Series Solution of Linear Ordinary Differential Equations

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Aim: To study methods for determining series expansions for solutions to linear ODE with variable coefficients.

In particular, we shall obtain

- the form of the series expansion,
- a recurrence relation for determining the coefficients, and
- the interval of convergence of the expansion.
Review of power series

A series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots,$$  \hspace{1cm} (1)

is called a power series about the point $x_0$. Here, $x$ is a variable and $a_n$’s are constants.

The series (1) converges at $x = c$ if $\sum_{n=0}^{\infty} a_n(c - x_0)^n$ converges. That is, the limit of partial sums

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n(c - x_0)^n < \infty.$$ 

If this limit does not exist, the power series is said to diverge at $x = c$. 

Note that \( \sum_{n=0}^{\infty} a_n(x-x_0)^n \) converges at \( x = x_0 \) as
\[
\sum_{n=0}^{\infty} a_n(x_0 - x_0)^n = a_0.
\]

Q. What about convergence for other values of \( x \)?

**Theorem: (Radius of convergence)**
For each power series of the form (1), there is a number \( R \) (0 \( \leq \) \( R \) \( \leq \) \( \infty \)), called the radius of convergence of the power series, such that the series converges absolutely for \( |x - x_0| < R \) and diverge for \( |x - x_0| > R \).

If the series (1) converges for all values of \( x \), then \( R = \infty \).
When the series (1) converges only at \( x_0 \), then \( R = 0 \).
Theorem: (Ratio test) If

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L, \]

where \( 0 \leq L \leq \infty \), then the radius of convergence (R) of the power series \( \sum_{n=0}^{\infty} a_n(x - x_0)^n \) is

\[ R = \begin{cases} \frac{1}{L} & \text{if } 0 < L < \infty, \\ \infty & \text{if } L = 0, \\ 0 & \text{if } L = \infty. \end{cases} \]

Remark. If the ratio \( \left| \frac{a_{n+1}}{a_n} \right| \) does not have a limit, then methods other than the ratio test (e.g. root test) must be used to determine \( R \).
Example: Find $R$ for the series $\sum_{n=0}^{\infty} \frac{(-2)^n}{n+1} (x - 3)^n$.

Note that $a_n = \frac{(-2)^n}{n+1}$. We have

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-2)^{n+1}(n+1)}{(-2)^n(n+2)} \right| = \lim_{n \to \infty} \frac{2(n+1)}{(n+2)} = 2 = L.$$ 

Thus, $R = 1/2$. The series converges absolutely for $|x - 3| < \frac{1}{2}$ and diverge for $|x - 3| > \frac{1}{2}$.

Next, what happens when $|x - 3| = 1/2$?

At $x = 5/2$, the series becomes the harmonic series $\sum_{n=0}^{\infty} \frac{1}{n+1}$, and hence diverges. When $x = 7/2$, the series becomes an alternating harmonic series, which converges.

Thus, the power series converges for each $x \in (5/2, 7/2]$. 
Given two power series

\[ f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \quad g(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n, \]

with nonzero radii of convergence. Then

\[ f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n \]

has common interval of convergence. The formula for the product is

\[ f(x)g(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n, \quad \text{where} \quad c_n := \sum_{k=0}^{n} a_k b_{n-k}. \]  

(2)

This power series in (2) is called the \textbf{Cauchy product} and will converge for all \( x \) in the common interval of convergence for the power series of \( f \) and \( g \).
Differentiation and integration of power series

**Theorem:** If \( f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \) has a positive radius of convergence \( R \), then \( f \) is differentiable in the interval \( |x - x_0| < R \) and termwise differentiation gives the power series for the derivative:

\[
f'(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1} \quad \text{for} \quad |x - x_0| < R.
\]

Furthermore, termwise integration gives the power series for the integral of \( f \):

\[
\int f(x) \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1} + C \quad \text{for} \quad |x - x_0| < R.
\]
Example: A power series for

\[ \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots. \]

Since \( \frac{d}{dx} \left\{ \frac{1}{1 - x} \right\} = \frac{1}{(1-x)^2} \), we obtain a power series for

\[ \frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots. \]

Since \( \tan^{-1} x = \int_0^x \frac{1}{1+t^2} \, dt \), integrate the series for \( \frac{1}{1+x^2} \) termwise to obtain

\[ \tan^{-1} x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \cdots + \frac{(-1)^n x^{2n+1}}{2n + 1} + \cdots. \]
Shifting the summation index

The index of a summation in a power series is a dummy index and hence

\[ \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{k=0}^{\infty} a_k(x - x_0)^k = \sum_{i=0}^{\infty} a_i(x - x_0)^i. \]

Shifting the index of summation is particularly important when one has to combine two different power series.

Example:

\[ \sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2} = \sum_{k=0}^{\infty} (k + 2)(k + 1)a_{k+2}x^k. \]

\[ x^3 \sum_{n=0}^{\infty} n^2(n - 2)a_n x^n = \sum_{n=3}^{\infty} (n - 3)^2(n - 5)a_{n-3}x^n. \]
Definition: (Analytic function)
A function $f$ is said to be analytic at $x_0$ if it has a power series representation $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ in an neighborhood about $x_0$, and has a positive radius of convergence.

Example: Some analytic functions and their representations:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1}.$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x - 1)^n, \quad x > 0.$$
Power series solutions to linear ODEs

Consider linear ODE of the form:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \quad a_2(x) \neq 0. \quad (\star)$$

Writing in the standard form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where $p(x) := \frac{a_1(x)}{a_2(x)}$ and $q(x) := \frac{a_0(x)}{a_2(x)}$.

**Definition:** A point $x_0$ is called an ordinary point of $(\star)$ if both $p(x) = \frac{a_1(x)}{a_2(x)}$ and $q(x) = \frac{a_0(x)}{a_2(x)}$ are analytic at $x_0$. If $x_0$ is not an ordinary point, it is called a singular point of $(\star)$. 
Example: Find all the singular point points of

\[ xy''(x) + x(x - 1)^{-1}y'(x) + (\sin x)y = 0, \quad x > 0 \]

Here,

\[ p(x) = \frac{1}{(1 - x)}, \quad q(x) = \frac{\sin x}{x}. \]

Note that \( p(x) \) is analytic except at \( x = 1 \). \( q(x) \) is analytic everywhere as it has power series representation

\[ q(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots. \]

Hence, \( x = 1 \) is the only singular point of the given ODE.
Power series method about an ordinary point

Consider the equation

$$2y'' + xy' + y = 0. \quad (**)$$

Let’s find a power series solution about $x = 0$. Seek a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and then attempt to determine the coefficients $a_n$’s. Differentiate termwise to obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n - 1) a_n x^{n-2}. $$
Substituting these power series in (**), we find that

\[ \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0. \]

By shifting the indices, we rewrite the above equation as

\[ \sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2} x^k + \sum_{k=1}^{\infty} ka_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0. \]

Combining the like powers of \( x \) in the three summation to obtain

\[ 4a_2 + a_0 + \sum_{k=1}^{\infty} [2(k+2)(k+1)a_{k+2} + ka_k + a_k] x^k = 0. \]
Equating the coefficients of this power series equal to zero yields

\[ 4a_2 + a_0 = 0 \]
\[ 2(k + 2)(k + 1)a_{k+2} + (k + 1)a_k = 0, \quad k \geq 1. \]

This leads to the recurrence relation

\[ a_{k+2} = \frac{-1}{2(k + 2)}a_k, \quad k \geq 1. \]

Thus,

\[ a_2 = \frac{-1}{2^2}a_0, \quad a_3 = \frac{-1}{2 \cdot 3}a_1 \]
\[ a_4 = \frac{-1}{2 \cdot 4}a_2 = \frac{1}{2^2 \cdot 2 \cdot 4}a_0, \quad a_5 = \frac{-1}{2 \cdot 5}a_3 = \frac{1}{2^2 \cdot 3 \cdot 5}a_1 \]
\[ \ldots \]
\[ \ldots \]
With $a_0$ and $a_1$ as arbitrary constants, we find that

$$a_{2n} = \frac{(-1)^n}{2^{2n}n!}a_0, \quad n \geq 1,$$

and

$$a_{2n+1} = \frac{(-1)^n}{2^n[1 \cdot 3 \cdot 5 \cdots (2n + 1)]}a_1, \quad n \geq 1.$$

From this, we have two linearly independent solutions as

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}n!}x^{2n},$$

and

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n[1 \cdot 3 \cdot 5 \cdots (2n + 1)]}x^{2n+1}.$$
Hence the general solution is

\[ y(x) = a_0 y_1(x) + a_1 y_2(x). \]

**Remark.** Suppose we are given the value of \( y(0) \) and \( y'(0) \), then \( a_0 = y(0) \) and \( a_1 = y'(0) \). These two coefficients leads to a unique power series solution for the IVP.

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