.  

*** End ***