Model dependence of transonic properties of accretion flows around black holes

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ABSTRACT

We analytically study how the behaviour of accretion flows changes when the flow model is varied. We study the transonic properties of the conical flow, a flow of constant height and a flow in vertical equilibrium, and show that all these models are basically identical, provided that the polytropic constant is suitably changed from one model to another. We show that this behaviour is extendible even when standing shocks are produced in the flow. The parameter space where shocks are produced remains roughly identical in all these models when the same transformation among the polytropic indices is used. We present applications of these findings.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – shock waves.

1 INTRODUCTION

Fully self-consistent study of any astrophysical system is generally prohibitive. Very often, for simplicity, it is necessary to construct models which have all the salient features of the original problem. However, these models need not be unique. In the present paper we make a pedagogical review of three different models of rotating accretion flows, and show that even though they are based on fundamentally different assumptions, they have identical physical properties. What is more, results of one model could be obtained from the other by changing a *physical* parameter, namely the polytropic constant. In other words, all these models are *identical*.

Accretion disc physics has undergone major changes in the last 50 years. Bondi (1952) studied spherical accretion and found the existence of only one saddle-type sonic point in an adiabatic flow. Later, for the Keplerian disc model of Shakura & Sunyaev (1973) and the thick disc model of Paczyński & Wiita (1980) the disc solutions became more realistic, although none of them was transonic, i.e., none was passing through any sonic point. Meanwhile, Liang & Thompson (1980) generalized this work for a flow which included angular momentum, and discovered that there could be three sonic points. Matsumoto et al. (1984) tried to let the flow pass through the inner sonic point only, and found that the flow could pass through nodal type sonic points.

Chakrabarti (1989, 1990, hereafter C89 and C90 respectively) studied transonic properties of accretion flows which are conical in shape in the meridional plane ('Wedge-shaped Flow'), and also flows which are in vertical equilibrium. Subsequently, Chakrabarti

& Molteni (1993) studied flows of constant height, and also verified by time-dependent numerical simulations that the flow indeed allows standing shocks in it. In a Bondi (1952) flow, to specify a solution one requires exactly one parameter, namely the specific energy $\mathcal{E}$ of the flow. This is in turn determined by the temperature of the flow at a large distance. In an inviscid, rotating axisymmetric accretion flow, one requires two parameters, namely specific energy $\mathcal{E}$ and specific angular momentum λ. Once they are specified, all the crucial properties of the flow, namely the locations of the sonic points, the shocks, as well as the complete global solution, are determined. C89 numerically studied the properties of the parameter space rather extensively, and divided the parameter space in terms of whether standing shocks can form or not. In the present paper we compare these models completely analytically and show, very interestingly, that one could easily 'map' one model on to another by suitably changing the polytropic index of the flow. In other words, we show that these models are roughly identical to one another as far as the transonic properties go.

In the next section we present a set of equations which govern the steady state flow in all the three models. In Section 3, we present the sonic point analysis, and provide the expressions for the energy of the flow in terms of the sonic points. We observe that these expressions are identical, provided that there is a unique relation among the polytropic indices of these model flows. In Section 4 we compare shock locations in all the three models. We also compare the parameter space which allows shock formation in these models with the regions obtained using purely numerical methods. In Section 5 we show that in fact if the relations between the polytropic indices are used, the shock locations in all these models are also roughly identical. Consequently, the apparently disjoint parameter spaces drawn with the same polytropic index overlap almost completely when the above-mentioned relations among polytropic indices is used. This remarkable behaviour

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shows an underlying unity in these apparently diverse models. Finally, in Section 6 we draw our conclusions.

2 MODEL EQUATIONS

As discussed in the introduction, we shall be concerned with three axisymmetric and inviscid models: (a) *Model H*: the flow has a constant height everywhere; (b) *Model C*: the flow cross-section in meridional plane is conical in shape, and (c) *Model V*: the flow is in equilibrium in the transverse direction. Fig. 1 shows a cartoon diagram of these three models. The filled circle at the centre corresponds to the black hole. The lightly shaded region of the disc corresponds to the pre-shock flow, while the heavily shaded region corresponds to the post-shock flow. We also assume that the distances are measured in units of $r_S = 2GM_{BH}/c^2$, where G is the gravitational constant, c is the velocity of light, and M_{BH} is the mass of the black hole. Velocities and angular momenta are measured in units of c and $cr_S = 2GM_{BH}/c$ respectively. In all the three models, the dimensionless energy conservation law can be written as

$$\mathcal{E} = \frac{\vartheta_e^2}{2} + \frac{a_e^2}{\gamma - 1} + \frac{\lambda^2}{2x^2} + g(x), \tag{1}$$

where $g(x)$ is the pseudo-Newtonian potential introduced by Paczyński & Wiita (1980) and is given by $g(x) = -\frac{1}{2(x-1)}$. Here ϑ_e and $a_e = \sqrt{\gamma p/\rho}$ are the non-dimensional radial and the sound velocities respectively, x is the non-dimensional radial distance, and the subscript e refers to the quantities measured on the equatorial plane. The flow has been chosen to be adiabatic with equation of state, $P = K\rho^\gamma$, where K is a constant which measures the entropy of the flow, and γ is the polytropic exponent. The energy equation is the integral form of the radial momentum balance equation.

The mass flux conservation equation, which comes directly from the continuity equation, depends on specific geometry of the models. Apart from a geometric constant, the conservation equation is given by,

$$\dot{M} = \vartheta_e \rho a^\zeta x^\beta (x-1)^\delta, \tag{2}$$

where β, ζ and δ are constants. For Model V (see C89), $\beta = 3/2$, $\zeta = 1$, $\delta = 1$. For Model C (see C90), $\beta = 2$, $\zeta = 0$, $\delta = 0$. For Model H (Chakrabarti 1992; Chakrabarti & Molteni 1993), $\beta = 1$, $\zeta = 0$, $\delta = 0$. Note that since the local disc height $h(x)$ depends on sound speed, $h(x) \sim a_e x^{1/2}(x-1)$, and so a factor of a_e^ζ is applicable for this model.
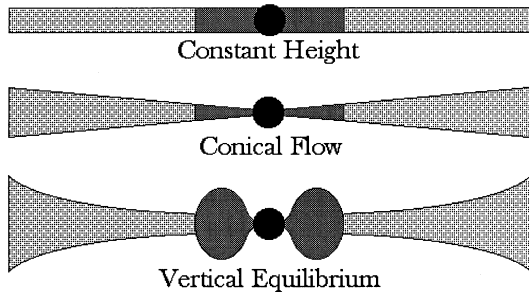
Figure 1. Cartoon diagram of three different models discussed in the text. In a constant-height flow (H), the disc thickness is constant (top). In a conical flow (C), the cross-section in the meridional plane is conical (middle). In a vertical equilibrium flow (V), matter is locally in vertical equilibrium at every point of the disc (bottom).

Although it is customary to deal with the conserved mass accretion rate of the flow, since we incorporate shock formation where entropy is increased, it is more convenient to rewrite the mass flux conservation equation in terms of ϑ_e and a_e in the following way:

$$\dot{\mathcal{M}} = \vartheta_e a_e^\alpha x^\beta (x-1)^\delta = \vartheta_e a_e^\alpha f(x), \tag{3}$$

where $\alpha = 2n + \zeta$, $\alpha = 2n$ and $\alpha = 2n$ for Models V, C and H respectively, and $f(x) = x^\beta(x-1)^\delta$. We shall use the phrase 'entropy-accretion rate' for the quantity $\dot{\mathcal{M}} = \dot{M}K^n\gamma^n$. In a flow without a shock this quantity remains constant, but in the presence of a shock it changes because of the generation of entropy.

3 SONIC POINT ANALYSIS AND RELATION BETWEEN MODELS

Since the flow is expected to be subsonic at a long distance and supersonic on the horizon, the flow must pass through sonic points. At the sonic point a few conditions are to be satisfied. They can be derived in the following way.

First, we differentiate the energy equation and the mass conservation equation, and eliminate da/dx from them to obtain

$$\frac{d\vartheta}{dx} = \frac{\frac{2na^2}{\alpha}\left[\frac{\beta'x - \beta}{x(x-1)}\right] - \frac{dG}{dx}}{\left(\vartheta - \frac{2na^2}{\alpha\vartheta}\right)}. \tag{4}$$

Here, $G(x) = \frac{\lambda^2}{2x^2} - \frac{1}{2(x-1)}$ is the effective potential and $\beta' = \beta + \delta$. Since the flow is assumed to be smooth everywhere if at any point of the flow denominator vanishes, the numerator must also vanish there. The vanishing of the denominator gives

$$\vartheta_c^2(x_c) = \frac{2n}{\alpha}a_c^2(x_c). \tag{5}$$

The vanishing of the numerator gives

$$a_c^2(x_c) = \frac{\alpha(x_c - 1)}{2nx_c^2}\frac{[\lambda_K^2(x_c) - \lambda^2]}{(\beta'x_c - \beta)}. \tag{6}$$

The subscript c denotes quantities at the critical points. Here, λ_K is the Keplerian angular momentum defined as $\lambda_K^2 = x_c^3/[2(x_c - 1)^2]$. It is to be noted that since the square of the sound speed (equation 6) is always positive, the angular momentum at the sonic point must be sub-Keplerian, i.e., $\lambda < \lambda_K$. When the above expression for the velocity of sound is inserted in the expression for the specific energy, we get, for the *Vertical Equilibrium (V) Model*,

$$\mathcal{E}_V = \frac{n_V + 1}{5}\frac{x_c}{(x_c - 3/5)(x_c - 1)}$$

$$- \left[\frac{4(n_V + 1)}{5}\frac{(x_c - 1)}{x_c - 3/5} - 1\right]\frac{\lambda^2}{2x_c^2} - \frac{1}{2(x_c - 1)}; \tag{7a}$$

for *the Conical Flow (C) Model*,

$$\mathcal{E}_C = \frac{2n_C + 1}{8}\frac{x_c}{(x_c - 1)^2} - \frac{2n_C - 1}{2}\frac{\lambda^2}{2x_c^2} - \frac{1}{2(x_c - 1)}, \tag{7b}$$

and *for Constant Height Flow (H) Model*,

$$\mathcal{E}_H = \frac{2n_H + 1}{4}\frac{x_c}{(x_c - 1)^2} - 2n_H\frac{\lambda^2}{2x_c^2} - \frac{1}{2(x_c - 1)}. \tag{7c}$$

Here, we have used the subscripts V, C and H under specific energy

and polytropic index n to denote specific models. For a given angular momentum and at the same sonic point, the energy expression will be the same, provided that

$$\frac{2n_C + 1}{8} = \frac{2n_H + 1}{4} = \frac{n_V + 1}{5}, \tag{8}$$

where we have used

$$(x_c - 1)/(x_c - 3/5) \sim 1 \tag{8a}$$

for Model V.

The relations in equation (8) are very important. If these relations are satisfied, then transonic properties of Model C with polytropic index n_C would be identical to those of Model H with index n_H and those of Model V with index n_V respectively.

4 SHOCK INVARIANTS AND LOCATIONS IN DIFFERENT MODELS

In between two sonic points, the flow can undergo a standing shock transition. For an inviscid flow, at the shock, a set of conditions are to be satisfied. These are known as the Rankine–Hugoniot conditions (Landau & Lifshitz 1959). These conditions are different for different models (C89; C90). For Model V, the shock conditions are as follows: the energy flux conservation equation,

$$\mathcal{E}_+ = \mathcal{E}_-, \tag{9a}$$

the pressure balance condition,

$$W_+ + \Sigma_+ \vartheta_{e+}^2 = W_- + \Sigma_- \vartheta_{e-}^2, \tag{9b}$$

and the baryon flux conservation equation,

$$\dot{M}_+ = \dot{M}_-, \tag{9c}$$

where subscripts '−' and '+' refer, respectively, to quantities before and after the shock. Here, W and Σ denote the pressure and the density, integrated in the vertical direction (see, e.g., Matsumoto et al. 1984), i.e.,

$$\Sigma = \int_{-h}^{h} \rho \, dz = 2\rho_e I_n h, \tag{10a}$$

and

$$W = \int_{-h}^{h} P \, dz = 2P_e I_{n+1} h, \tag{10b}$$

where, $I_n = \frac{(2^n n!)^2}{(2n+1)!}$, n being the polytropic index as defined previously.

For Models C and H, the shock conditions are as follows: the energy flux conservation equation,

$$\mathcal{E}_+ = \mathcal{E}_-, \tag{11a}$$

the pressure balance condition,

$$P_+ + \rho_+ \vartheta_{e+}^2 = P_- + \rho_- \vartheta_{e-}^2, \tag{11b}$$

and the baryon number conservation equation,

$$\dot{M}_+ = \dot{M}_-. \tag{11c}$$

The subscripts '−' and '+' have the same interpretation as before. Here, P and ρ denote the local pressure and the local density. In the subsequent analysis we drop the subscript e if no confusion arises in doing so.

The expressions for the conserved quantities could be combined to obtain the so-called Mach number relation, which must be satisfied at the shock. For Model V, we obtain this relation as follows. We rewrite the energy conservation equation (9a) and the pressure balance equation (9b) in terms of the Mach number $M = \vartheta/a$ of the flow:

$$\frac{1}{2} M_+^2 a_+^2 + \frac{a_+^2}{\gamma - 1} = \frac{1}{2} M_-^2 a_-^2 + \frac{a_-^2}{\gamma - 1}, \tag{12a}$$

$$\dot{\mathcal{M}}_+ = M_+ a_+^{\nu'} f(x_s), \tag{12b}$$

$$\dot{\mathcal{M}}_- = M_- a_-^{\nu'} f(x_s), \tag{12c}$$

where $\nu' = \frac{2\gamma}{\gamma-1}$, and

$$\frac{a_+^\nu}{\dot{\mathcal{M}}_+} \left(\frac{2}{3\gamma - 1} + M_+^2 \right) = \frac{a_-^\nu}{\dot{\mathcal{M}}_-} \left(\frac{2}{3\gamma - 1} + M_-^2 \right), \tag{12d}$$

where, $\nu = \frac{3\gamma-1}{\gamma-1}$, and x_s is the location of the shock. $f(x_s) = x_s^{3/2}(x_s - 1)$ is the term in accretion rate which is explicitly a function of x and is the same both before and after the shock. From equations 12(a)–(d) one obtains the following equation relating the pre- and post-shock Mach numbers of the flow of Model V at the shock (C89a):

$$C = \frac{[M_+(3\gamma - 1) + (2/M_+)]^2}{2 + (\gamma - 1)M_+^2}$$

$$= \frac{[M_-(3\gamma - 1) + (2/M_-)]^2}{2 + (\gamma - 1)M_-^2}. \tag{13}$$

The constant C is invariant across the shock. The Mach number of the flow just before and after the shock can be written down in terms of C as

$$M_{\mp}^2 = \frac{2(3\gamma - 1) - C \pm \sqrt{C^2 - 8C\gamma}}{(\gamma - 1)C - (3\gamma - 1)^2}. \tag{14}$$

The product of the Mach number is given by

$$M_+ M_- = -\frac{2}{[(3\gamma - 1)^2 - (\gamma - 1)C]^{1/2}}. \tag{15}$$

Similarly, one can obtain the Mach-number relations and the expression for Mach numbers for the other two models. The relation between the pre- and the post-shock Mach numbers of the flow at the shock for Models C and H are given by,

$$C = \frac{[\gamma M_+ + (1/M_+)]^2}{2 + (\gamma - 1)M_+^2} = \frac{[\gamma M_- + (1/M_-)]^2}{2 + (\gamma - 1)M_-^2}. \tag{16}$$

So far, in the literature, analytical shock studies have been carried out in models of vertical equilibrium (Das, Chattopadhyay & Chakrabarti 2001) by using the Mach-invariant relations (equation 13) when two parameters, namely the specific energy and the specific angular momentum, are given. We carried out the same analysis using equation (16) for Models H and C respectively, and obtained shock locations and parameter space boundaries for all the three models. Fig. 2 compares these results where plots of specific energy (y-axis) is given as function of specific angular momentum (x-axis). Solid boundaries mark regions for which standing shocks form in different models. Shaded regions are obtained from the analytical method (Das et al. 2001), and the results of these two methods roughly agree. We note that constant height flows occupy much a larger region than that of the conical or vertical equilibrium.
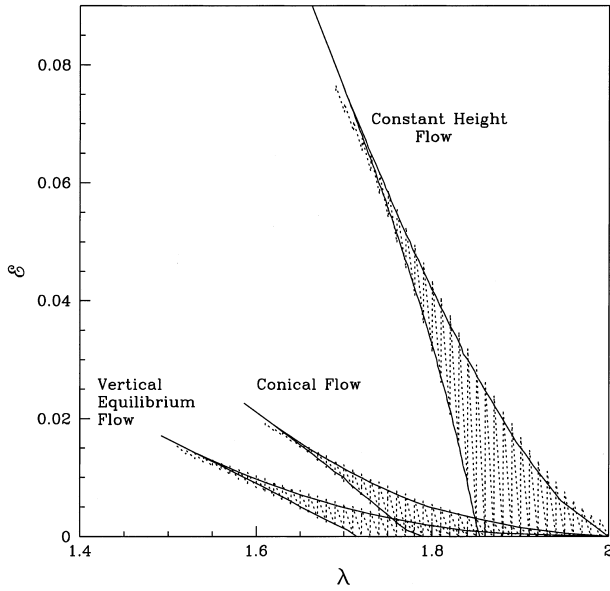
Figure 2. Comparison of the parameter space in three different models in which shocks form. $n_V = n_H = n_C = 3$ is chosen throughout. Solid boundaries are obtained using the numerical method, and shaded regions are obtained using the analytical method.

# 5 RESULTS OF MAPPING OF ONE MODEL TO ANOTHER

We have already noticed that one could use a relation (equation 8) which maps one model on to another, as far as the transonic properties go. If, for instance, we choose $n_H = 3$, we find that $n_V = 31/4$ and $n_C = 13/2$ respectively. This means that for a given energy and angular momentum a model H flow of polytropic index 3 would have sonic points exactly at the same place as a Model V flow of polytropic index 31/4 and Model C flow of polytropic index 13/2 respectively. Corresponding polytropic exponents are $\gamma_H = 4/3$, $\gamma_V = 1.129$ and $\gamma_C = 1.15385$ respectively. In physical terms, a relativistic flow of constant height would have same properties as more or less isothermal flows in a conical flow and a flow in vertical equilibrium.

What about the shock locations? In Fig. 3 we compare the locations of the standing shocks around a black hole in these three models. Solid curves are for Model V, small circles are for Model H, and crosses are for Model C. The polytropic indices n_V, n_C and n_H are as above. We note that although the models are different, the shock locations are also remarkably close to one another. In Fig. 4 a comparison of the parameter space is shown once more (cf. Fig. 2). However, the polytropic indices are chosen as above. Unlike disjoint regions in Fig. 2, we find that the regions are almost completely overlapping when equation (8) was used. This also shows that the equation (8) is valid even for the study of shock waves. We thus believe that, generally speaking, the three models are identical when the equation (8) is taken into account.

What could be possible applications of the pedagogical exercise we carried out? One could imagine that certain models are easier to study (say, using numerical simulations) than the others. For instance, Chakrabarti & Molteni (1993), Molteni, Gerardi & Chakrabarti (1994) and Chakrabarti & Molteni (1995) studied constant-height discs using smoothed particle hydrodynamics. This was done because a flow in vertical equilibrium cannot be forced on a time-dependent study. However, one could question
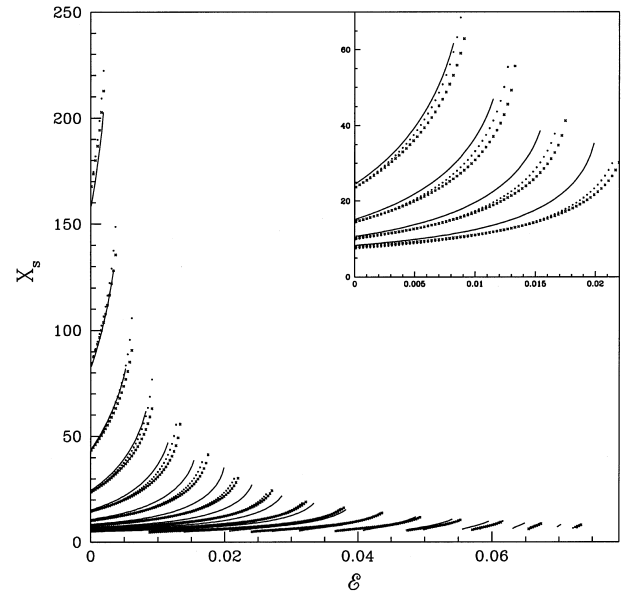
Figure 3. Variations of shock locations (y-axis) as functions of the specific energy (x-axis) and angular momenta. The leftmost curve is drawn for $\lambda = 2$, and other curves are for decreasing angular momentum with an interval of $\delta\lambda = 0.02$. Solid curves, filled circles and crosses are drawn for Models V, H and C respectively with $n_V = 31/4$, $n_C = 13/2$ and $n_H = 3$, which obey equation (8).
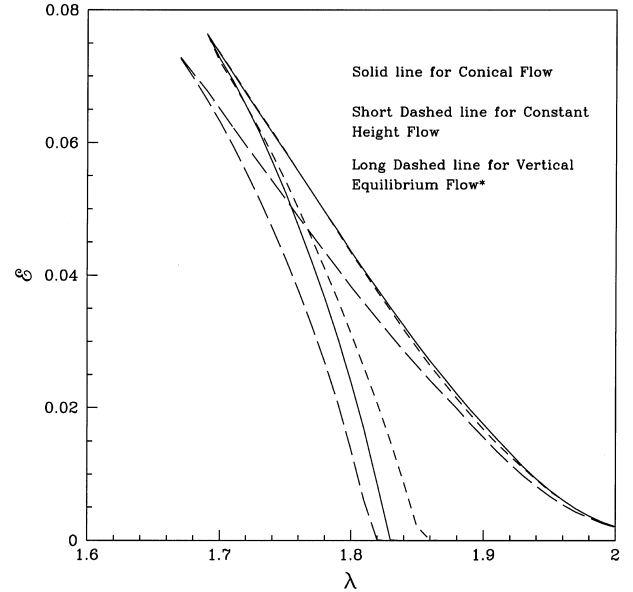
Figure 4. Nature of the boundary of the parameter space for the three models of the accretion flows. Solid, short-dashed, and long-dashed curves are drawn for Models C, H and V respectively with polytropic indices $n_V = 31/4$, $n_C = 13/2$ and $n_H = 3$ which obey equation (8). The roughly similar parameter space shows that the mapping of the indices based on the transonic properties remain roughly the same even when standing shocks are considered. The asterisk mark on 'Vertical Equilibrium Flow' indicates that condition (8a) has been utilized.

whether one can draw any conclusion about the behaviour of flows in vertical equilibrium using a simulation of constant height. Our present study shows that it does. Since the three models are shown to be identical, running a simulation for one model would give results for other models in a straightforward manner. Similarly, a

study of the stability of a model of constant height may be simpler, and stability of one model would imply stability of others.

6 CONCLUDING REMARKS

In this paper we discovered a unique relation among the polytropic indices of three different models of the axisymmetric accretion flows which ensures identical transonic properties in the sense that if all these models have the same conserved energies and angular momenta, then the sonic points also form exactly at the same place. When we proceeded further to compute the shock locations, we found that even the shocks form roughly at the same places. Apparently, disjoint parameter spaces for shock formation with the same value of polytropic index in three different models exhibit considerable overlap when the same unique relation (equation 8) was used. This shows that the models are virtually identical in properties, and various disc models belong to a one-parameter family. Our finding has given some insight into the relation between the nature of a flow with its equation of state. It seems that the relativistic equation of state in a flow in vertical equilibrium behaves similarly to a roughly isothermal flow in a disc of constant height or in a conical flow. It is possible that in the latter models (Models C and H) the geometric compression is smaller, and hence it is easier to keep them roughly isothermal while conserving energy as well.

Although our work has been mainly pedagogical, we believe that it could have several applications. For instance, linear and non-linear stability analysis and time-dependent calculations (numerical simulations) are easier to perform when the disc is of constant thickness. Our work indicates that once certain properties regarding stability are established in one flow model, they would remain valid in other models also, provided that the relation among the polytropic indices is incorporated.

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