Systems of First Order Differential Equations

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A first order system of $n$ (not necessarily linear) equations in $n$ unknown functions $x_1(t), x_2(t), \ldots, x_n(t)$ in normal form is given by

\[
\begin{align*}
x'_1(t) &= f_1(t, x_1, x_2, \ldots, x_n), \\
x'_2(t) &= f_2(t, x_1, x_2, \ldots, x_n), \\
&\vdots \\
x'_n(t) &= f_n(t, x_1, x_2, \ldots, x_n).
\end{align*}
\]

Higher-order differential equations often can be rewritten as first-order system. We can convert the $n$th order ODE

\[
y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}) \tag{1}
\]

into a first-order system as follows.
Setting

\[ x_1(t) := y(t), \ x_2(t) := y'(t), \ldots, \ x_n(t) := y^{(n-1)}(t). \]

we obtain \( n \) first-order equations:

\[
\begin{align*}
    x'_1(t) &= y'(t) = x_2(t), \\
    x'_2(t) &= y''(t) = x_3(t), \\
    &\vdots \\
    x'_{n-1}(t) &= y^{(n-1)}(t) = x_n(t), \\
    x'_n(t) &= y^{(n)}(t) = f(t, x_1, x_2, \ldots, x_n).
\end{align*}
\]

(2)

If (1) has \( n \) initial conditions:

\[
y(t_0) = \alpha_1, \ y'(t_0) = \alpha_2, \ldots, \ y^{(n-1)}(t_0) = \alpha_n,
\]

then the system (2) has initial conditions:

\[
x_1(t_0) = \alpha_1, \ x_2(t_0) = \alpha_2, \ldots, \ x_n(t_0) = \alpha_n.
\]
Example: $y''(t) + 3y'(t) + 2y(t) = 0; \quad y(0) = 1, \quad y'(0) = 3$.

Setting

$$x_1(t) := y(t) \quad \text{and} \quad x_2(t) := y'(t)$$

we obtain

$$x'_1(t) = x_2(t),$$
$$x'_2(t) = -3x_2(t) - 2x_1(t).$$

The ICs transform to $x_1(0) = 1, \ x_2(0) = 3$.

We shall consider only linear systems of first-order ODEs.
Consider the linear system in the normal form:

\[
\begin{align*}
    x'_1(t) &= a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t), \\
    x'_2(t) &= a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t), \\
    &\vdots \\
    x'_n(t) &= a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t).
\end{align*}
\]

In matrix and vector notations, we write it as

\[
x'(t) = A(t)x(t) + f(t),
\]

where \( x(t) = [x_1(t), \ldots, x_n(t)]^T \), \( f(t) = [f_1(t), \ldots, f_n(t)]^T \), and \( A(t) = [a_{ij}(t)] \) is a \( n \times n \) matrix.

When \( f = 0 \) the linear system (3) is said to be homogeneous.
Definition: The IVP for the system

\[ x'(t) = A(t)x(t) + f(t) \quad (4) \]

is to find a vector function \( x(t) \in C^1 \) that satisfies the system (4) on an interval \( I \) and the initial conditions \( x(t_0) = x_0 = (x_{1,0}, \ldots, x_{n,0})^T \), where \( t_0 \in I \) and \( x_0 \in \mathbb{R}^n \).

Theorem: (Existence and Uniqueness)

Let \( A(t) \) and \( f(t) \) are continuous on \( I \) and \( t_0 \in I \). Then, for any choice of \( x_0 = (x_{1,0}, \ldots, x_{n,0})^T \in \mathbb{R}^n \), there exists a unique solution \( x(t) \) to the IVP

\[ x'(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0 \]

on the whole interval \( I \).
Example: Consider the IVP:

\[ x'(t) = \begin{bmatrix} t^3 & \tan t \\ t & \sin t \end{bmatrix} x(t) + \begin{bmatrix} \sqrt{1-t} \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \]

This IVP has a unique solution on the interval \((-\pi/2, 1)\).

Definition: The Wronskian of \(n\) vector functions

\[ x_1(t) = (x_{1,1}, \ldots, x_{1,n})^T, \ldots, x_n(t) = (x_{1,n}, \ldots, x_{n,n})^T \]

is defined as

\[
W(x_1, \ldots, x_n)(t) := \begin{vmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{vmatrix} = \det [x_1 \ x_2 \ \ldots \ x_n].
\]
Theorem: Let $A(t)$ is an $n \times n$ matrix of continuous functions. If $x_1, x_2, \ldots, x_n$ are linearly independent solutions to $x'(t) = A(t)x$ on $I$, then $W(t) := \det[x_1 \ x_2 \ldots \ x_n] \neq 0$ on $I$.

Proof. Suppose $W(t_0) = 0$ at some point $t_0 \in I$. Now, $W(t_0) = 0 \implies x_1(t_0), x_2(t_0), \ldots, x_n(t_0)$ are L.D.
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$$c_1x_1(t_0) + c_2x_2(t_0) + \ldots + c_nx_n(t_0) = 0.$$
Theorem: Let $A(t)$ is an $n \times n$ matrix of continuous functions. If $x_1, x_2, \ldots, x_n$ are linearly independent solutions to $x'(t) = A(t)x$ on $I$, then $W(t) := \det[x_1 \ x_2 \ldots \ x_n] \neq 0$ on $I$.

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$$c_1x_1(t_0) + c_2x_2(t_0) + \ldots + c_nx_n(t_0) = 0.$$ 

Note that $c_1x_1(t) + c_2x_2(t) + \ldots + c_nx_n(t)$ and $z(t) = 0$ are both solutions to $x'(t) = A(t)x(t), \ x(t_0) = 0$ on $I$ and $\sum_{i=1}^{n} c_ix_i(t_0) = z(t_0) = 0$. By the existence and uniqueness theorem

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Theorem: Let $A(t)$ is an $n \times n$ matrix of continuous functions. If $x_1, x_2, \ldots, x_n$ are linearly independent solutions to $x'(t) = A(t)x$ on $I$, then $W(t) := \det[x_1 \ x_2 \ \ldots \ x_n] \neq 0$ on $I$.

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$$c_1x_1(t) + c_2x_2(t) + \ldots + c_nx_n(t) = 0, \ \forall t \in I$$

which contradicts to the fact that $x_1, \ldots, x_n$ are L.I. . Hence, $W(t_0) \neq 0$. Since $t_0 \in I$ is arbitrary, the result follows.
**Theorem:** (Abel’s formula)

If $x_1, \ldots, x_n$ are $n$ solutions to $x'(t) = A(t)x(t)$ on an interval $I$ and $t_0$ is any point of $I$, then for all $t \in I$,

$$W(t) = W(t_0) \exp \left( \int_{t_0}^{t} \left\{ \sum_{i=1}^{n} a_{ii}(s) \right\} ds \right),$$

where $a_{ii}$’s are the main diagonal elements of $A$.

**Proof:** Prove for $n = 3$. (See Theorem 11.12 of Ross’s book.)

**Fact:**

- The Wronskian of solutions to $x'(t) = A(t)x(t)$ is either zero or never zero on $I$.
- A set of $n$ solutions to $x'(t) = A(t)x(t)$ on $I$ is linearly independent on $I$ if and only if $W(x_1, \ldots, x_n)(t) \neq 0$ on $I$. 
Representation of Solutions

**Theorem:** (Homogeneous case)

Let \( x_1, \ldots, x_n \) be \( n \) linearly independent solutions to

\[
x'(t) = A(t)x(t), \quad t \in I,
\]

where \( A(t) \) is continuous on \( I \). Then, every solution to \( x'(t) = A(t)x(t) \) can be expressed in the form

\[
x(t) = c_1x_1(t) + \cdots + c_nx_n(t),
\]

where \( c_i \)'s are constants.

**Definition:** A set \( \{x_1, \ldots, x_n\} \) of \( n \) linearly independent solutions to

\[
x'(t) = A(t)x(t), \quad t \in I
\]

is called a **fundamental solution set** for \((*)\) on \( I \).
The matrix $\Phi(t)$ defined by

$$\Phi(t) := \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \\ x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{bmatrix}$$

is called a **fundamental matrix** for $(\ast)$.  

**Note:** 1. We can use $\Phi(t)$ to express the general solution $\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + \cdots + c_n \mathbf{x}_n(t) = \Phi(t) \mathbf{c}$, where $\mathbf{c} = (c_1, \ldots, c_n)^T$.  

2. Since $\det \Phi(t) = W(x_1, \ldots, x_n) \neq 0$ on $I \implies \Phi(t)$ is invertible for every $t \in I$. 
Theorem: There exists fundamental sets of solutions of the homogeneous linear system of differential equations

\[ \frac{d}{dt}x = A(t)x. \]
Example: The set \( \{ \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \} \), where

\[
\mathbf{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix},
\]

is a fundamental solution set for the system \( \mathbf{x}'(t) = A(t)\mathbf{x}(t) \) on \( \mathbb{R} \), where \( A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \).

Note that \( A\mathbf{x}_i(t) = \mathbf{x}'_i(t), \ i = 1, 2, 3. \) Further,

\[
W(t) = \begin{vmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{vmatrix} = -3 \neq 0.
\]
The fundamental matrix $\Phi(t) = \begin{bmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{bmatrix}$.

Thus, the GS is

$x(t) = \Phi(t)c = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}$. 
**Theorem:** (Non-homogeneous case)

Let $x_p$ be a particular solution to

$$x'(t) = A(t)x(t) + f(t), \quad t \in I,$$

and let $\{x_1, \ldots, x_n\}$ be a fundamental solution set on $I$ for the corresponding homogeneous system $x'(t) = A(t)x(t)$.

Then every solution to $(**)$ can be expressed in the form

$$x(t) = c_1x_1(t) + \cdots + c_nx_n(t) + x_p(t)$$

$$= \Phi(t)c + x_p(t).$$

*** End ***