We know that $\sqrt{2}$ is not a rational number, but we can find rational numbers as close as we wish to $\sqrt{2}$. For instance, the sequence of rationals

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

seem to get closer and closer to $\sqrt{2}$, as their squares indicate:

$$1.96, 1.9881, 1.99396, 1.99996164, 1.9999899241, \ldots$$

Thus it seems that we can create a square root of 2 by taking a “limit” of a sequence of rational numbers. This is how we shall construct the reals.

**Definition 1** (Sequence). A sequence of real numbers or a sequence in $\mathbb{R}$ is a mapping $f : \mathbb{N} \to \mathbb{R}$. We write $x_n$ for $f(n), n \in \mathbb{N}$ and it is customary to denote a sequence as $\langle x_n \rangle$ or $(x_n)$ or $\{x_n\}$.

**Example 1.** There are different ways of expressing a sequence. For example:

1. (Constant sequence): $(a, a, a, \ldots)$, where $a \in \mathbb{R}$
2. (Sequence defined by listing): $(1, 4, 8, 11, 52, \ldots)$
3. (Sequence defined by rule): $(x_n)$, where $x_n = 3n^2$ for all $n \in \mathbb{N}$
4. (Sequence defined recursively): $(x_n)$, where $x_1 = 4$ and $x_{n+1} = 2x_n - 5$ for all $n \in \mathbb{N}$

**Convergence:** What does it mean?

Think of the examples:

1. $(2, 2, 2, \ldots)$
2. $(\frac{1}{n})$
3. $((-1)^n \frac{1}{n})$
4. $(1, 2, 1, 2, \ldots)$
5. $(\sqrt{n})$
6. $((-1)^n(1 - \frac{1}{n}))$
7. $(n^2 - 1)$

**Definition 2** (Convergent sequence). A sequence $(x_n)$ is said to be convergent if there exists $\ell \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying $|x_n - \ell| < \varepsilon$ for all $n \geq n_0$. We say that $\ell$ is a limit of $(x_n)$. 
Notation: We write \( \lim_{n \to \infty} x_n = \ell \) or \( x_n \to \ell \).

Theorem 1. Limit of a convergent sequence is unique.

Proof. Let \((x_n)\) be a convergent sequence. Assume that \( \lim_{n \to \infty} x_n = \ell_1 \) and \( \lim_{n \to \infty} x_n = \ell_2 \). We claim that \( \ell_1 = \ell_2 \). To see this, by way of contradiction assume that \( \ell_1 \neq \ell_2 \). Then \( \varepsilon = \frac{|\ell_1 - \ell_2|}{3} > 0 \). By definition of convergence of a sequence, there are positive integers \( n_1 \) and \( n_2 \) such that

\[
|x_n - \ell_1| < \varepsilon \quad \text{for all} \quad n \geq n_1 \quad \text{and} \quad |x_n - \ell_2| < \varepsilon \quad \text{for all} \quad n \geq n_2.
\]

Let \( n_0 = \max\{n_1, n_2\} \). Then

\[
|x_n - \ell_1| < \varepsilon \quad \text{and} \quad |x_n - \ell_2| < \varepsilon \quad \text{for all} \quad n \geq n_0.
\]

Using triangle inequality, we have

\[
3\varepsilon = |\ell_1 - \ell_2| = |\ell_1 - x_{n_0} + x_{n_0} - \ell_2| \leq |\ell_1 - x_{n_0}| + |x_{n_0} - \ell_2| < 2\varepsilon,
\]

which is a contradiction. Hence, we must have \( \ell_1 = \ell_2 \). \( \square \)

Example 2. Using the definition of convergence of a sequence, show that \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

Solution. Let \( \varepsilon > 0 \). By using the Archimedean property, we can find a positive integer \( n_0 \) such that \( n_0 \cdot \varepsilon > 1 \), that is, \( \varepsilon > \frac{1}{n_0} \). Now, for all \( n \geq n_0 \), we have

\[
|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.
\]

This proves that \( \lim_{n \to \infty} \frac{1}{n} = 0 \). \( \square \)

Example 3. Consider the sequence \((x_n)\) where \( x_n = (-1)^n \). The terms of the sequence are \(-1, 1, -1, 1, -1, 1, \ldots\). It is intuitively clear that this sequence does not approach to any real number. Therefore, the sequence does not converge. We now establish this fact by using the definition.

Solution. By way of contradiction assume that the given sequence converges to \( \ell \). Then for \( \varepsilon = \frac{1}{2} \), there exists a natural number \( n_0 \) such that

\[
|(-1)^n - \ell| < \frac{1}{2} \quad \text{for all} \quad n \geq n_0.
\]

This gives \( |1 - \ell| < \frac{1}{2} \) and \( |1 + \ell| < \frac{1}{2} \). Now, using triangle inequality we have

\[
2 = |1 + 1| = |1 - \ell + \ell + 1| \leq |1 - \ell| + |1 + \ell| < 1,
\]

which is a contradiction. This proves that \(((-1)^n)\) does not converge. \( \square \)

Example 4. If \( |\alpha| < 1 \), then the sequence \((\alpha^n)\) converges to 0.
Solution. If $\alpha = 0$, then $\alpha^n = 0$ for all $n \in \mathbb{N}$ and so $(\alpha^n)$ converges to 0. Now we assume that $\alpha \neq 0$. Since $|\alpha| < 1$, $\frac{1}{|\alpha|} > 1$ and so $\frac{1}{|\alpha|} = 1 + h$ for some $h > 0$. For all $n \in \mathbb{N}$, we have $(1 + h)^n = 1 + nh + \frac{n(n-1)h^2}{2!} + \cdots + h^n > nh \Rightarrow |\alpha|^n = \frac{1}{(1+h)^n} < \frac{1}{nh}$ for all $n \in \mathbb{N}$.

Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ satisfying $n_0 > \frac{1}{h\varepsilon}$. Then $|\alpha^n - 0| = |\alpha|^n < \frac{1}{n_0h} < \varepsilon$ for all $n \geq n_0$ and hence $(\alpha^n)$ converges to 0.

Alternative proof: Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ satisfying $n_0 > \frac{\log \varepsilon}{\log |\alpha|}$. Then for all $n \geq n_0$, we have $|\alpha^n - 0| = |\alpha|^n \leq |\alpha|^{n_0} < \varepsilon$ and hence $(\alpha^n)$ converges to 0.

Bounded sequence: Given a sequence $(x_n)$, we can ask whether the set $\{x_1, x_2, x_3, \ldots\}$ is bounded or not. If this set is bounded then we call that the sequence $(x_n)$ is bounded. Equivalently, the sequence $(x_n)$ is bounded if there is a positive number $M$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. If $(x_n)$ is not bounded then it is said to be unbounded. For example, $(a), ((-1)^n), \left(\frac{1}{n}\right)$ are bounded sequences; whereas $(n^2)$ and $(2\sqrt{n})$ are unbounded sequences.

**Theorem 2.** Every convergent sequence is bounded.

**Proof.** Let $(x_n)$ be a convergent sequence and let $\lim_{n \to \infty} x_n = \ell$. By taking $\varepsilon = 1$, we find a positive integer $n_0$ such that $|x_n - \ell| < 1$ for all $n \geq n_0$. Equivalently,

$$\ell - 1 < x_n < \ell + 1 \quad \text{for all} \quad n \geq n_0.$$ 

Thus, the set $\{x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots\}$ is bounded from below by $\ell - 1$ and bounded from above by $\ell + 1$. Using the triangle inequality, we have

$$|x_n| = |x_n - \ell + \ell| \leq |x_n - \ell| + |\ell| < 1 + |\ell| \quad \text{for all} \quad n \geq n_0.$$ 

Now, there are only finitely many elements left, namely $x_1, x_2, \ldots, x_{n_0-1}$. Let

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{n_0-1}|, |\ell| + 1\}.$$ 

Then we have $|x_n| \leq M$ for all $n \geq 1$. Hence, $(x_n)$ is a bounded sequence. \(\square\)

**Remark 1.** From the above theorem, it follows that if a sequence is not bounded then it is not convergent. For example, the sequence $(\sqrt{n})$ is unbounded and hence is not convergent. However, every bounded sequence is not convergent. For example, $((−1)^n)$ is a bounded sequence but it does not converge.

**Limit rules for convergent sequences:**

**Theorem 3.** Let $x_n \to x$ and $y_n \to y$. Then

(a) $x_n + y_n \to x + y$.

(b) $\alpha x_n \to \alpha x$ for all $\alpha \in \mathbb{R}$.

(c) $|x_n| \to |x|$.

(d) $x_n y_n \to xy$.

(e) $\frac{x_n}{y_n} \to \frac{x}{y}$ if $y_n \neq 0$ for all $n \in \mathbb{N}$ and $y \neq 0$. 
Example 5. The sequence \((\frac{2n^2 - 3n}{3n^2 + 5n + 3})\) is convergent with limit \(\frac{2}{3}\).

Solution. We have \(\frac{2n^2 - 3n}{3n^2 + 5n + 3} = \frac{2 - \frac{3}{n}}{3 + \frac{5}{n} + \frac{3}{n^2}}\) for all \(n \in \mathbb{N}\). Since \(\frac{1}{n} \to 0\), the limit rules for algebraic operations on sequences imply that the given sequence is convergent with limit \(\frac{2}{3 + 0 + 0} = \frac{2}{3}\). \(\Box\)

Example 6. The sequence \((\sqrt{n + 1} - \sqrt{n})\) is convergent with limit 0.

Solution. For all \(n \in \mathbb{N}\), \(\sqrt{n + 1} - \sqrt{n} = \frac{1}{\sqrt{n + 1} + \sqrt{n}} = \frac{1}{\sqrt{1 + \frac{1}{n} + 1}}\). Since \(\frac{1}{n} \to 0\), the limit rules for algebraic operations on sequences imply that the given sequence is convergent with limit \(\frac{0}{\sqrt{1 + 0 + 1}} = 0\). \(\Box\)

Example 7. Consider the sequence \((\frac{\cos n}{n})\). Since \(-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}\) for all \(n \in \mathbb{N}\), so by applying Sandwich theorem we find that \(\lim_{n \to \infty} \frac{\cos n}{n} = 0\).

Example 8. If \(\alpha > 0\), then the sequence \((\alpha^\frac{1}{n})\) converges to 1.

Solution. We first assume that \(\alpha \geq 1\) and let \(x_n = \alpha^\frac{1}{n} - 1\) for all \(n \in \mathbb{N}\). Then \(x_n \geq 0\) and \(\alpha = (1 + x_n)^n = 1 + nx_n + \frac{n(n-1)}{2!}x_n^2 + \cdots + x_n^n > nx_n\) for all \(n \in \mathbb{N}\). So \(0 \leq x_n < \frac{\alpha}{n}\) for all \(n \in \mathbb{N}\). Since \(\frac{\alpha}{n} \to 0\), by Sandwich theorem, it follows that \(x_n \to 0\). Consequently \(\alpha^\frac{1}{n} \to 1\). If \(\alpha < 1\), then \(\frac{1}{\alpha} > 1\) and as proved above, \((\frac{1}{\alpha})^{\frac{1}{n}} \to 1\). It follows that \(\alpha^\frac{1}{n} \to 1\).

Alternative proof: We first assume that \(\alpha \geq 1\). For each \(n \in \mathbb{N}\), applying the A.M. \(\geq\) G.M. inequality for the numbers 1, ..., 1, \(\alpha\) (1 is repeated \(n - 1\) times), we get \(1 \leq \alpha^{\frac{1}{n}} \leq 1 + \frac{\alpha - 1}{n}\). Since \(\frac{\alpha - 1}{n} \to 0\), by Sandwich theorem, it follows that \(\alpha^{\frac{1}{n}} \to 1\). The case for \(\alpha < 1\) is same as given in the above proof. \(\Box\)

Example 9. The sequence \((n^\frac{1}{n})\) converges to 1.

Solution. For all \(n \in \mathbb{N}\), let \(a_n = n^\frac{1}{n} - 1\). Then for all \(n \in \mathbb{N}\),

\[ n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2!}a_n^2 + \cdots + a_n^n > \frac{n(n-1)}{2!}a_n^2. \]

This implies \(0 \leq a_n^2 < \frac{2}{n-1}\) for all \(n \geq 2\). Since \(\frac{2}{n-1} \to 0\), by Sandwich theorem, it follows that \(a_n^2 \to 0\) and so \(a_n \to 0\). Consequently \(n^\frac{1}{n} \to 1\). \(\Box\)
Theorem 5. Let $r \in \mathbb{R}$. Then there exists a sequence $(x_n)$ of rational numbers such that $\lim_{n \to \infty} x_n = r$.

Proof. We know that between two real numbers, there is a rational number. For each $n \in \mathbb{N}$, consider the real numbers $r - \frac{1}{n}$ and $r + \frac{1}{n}$. Let $x_n$ be a rational number such that $r - \frac{1}{n} < x_n < r + \frac{1}{n}$. Then $(x_n)$ is a sequence of rational numbers, and by Sandwich theorem $(x_n)$ converges to $r$.

**Divergent sequence:**

**Definition 3.** A sequence $(x_n)$ is said to be divergent if it has no limit.

**Example 10.** If $(x_n)$ is unbounded then it is divergent. For example, $(\sqrt{n})$, $(3n^2)$, $((-1)^n n^3)$ are all divergent. We have seen that the sequence $((-1)^n)$ is not convergent, and so it is a divergent sequence although it is bounded.

**Definition 4.** A sequence $(x_n)$ is said to approach infinity or diverges to infinity if for any real number $M > 0$, there is a positive integer $n_0$ such that $a_n \geq M$ for all $n \geq n_0$. Similarly, $(x_n)$ is said to approach $-\infty$ or diverges to $-\infty$ if for any real number $M > 0$, there is a positive integer $n_0$ such that $a_n \leq -M$ for all $n \geq n_0$.

**Remark 2.** Let $(x_n)$ and $(y_n)$ be two sequences of real numbers.

- If $(x_n)$ and $(y_n)$ both diverge to $\infty$, then the sequences $(x_n + y_n)$ and $(x_n y_n)$ also diverge to $\infty$.

- If $(x_n)$ diverges to $\infty$ and $(y_n)$ converges, then $(x_n + y_n)$ diverges to $\infty$.

**Monotone sequence:**

**Definition 5.** A sequence $(x_n)$ is said to be increasing if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Similarly, $(x_n)$ is said to be decreasing if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. We say that $(x_n)$ is monotonic if it is either increasing or decreasing.

**Example 11.** The sequence $(1 - \frac{1}{n})$ is increasing.

Solution. For all $n \in \mathbb{N}$, $\frac{1}{n+1} < \frac{1}{n}$ and so $1 - \frac{1}{n+1} > 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore the given sequence is increasing.

**Example 12.** The sequence $(n + \frac{1}{n})$ is increasing.

Solution. For all $n \in \mathbb{N}$, $(n + 1 + \frac{1}{n+1}) - (n + \frac{1}{n}) = 1 - \frac{1}{n(n+1)} > 0 \Rightarrow n + 1 + \frac{1}{n+1} > n + \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore the given sequence is increasing.

**Example 13.** The sequence $(\cos \frac{\pi n}{3})$ is not monotonic.

Solution. Since $\cos \frac{\pi}{3} = \frac{1}{2}$, $\cos \frac{2\pi}{3} = -1$ and $\cos \frac{6\pi}{3} = 1$, we have $\cos \frac{\pi}{3} > \cos \frac{2\pi}{3} < \cos \frac{6\pi}{3}$ and hence the given sequence is neither increasing nor decreasing. Consequently the given sequence is not monotonic.

**Theorem 6.** If $(x_n)$ is increasing and not bounded above then $(x_n)$ diverges to $\infty$. If $(x_n)$ is decreasing and not bounded below then $(x_n)$ diverges to $-\infty$. 


Theorem 7 (Monotone convergence theorem). Let \((x_n)\) be a sequence of real numbers.
(a) If \((x_n)\) is increasing and bounded above then \((x_n)\) converges to \(\sup\{x_n : n \in \mathbb{N}\}\).
(b) If \((x_n)\) is decreasing and bounded below then \((x_n)\) converges to \(\inf\{x_n : n \in \mathbb{N}\}\).
(c) A monotonic sequence converges if and only if it is bounded.

Proof. We only prove (a). Since \((x_n)\) is increasing, so it is bounded below by \(x_1\). Also, \((x_n)\) is bounded above and hence \((x_n)\) is bounded. Let \(s = \sup\{x_n : n \in \mathbb{N}\}\). We claim that \(x_n \to s\). To prove this, let \(\varepsilon > 0\). Then \(s - \varepsilon\) is not an upper bound and so there exists some \(n_0\) such that \(s - \varepsilon < x_{n_0}\). Since \((x_n)\) is increasing so \(x_n \geq x_{n_0}\) for all \(n \geq n_0\). Therefore

\[
s - \varepsilon < x_{n_0} \leq x_n \leq s < s + \varepsilon \quad \text{for all} \quad n \geq n_0.
\]

This proves that \(x_n \to s\).

Example 14. Let \(x_1 = 1\) and \(x_{n+1} = \frac{1}{3}(x_n + 1)\) for all \(n \in \mathbb{N}\). Then the sequence \((x_n)\) is convergent and \(\lim_{n \to \infty} x_n = \frac{1}{2}\).

Solution. For all \(n \in \mathbb{N}\), we have \(x_{n+1} - x_n = \frac{1}{3}(1 - 2x_n)\). Also, \(x_1 > \frac{1}{2}\) and if we assume that \(x_k > \frac{1}{2}\) for some \(k \in \mathbb{N}\), then \(x_{k+1} = \frac{1}{3}(x_k + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2}\). Hence by the principle of mathematical induction, \(x_n > \frac{1}{2}\) for all \(n \in \mathbb{N}\). So \((x_n)\) is bounded below. Again, from above, we get \(x_{n+1} - x_n < 0\) for all \(n \in \mathbb{N} \Rightarrow x_{n+1} < x_n\) for all \(n \in \mathbb{N} \Rightarrow (x_n)\) is decreasing. Therefore \((x_n)\) is convergent. Let \(\ell = \lim x_n\). Then \(\lim_{n \to \infty} x_{n+1} = \ell\) and since \(x_{n+1} = \frac{1}{3}(x_n + 1)\) for all \(n \in \mathbb{N}\), we get \(\ell = \frac{1}{3}(\ell + 1) \Rightarrow \ell = \frac{1}{2}\).

Example 15. The sequence \(((1 + 1/n)^n)\) is convergent.

Solution. Let \(a_n = (1 + 1/n)^n\). Then

\[
a_n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n}
\]

\[
= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right).
\]

Similarly, we have

\[
a_{n+1} = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n+1}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{n}{n+1}\right)
\]

\[
+ \frac{1}{(n+1)!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{n}{n+1}\right).
\]

Note that the expression for \(a_n\) contains \(n + 1\) terms, while that for \(a_{n+1}\) contains \(n + 2\) terms. Moreover, each term appearing in \(a_n\) is less than or equal to the corresponding term in \(a_{n+1}\), and \(a_{n+1}\) has one more positive term. Therefore, we have

\[
2 \leq a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots,
\]

so that the sequence \((a_n)\) is increasing. For \(n > 1\), we have

\[
2 < a_n < 1 + 1 + \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right) = 2 + \left(1 - \frac{1}{2^{n-1}}\right) < 3.
\]

By Monotone convergence theorem, the sequence \((a_n)\) converges to a real number that lies between 2 and 3. We define the number \(e\) to be the limit of this sequence.
Theorem 8. Let $x_n \neq 0$ for all $n \in \mathbb{N}$ and let $L = \lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right|$ exist.

(a) If $L < 1$, then $x_n \to 0$.

(b) If $L > 1$, then $(x_n)$ is divergent.

Proof. Note that $L \geq 0$. Let $r$ be such that $L < r < 1$. Let $\varepsilon = r - L$. Then $\varepsilon > 0$, and hence there is some $n_0 \in \mathbb{N}$ such that

$$0 \leq \frac{|x_{n+1}|}{|x_n|} < L + \varepsilon = r \quad \text{for all} \quad n \geq n_0.$$

Using the above inequality for $n_0, n_0 + 1, \ldots, n$, we have

$$0 \leq \frac{|x_{n_0+1}|}{|x_{n_0}|} \frac{|x_{n_0+2}|}{|x_{n_0+1}|} \ldots \frac{|x_{n}|}{|x_{n+1}|} < r^{n-n_0+1}$$

$$\Rightarrow 0 \leq |x_{n+1}| < \left( \frac{|x_{n_0}|}{r^{n_0-1}} \right) \cdot r^n$$

Since $0 < r < 1$, so $r^n \to 0$. Now using Sandwich theorem, we have $x_n \to 0$. This completes the proof of (a).

Suppose that $L > 1$. Let $r$ be such that $1 < r < L$. Let $\varepsilon = L - r$. Since $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}$ such that

$$r = L - \varepsilon < \frac{|x_{n+1}|}{|x_n|} \quad \text{for all} \quad n \geq n_0.$$

This yields (as shown before)

$$|x_{n+1}| > \left( \frac{|x_{n_0}|}{r^{n_0-1}} \right) \cdot r^n$$

Since $r > 1$, so $r^n$ diverges to infinity and hence $(x_n)$ also diverges.

Remark 3. If $L = \lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right| = 1$, then $(x_n)$ may converge or diverge. For example, the sequence $((-1)^n)$ diverges and $L = 1$. For any nonzero constant sequence, $L = 1$ and constant sequences are convergent.

Example 16. If $\alpha \in \mathbb{R}$, then the sequence $\left( \frac{\alpha^n}{n!} \right)$ is convergent.

Solution. Let $x_n = \frac{\alpha^n}{n!}$ for all $n \in \mathbb{N}$. If $\alpha = 0$, then $x_n = 0$ for all $n \in \mathbb{N}$ and so $(x_n)$ converges to 0. If $\alpha \neq 0$, then $\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lim_{n \to \infty} \frac{|\alpha|}{n+1} = 0 < 1$ and so $(x_n)$ converges to 0.

Example 17. The sequence $\left( \frac{2^n}{n^2} \right)$ is not convergent.

Solution. If $x_n = \frac{2^n}{n^2}$ for all $n \in \mathbb{N}$, then $\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lim_{n \to \infty} \frac{2}{n+1} = 2 > 1$. Therefore the sequence $(x_n)$ is not convergent.
Definition 6 (Subsequence). Let \((x_n)\) be a sequence in \(\mathbb{R}\). If \((n_k)\) is a sequence of positive integers such that \(n_1 < n_2 < n_3 < \cdots\), then \((x_{n_k})\) is called a subsequence of \((x_n)\).

Example 18. Think of some divergent sequences and their convergent subsequences.

Theorem 9. If a sequence \((x_n)\) converges to \(\ell\), then every subsequence of \((x_n)\) must converge to \(\ell\).

Remark 4. From the above theorem, we have the following:

- If \((x_n)\) has a subsequence \((x_{n_k})\) such that \(x_{n_k} \not\to \ell\), then \(x_n \not\to \ell\).
- If \((x_n)\) has two subsequences converging to two different limits, then \((x_n)\) cannot be convergent.

Example 19. If \(x_n = (-1)^n(1 - \frac{1}{n})\) for all \(n \in \mathbb{N}\), then \((x_n)\) is not convergent.

Solution. We have \(x_{2n-1} = (-1)^{2n-1}(1 - \frac{1}{2n-1}) = 1 - \frac{1}{2n-1} \to 1 \neq -1\). Hence, \((x_n)\) is not convergent.

Example 20. Let \((x_n)\) be a sequence in \(\mathbb{R}\). Then \((x_{2n})\) and \((x_{2n-1})\) are two subsequences of \((x_n)\). Suppose that \(x_{2n} \to \ell \in \mathbb{R}\) and \(x_{2n-1} \to \ell\). Then \(x_n \to \ell\).

Solution. Let \(\varepsilon > 0\). Since \(x_{2n} \to \ell\) and \(x_{2n-1} \to \ell\), there exist \(n_1, n_2 \in \mathbb{N}\) such that \(|x_{2n} - \ell| < \varepsilon\) for all \(n \geq n_1\) and \(|x_{2n-1} - \ell| < \varepsilon\) for all \(n \geq n_2\). Taking \(n_0 = \max\{2n_1, 2n_2 - 1\} \in \mathbb{N}\), we find that \(|x_n - \ell| < \varepsilon\) for all \(n \geq n_0\). Hence \(x_n \to \ell\).

Example 21. The sequence \((1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, \ldots)\) converges to 1.

Solution. If \((x_n)\) denotes the given sequence, then \(x_{2n} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1\) and \(x_{2n-1} = 1 \to 1\). Therefore \((x_n)\) converges to 1.

Theorem 10. Every sequence of real numbers has a monotone subsequence.

Proof. Let \((x_n)\) be a sequence of real numbers. A term \(x_p\) is called a peak in \((x_n)\) if \(x_p > x_m\) for all \(m > p\). That is, a peak in \((x_n)\) is a term which is greater than all the succeeding terms. Let \(\mathcal{P}\) be the set of all the peaks of \((x_n)\). We now consider the following two cases:

- \(\mathcal{P}\) is finite: Note that in this case \(\mathcal{P}\) can be empty also. Let \(p_1 < p_2 < \cdots < p_t\) so that \(x_{p_1}, x_{p_2}, \ldots, x_{p_t}\) are the only peaks of \((x_n)\). Let \(n_1 > p_t\). Then \(x_{n_1}\) is not a peak. Hence there is some \(n_2 \in \mathbb{N}\) such that \(n_2 > n_1\) and \(x_{n_1} \leq x_{n_2}\). Again, since \(n_2 > p_t\) so \(x_{n_2}\) is not a peak. Hence, there is some \(n_3 > n_2\) such that \(x_{n_2} \leq x_{n_3}\). In this way, using the principle of mathematical induction, we have an increasing sequence \(n_1 < n_2 < n_3 < \cdots < n_k < \cdots\) in \(\mathbb{N}\) such that \(x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots \leq x_{n_k} \leq \cdots\). This proves that \((x_{n_k})\) is an increasing subsequence of \((x_n)\).

- \(\mathcal{P}\) is infinite: In this case, we have \(p_1 < p_2 < \cdots < p_k < \cdots\) so that \(x_{p_1}, x_{p_2}, \ldots, x_{p_k}, \ldots\) are the peaks of \((x_n)\). Clearly, \(x_{p_1} > x_{p_2} > \ldots > x_{p_k} > \ldots\). Hence, \((x_{p_k})\) is a decreasing sequence of \((x_n)\).
Theorem 11 (Bolzano-Weierstrass Theorem). Every bounded sequence in \( \mathbb{R} \) has a convergent subsequence.

Proof. Let \((x_n)\) be a sequence of real numbers. By the previous theorem, \((x_n)\) has a monotone subsequence, say \((x_{n_k})\). Since \((x_n)\) is bounded, so \((x_{n_k})\) is also bounded. By the Monotone convergence theorem, \((x_{n_k})\) is convergent. \(\square\)

Cauchy sequence:

Definition 7 (Cauchy sequence). A sequence \((x_n)\) is called a Cauchy sequence if for each \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(|x_m - x_n| < \varepsilon\) for all \(m, n \geq n_0\).

Theorem 12. Every Cauchy sequence is bounded.

Theorem 13 (Cauchy’s criterion for convergence). A sequence in \( \mathbb{R} \) is convergent if and only if it is a Cauchy sequence.

Proof. Let \((x_n)\) be convergent and \(x_n \to \ell\). Let \(\varepsilon > 0\). Then there is some \(n_0 \in \mathbb{N}\) such that \(|x_n - \ell| < \varepsilon/2\) for all \(n, m \geq n_0\). Now,

\[
|x_n - x_m| = |x_n - \ell + \ell - x_m| \leq |x_n - \ell| + |x_m - \ell| < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]

for all \(m, n \geq n_0\). Hence, \((x_n)\) is a Cauchy sequence. Conversely, suppose that \((x_n)\) is a Cauchy sequence. Then \((x_n)\) is bounded, and by Bolzano Weierstrass theorem, \((x_n)\) has a convergent subsequence, say \((x_{n_k})\). Suppose that \(x_{n_k} \to \ell\). We claim that \(x_n \to \ell\). To prove this, let \(\varepsilon > 0\). Since \((x_n)\) is Cauchy, so there is some \(n_0 \in \mathbb{N}\) such that \(|x_n - x_m| < \varepsilon/2\) for all \(n, m \geq n_0\). Also, since \(x_{n_k} \to \ell\), there is \(k_0 \in \mathbb{N}\) such that \(|x_{n_k} - \ell| < \varepsilon/2\) for all \(k \geq k_0\). Let \(j = \max\{k_0, n_0\}\). Since \(n_j \geq j\), so \(n_j \geq n_0\). Also, \(j \geq k_0\). Therefore

\[
|x_n - \ell| \leq |x_n - x_{n_j}| + |x_{n_j} - \ell| < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]

for all \(n \geq n_0\). Hence the sequence \((x_n)\) is convergent. \(\square\)

Example 22. Let \((x_n)\) satisfy either of the following conditions:

(a) \(|x_{n+1} - x_n| \leq \alpha^n\) for all \(n \in \mathbb{N}\)

(b) \(|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|\) for all \(n \in \mathbb{N}\),

where \(0 < \alpha < 1\). Then \((x_n)\) is a Cauchy sequence.

Solution. (a) For all \(m, n \in \mathbb{N}\) with \(m > n\), we have

\[
|x_m - x_n| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|
\]

\[
\leq \alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}
\]

\[
= \frac{\alpha^n}{1 - \alpha} (1 - \alpha^{m-n})
\]

\[
< \frac{\alpha^n}{1 - \alpha}
\]

Since \(0 < \alpha < 1\), \(\alpha^n \to 0\) and so given any \(\varepsilon > 0\), we can choose \(n_0 \in \mathbb{N}\) such that \(\frac{\alpha^{n_0}}{1 - \alpha} < \varepsilon\). Hence for all \(m, n \geq n_0\), we have \(|x_m - x_n| < \frac{\alpha^{n_0}}{1 - \alpha} < \varepsilon\). Therefore \((x_n)\) is a Cauchy sequence.
(b) For all \( m, n \in \mathbb{N} \) with \( m > n \), we have
\[
|x_m - x_n| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\
\leq (\alpha^{n-1} + \alpha^n + \cdots + \alpha^{m-2})|x_2 - x_1| \\
= \frac{\alpha^{n-1}}{1-\alpha} (1 - \alpha^{m-n})|x_2 - x_1| \\
\leq \frac{\alpha^{n-1}}{1-\alpha} |x_2 - x_1|
\]
Since \( 0 < \alpha < 1 \), \( \alpha^{n-1} \to 0 \) and so given any \( \varepsilon > 0 \), we can choose \( n_0 \in \mathbb{N} \) such that \( \frac{\alpha^{n_0-1}}{1-\alpha} |x_2 - x_1| < \varepsilon \). Hence for all \( m, n \geq n_0 \), we have \( |x_m - x_n| \leq \frac{\alpha^{n_0-1}}{1-\alpha} |x_2 - x_1| < \varepsilon \). Therefore \( (x_n) \) is a Cauchy sequence.

Example 23. Let \( (x_n) \) be a sequence defined as \( x_1 = 1 \) and \( x_{n+1} = 1 + \frac{1}{x_n} \) for \( n \in \mathbb{N} \). Then \( x_{n+1}x_n = 1 + x_n > 2 \). Now,
\[
|x_{n+2} - x_{n+1}| = \left| \frac{x_{n+1} - x_n}{x_{n+1}x_n} \right| < \frac{1}{2} |x_{n+1} - x_n|.
\]
Hence, \( (x_n) \) is a Cauchy sequence.

Limit superior and limit inferior:

Let \( (x_n) \) be a bounded sequence. Let \( y_1 = \sup\{x_1, x_2, \ldots\} \), \( y_2 = \sup\{x_2, x_3, \ldots\} \), and so on. That is, for \( n \in \mathbb{N} \),
\[
y_n = \sup\{x_n, x_{n+1}, \ldots\} = \sup_{k \geq n} x_k.
\]
Let \( A \) and \( B \) be two nonempty subsets of \( \mathbb{R} \) such that \( A \subseteq B \). Then clearly, \( \sup(A) \leq \sup(B) \) and \( \inf(B) \leq \inf(A) \). Hence,
\[
y_1 \geq y_2 \geq y_3 \geq \cdots.
\]
Since \( (x_n) \) is bounded so the sequence \( (y_n) \) is bounded below. By Monotone convergence theorem, \( (y_n) \) is convergent and converges to the infimum of \( \{y_1, y_2, \ldots\} \). The limit of the sequence \( (y_n) \) is called the limit superior of the sequence \( (x_n) \), and is denoted by \( \lim \sup x_n \). Thus,
\[
\lim \sup x_n := \lim_{n \to \infty} y_n = \inf_{n} \sup_{k \geq n} x_k.
\]
Similarly, let \( z_1 = \inf\{x_1, x_2, \ldots\} \), \( z_2 = \inf\{x_2, x_3, \ldots\} \), and so on. That is, for \( n \in \mathbb{N} \),
\[
z_n = \inf\{x_n, x_{n+1}, \ldots\} = \inf_{k \geq n} x_k.
\]
We have \( z_1 \leq z_2 \leq z_3 \leq \cdots \). Since \( (x_n) \) is bounded so the sequence \( (z_n) \) is bounded above. By Monotone convergence theorem, \( (z_n) \) is convergent and converges to the supremum of \( \{z_1, z_2, \ldots\} \). The limit of the sequence \( (z_n) \) is called the limit inferior of the sequence \( (x_n) \), and is denoted by \( \lim \inf x_n \). Thus,
\[
\lim \inf x_n := \lim_{n \to \infty} z_n = \sup_{n} \inf_{k \geq n} x_k.
\]
Example 24. Consider the sequence \((x_n)\), where \(x_n = (-1)^n\). Clearly, for any \(n\), \(y_n = \sup\{x_n, x_{n+1}, \ldots\} = 1\) and \(z_n = \inf\{x_n, x_{n+1}, \ldots\} = -1\). Hence, lim sup \(x_n = 1\) and lim inf \(x_n = -1\).

Example 25. Consider the sequence \((x_n)\), where \(x_n = \frac{1}{n}\). Clearly, for any \(n\), \(y_n = \sup\{\frac{1}{k} : k \geq n\} = \frac{1}{n}\) and \(z_n = \inf\{\frac{1}{k} : k \geq n\} = 0\). Hence, lim sup \(x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{1}{n} = 0\) and lim inf \(x_n = 0\).

Remark 5. Suppose that \(|x_n| < M\) for \(n \in \mathbb{N}\). Then \(-M \leq z_n \leq y_n \leq M\) for all \(n\). Hence,

\[-M \leq \lim \inf x_n \leq \lim \sup x_n \leq M.\]

Theorem 14. Let \((a_n)\) and \((b_n)\) be two bounded sequences.

1. \(\lim \inf a_n \leq \lim \sup a_n.\)

2. If \(a_n \leq b_n\) for all \(n \in \mathbb{N}\), then \(\lim \sup a_n \leq \lim \sup b_n\) and \(\lim \inf a_n \leq \lim \inf b_n.\)

3. \(\lim \sup(a_n + b_n) \leq \lim \sup a_n + \lim \sup b_n\) and \(\lim \inf(a_n + b_n) \geq \lim \inf a_n + \lim \inf b_n.\)

Theorem 15. If \((a_n)\) is a convergent sequence, then

\(\lim \inf a_n = \lim a_n = \lim \sup a_n.\)

Proof. Let \(\ell = \lim a_n\). Let \(\varepsilon > 0\). Then there is some \(n_0 \in \mathbb{N}\) such that

\(\ell - \varepsilon/2 < a_n \leq \ell + \varepsilon/2\) for all \(n \geq n_0.\)

Let \(y_n = \sup\{a_k : k \geq n\}\) and \(z_n = \inf\{a_k : k \geq n\}\). Then, for all \(n \geq n_0\), we have

\(\ell - \varepsilon < \ell - \varepsilon/2 \leq z_n \leq a_n \leq y_n \leq \ell + \varepsilon/2 \leq \ell + \varepsilon.\)

Hence, \(\lim \sup a_n = \lim y_n = \ell\) and \(\lim \inf a_n = \lim z_n = \ell.\)

Theorem 16. Let \((a_n)\) be a bounded sequence. If \(\lim \sup a_n = \lim \inf a_n\), then \((a_n)\) is convergent and \(\lim_{n \to \infty} a_n = \lim \sup a_n.\)

Proof. Let \(\lim \sup a_n = \lim \inf a_n = \ell.\) Let \(y_n = \sup\{a_k : k \geq n\}\) and \(z_n = \inf\{a_k : k \geq n\}\). Then we have \(\lim \sup a_n = \lim y_n = \ell\) and \(\lim \inf a_n = \lim z_n = \ell.\) Let \(\varepsilon > 0.\) Then there are positive integers \(n_1\) and \(n_2\) such that

\(\ell - \varepsilon < y_n < \ell + \varepsilon\) for all \(n \geq n_1\) and \(\ell - \varepsilon < z_n < \ell + \varepsilon\) for all \(n \geq n_2.\)

Let \(n_0 = \max\{n_1, n_2\}.\) Then,

\(\ell - \varepsilon < z_n \leq a_n \leq y_n < \ell + \varepsilon\) for all \(n \geq n_0.\)

Hence, \(\lim a_n = \ell.\)

Alternative proof: Equivalently, we can directly apply Sandwich Theorem. We have \(z_n \leq a_n \leq y_n\) for all \(n.\) Since \(z_n \to \ell\) and \(y_n \to \ell,\) by using Sandwich Theorem, we have \(a_n \to \ell.\)