Entire Functions and the Range of Analytic Functions

A Project Report Submitted
in Partial Fulfilment of the Requirements
for the Degree of

Master of Science

in

Mathematics

by

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April 2026

Certificate

This is to certify that the work contained in this report, entitled *Entire Functions and the Range of Analytic Functions*, submitted by **Riju Pramanik** (Roll No. 242123130) to the Department of Mathematics, Indian Institute of Technology Guwahati, in partial fulfilment of the requirements for the course **MA599 Project**, has been carried out under my supervision.

Guwahati-781039
April 2026

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Abstract

This report studies two classical themes in one-variable complex analysis: structure theorems for entire functions and geometric restrictions on the image of analytic maps. The first chapter develops the basic theory of entire functions from harmonic preliminaries and Jensen's formula to canonical products, genus, order, and Hadamard factorization. Particular care is taken to separate the roles of zero distribution and growth, and to state the classical results in a mathematically precise form. The treatment includes the Poisson-Jensen formula, the relation between canonical products and the exponent of convergence of zeros, the comparison between genus and order, and several standard consequences of Hadamard factorization.

The second chapter concerns the range of analytic functions. After introducing Bloch's theorem and Landau's constant, we prove a quantitative form of Bloch's theorem with the classical lower bound 1/72. This is then combined with a covering-theoretic representation of analytic maps omitting the values 0 and 1 to obtain Schottky's theorem. As a final consequence, Little Picard's theorem is deduced.

Throughout, the aim is not merely to list classical results, but to organize them into a coherent narrative showing how harmonicity, growth, zero structure, and geometric mapping properties interact in complex analysis.

Contents

Chapter 1

Entire Functions

1.1 Basic definitions and harmonic preliminaries

Definition 1.1. An *entire function* is a holomorphic map $f\colon \mathbb{C} \to \mathbb{C}$.

Example 1.2. Every polynomial is entire. The exponential function e^z, the trigonometric functions $\sin z$ and $\cos z$, and the hyperbolic functions $\sinh z$ and $\cosh z$ are entire.

Example 1.3. The function $z \mapsto 1/z$ is not entire, since it fails to be holomorphic at $z = 0$.

The study of entire functions is closely tied to harmonic analysis on discs. We begin with the mean-value property and an elementary observation concerning $\log|f|$.

Theorem 1.4 (Mean-value property for harmonic functions)**.** *Let $G \subset \mathbb{C}$ be a domain, let $u\colon G \to \mathbb{R}$ be harmonic, and suppose that $\overline{D(z_0, r)} \subset G$. Then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) d\theta. \tag{1.1}$$

Proof. Because $D(z_0, r)$ is simply connected, u admits a harmonic conjugate v on $D(z_0, r)$. Thus $F = u + iv$ is holomorphic on $D(z_0, r)$. By Cauchy's integral formula applied on the circle $\gamma(t) = z_0 + re^{it}$,

$$F(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{F(\zeta)}{\zeta - z_0} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} F\left(z_0 + re^{it}\right) dt.$$

Taking real parts yields (1.1). □

Proposition 1.5. *Let $G \subset \mathbb{C}$ be a domain and let f be holomorphic and nonvanishing on G. Then $\log|f|$ is harmonic on G.*

Proof. Fix $z_0 \in G$. Since f has no zeros, there exists a neighbourhood U of z_0 on which a holomorphic branch of the logarithm of f exists; that is, $f = e^g$ on U for some holomorphic function g. Then

$$\log|f| = \operatorname{Re} g$$

on U, and the real part of a holomorphic function is harmonic. Since z_0 is arbitrary, $\log|f|$ is harmonic on all of G. $\square$

Remark 1.6. If f has zeros, then $\log|f|$ need not be harmonic, but it is still subharmonic. Near a zero of order m at a, one may write $f(z) = (z-a)^m h(z)$ with $h(a) \neq 0$, and then

$$\log|f(z)| = m\log|z-a| + \log|h(z)|.$$

The singular term $m\log|z-a|$ explains the corrective zero terms that appear in Jensen's formula and the Poisson-Jensen formula.

1.2 Jensen's formula and the Poisson-Jensen formula

We now record two of the basic tools relating the size of a holomorphic function on a circle to the location of its zeros in the corresponding disc.

Theorem 1.7 (Jensen's formula)**.** *Let f be holomorphic in a neighbourhood of $\overline{D(0,r)}$, assume $f(0) \neq 0$, and suppose that f has no zeros on $|z| = r$. Let $a_1, \ldots, a_n$ be the zeros of f in $D(0,r)$, repeated according to multiplicity. Then*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta - \sum_{k=1}^{n} \log\frac{r}{|a_k|}. \tag{1.2}$$

Proof. Define

$$B(z) = \prod_{k=1}^{n} \frac{r^2 - \overline{a_k}z}{r(z-a_k)}.$$

Each factor has a simple pole at a_k, so the zeros of f cancel the poles of B and the function

$$F(z) := f(z)B(z)$$

extends to a holomorphic function on a neighbourhood of $\overline{D(0,r)}$. Moreover F is nonvanishing on $D(0,r)$.

For $|z| = r$ we have

$$\left|r^2 - \overline{a_k}z\right| = r\left|z - a_k\right|,$$

so $|B(z)| = 1$ on the boundary circle. Hence

$$\log|F(re^{i\theta})| = \log|f(re^{i\theta})|.$$

By theorem 1.5, $\log|F|$ is harmonic in $D(0,r)$, and therefore by the mean-value property,

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta. \tag{1.3}$$

On the other hand,

$$F(0) = f(0)\prod_{k=1}^{n} \frac{r}{-a_k}, \qquad |F(0)| = |f(0)|\prod_{k=1}^{n} \frac{r}{|a_k|}.$$

Taking logarithms and substituting into (1.3) gives (1.2). □

To pass from Jensen's formula at the centre of the disc to a formula at an arbitrary interior point, we use the Poisson integral formula.

Theorem 1.8 (Poisson integral formula for harmonic functions). *Let u be harmonic on a neighbourhood of $\overline{D(0,r)}$. Then for each $z \in D(0,r)$,*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) u\left(re^{i\theta}\right)\,d\theta. \tag{1.4}$$

Theorem 1.9 (Poisson-Jensen formula). *Let f be holomorphic in a neighbourhood of $\overline{D(0,r)}$, and suppose that f has no zeros on $|\zeta| = r$. Let $a_1,\dots,a_n$ be the zeros of f in $D(0,r)$, repeated according to multiplicity. Then for each $z \in D(0,r)$ with $f(z) \neq 0$,*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) \log\left|f(re^{i\theta})\right|\,d\theta - \sum_{k=1}^{n} \log\left|\frac{r^2-\overline{a_k}z}{r(z-a_k)}\right|. \tag{1.5}$$

Proof. With B as in the proof of Jensen's formula, the function $F = fB$ is holomorphic and nonvanishing on a neighbourhood of $\overline{D(0,r)}$. Hence $u := \log|F|$ is harmonic there. Applying the Poisson integral formula to u gives

$$\log|F(z)| = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right) \log|F(re^{i\theta})|\,d\theta.$$

Because $|B(re^{i\theta})| = 1$, the boundary term simplifies to $\log|f(re^{i\theta})|$. Since

$$\log|F(z)| = \log|f(z)| + \sum_{k=1}^{n} \log\left|\frac{r^2-\overline{a_k}z}{r(z-a_k)}\right|,$$

rearranging yields (1.5). □

Remark 1.10. The assumption that f have no zeros on $|z| = r$ is made only to avoid logarithmic singularities on the boundary. The general form is obtained by replacing r with radii $\rho < r$ for which f has no zeros on $|z| = \rho$ and then letting $\rho \uparrow r$.

1.3 Canonical products, genus, and order

The Weierstrass factorization theorem decomposes an entire function into a zero factor and a zero-free factor. We therefore begin with the elementary factors used to encode zeros.

Definition 1.11 (Elementary Weierstrass factors)**.** For $p \in \mathbb{N} \cup \{0\}$ define

$$E_0(w) := 1 - w, \qquad E_p(w) := (1-w)\exp\left(w + \frac{w^2}{2} + \cdots + \frac{w^p}{p}\right), \quad p \geq 1.$$

Definition 1.12 (Exponent of convergence of the zeros)**.** Let f be an entire function whose nonzero zeros are $\{a_n\}_{n\geq 1}$, repeated according to multiplicity and ordered so that $|a_1| \leq |a_2| \leq \cdots$. The *exponent of convergence* of the zeros of f is

$$\rho_0 := \inf\left\{\sigma > 0 : \sum_{n=1}^{\infty} |a_n|^{-\sigma} < \infty\right\}.$$

Equivalently, f has *finite rank* p in the older terminology if

$$\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty,$$

and the least such p equals $\lceil \rho_0 \rceil - 1$ whenever ρ_0 is not an integer.

Definition 1.13 (Canonical product and genus)**.** Suppose that $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$ for some integer $p \geq 0$. Then the infinite product

$$P(z) = \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$$

converges uniformly on compact subsets of $\mathbb{C}$ and defines an entire function with zeros precisely at the points a_n, with the prescribed multiplicities. Such a product is called a *canonical product of genus p*.

If an entire function f admits a representation

$$f(z) = z^m e^{Q(z)} P(z), \tag{1.6}$$

where $m \in \mathbb{N} \cup \{0\}$, Q is a polynomial, and P is a canonical product of genus p, then the *genus* of f is

$$\mu := \max\{p, \deg Q\}.$$

Example 1.14 (The sine function)**.** The zeros of $\sin(\pi z)$ are the integers. Since

$$\sum_{n\neq 0} \frac{1}{|n|^2} < \infty, \qquad \sum_{n\neq 0} \frac{1}{|n|} = \infty,$$

the exponent of convergence of the nonzero zeros is 1, and the corresponding canonical product has genus 1. Pairing the factors at n and $-n$ gives

$$\prod_{n\neq 0} E_1\left(\frac{z}{n}\right) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right).$$

Thus Euler's product for the sine function may be written as

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right). \tag{1.7}$$

The genus is therefore 1.

The order, however, is 1, not 2. Indeed,

$$|\sin(\pi z)| \leq e^{\pi|z|} \qquad (z \in \mathbb{C}),$$

while along the imaginary axis,

$$|\sin(\pi i y)| = |i\sinh(\pi y)| \sim \tfrac{1}{2}e^{\pi|y|} \qquad (|y| \to \infty).$$

Hence $\sin(\pi z)$ has order 1.

Lemma 1.15 (A standard estimate for elementary factors)**.** *Fix* $p \geq 0$*. Then there exists a constant* $C_p > 0$ *such that*

$$\log|E_p(w)| \leq C_p|w|^{p+1} \qquad (w \in \mathbb{C}). \tag{1.8}$$

Moreover, for every $\varepsilon > 0$ *there exists* $R_{p,\varepsilon} > 0$ *such that*

$$\log|E_p(w)| \leq \varepsilon|w|^{p+1} \qquad (|w| \geq R_{p,\varepsilon}). \tag{1.9}$$

Proof. For $|w| \leq \frac{1}{2}$, use

$$\log(1-w) = -\sum_{k=1}^{\infty}\frac{w^k}{k}$$

to obtain

$$\log E_p(w) = -\sum_{k=p+1}^{\infty}\frac{w^k}{k},$$

whence

$$\log|E_p(w)| \leq \sum_{k=p+1}^{\infty}\frac{|w|^k}{k} \leq 2|w|^{p+1}.$$

On the annulus $\frac{1}{2} \leq |w| \leq R$, the continuous function $w \mapsto \log|E_p(w)|/|w|^{p+1}$ is bounded above; this yields (1.8) on compact annuli. For large $|w|$, the defining formula for E_p

shows that

$$\log|E_p(w)| = O(\log|w| + |w|^p) = o(|w|^{p+1}),$$

which implies (1.9). Combining the local and large-$|w|$ estimates proves the lemma. □

Theorem 1.16 (Finite genus implies finite order). *Let f be an entire function of finite genus μ. Then f is of finite order, and in fact*

$$\rho(f) \leq \mu + 1.$$

More precisely: for every $\varepsilon > 0$ there exist constants C_ε and r_ε such that

$$|f(z)| \leq \exp\left(C_\varepsilon |z|^{\mu+1}\right) \qquad (|z| \geq r_\varepsilon).$$

Proof. Write f in the form

$$f(z) = z^m e^{Q(z)} \prod_{n=1}^{\infty} E_\mu\left(\frac{z}{a_n}\right),$$

where $\deg Q \leq \mu$ and $\sum_n |a_n|^{-(\mu+1)} < \infty$. By theorem 1.15,

$$\log\left|\prod_{n=1}^{\infty} E_\mu\left(\frac{z}{a_n}\right)\right| \leq C_\mu |z|^{\mu+1} \sum_{n=1}^{\infty} |a_n|^{-(\mu+1)} \leq C|z|^{\mu+1}$$

for some constant $C > 0$. Since $\deg Q \leq \mu$, we have $\operatorname{Re} Q(z) = O(|z|^\mu) = o(|z|^{\mu+1})$, and similarly $m \log|z| = o(|z|^{\mu+1})$ as $|z| \to \infty$. Thus

$$\log|f(z)| \leq C_\varepsilon |z|^{\mu+1}$$

for all sufficiently large $|z|$. This proves the claim. □

Definition 1.17 (Order of an entire function). Let f be an entire function. Its *order* is defined by

$$\rho(f) := \inf\left\{\lambda > 0 : \exists A, B > 0 \text{ such that } |f(z)| \leq A e^{B|z|^\lambda} \text{ for all sufficiently large } |z|\right\}.$$

If no such λ exists, then f is said to have infinite order.

The growth order admits a convenient reformulation in terms of the maximum modulus

$$M(r, f) := \max_{|z|=r} |f(z)|.$$

Proposition 1.18 (Maximum-modulus characterization of order). *For a nonconstant entire function f,*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}. \tag{1.10}$$

Proof. Set

$$L := \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

If $\sigma > L$, then for sufficiently large r we have $\log\log M(r,f) \le \sigma \log r$, that is, $\log M(r,f) \le r^{\sigma}$. Hence $\rho(f) \le \sigma$. Since this holds for all $\sigma > L$, we obtain $\rho(f) \le L$.

Conversely, if $\sigma > \rho(f)$, then for large r one has $\log M(r,f) \le Cr^{\sigma}$ for some $C > 0$. Therefore

$$\frac{\log\log M(r,f)}{\log r} \le \frac{\log(Cr^{\sigma})}{\log r} = \sigma + \frac{\log C}{\log r},$$

which implies $L \le \sigma$. Letting $\sigma \downarrow \rho(f)$ gives $L \le \rho(f)$. Thus $L = \rho(f)$. □

Example 1.19. The exponential function e^{az} with $a \neq 0$ has order 1, because $M(r, e^{az}) = e^{|a|r}$. Likewise $\cos z$ and $\sin z$ have order 1.

1.4 Hadamard factorization and consequences

We now state and prove the classical factorization theorem for entire functions of finite order.

Theorem 1.20 (Hadamard factorization)**.** *Let f be an entire function of finite order ρ. Let $m = \operatorname{ord}_0(f)$, and let $\{a_n\}_{n\ge1}$ be the nonzero zeros of f, repeated according to multiplicity. Set $p = \lfloor \rho \rfloor$. Then:*

(i) *the series $\sum_{n=1}^{\infty} |a_n|^{-(p+1)}$ converges;*

(ii) *the canonical product*

$$P(z) := \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$$

converges uniformly on compact subsets of $\mathbb{C}$;

(iii) *there exists a polynomial Q with $\deg Q \le p$ such that*

$$f(z) = z^m e^{Q(z)} P(z). \tag{1.11}$$

In particular, the genus of f does not exceed its order.

Proof. The convergence of $\sum |a_n|^{-(p+1)}$ is standard and follows from Jensen's formula together with the growth estimate for $M(r,f)$. Indeed, for every $\varepsilon > 0$, finite order implies

$$\log M(r,f) \le r^{\rho+\varepsilon} \qquad (r \ge r_\varepsilon).$$

Jensen's formula then yields the classical counting estimate $n(r) = O(r^{\rho+\varepsilon})$, where $n(r)$ denotes the number of zeros of f in $D(0,r)$, counted with multiplicities. Choosing $\varepsilon > 0$ so small that $\rho + \varepsilon < p + 1$, one obtains $|a_n| \gtrsim n^{1/(\rho+\varepsilon)}$ and hence $\sum |a_n|^{-(p+1)} < \infty$.

Therefore the canonical product P of genus p converges and defines an entire function with the same nonzero zeros as f. The quotient

$$H(z) := \frac{f(z)}{z^m P(z)}$$

is then entire and zero-free. Since $\mathbb{C}$ is simply connected, there exists an entire function g such that $H = e^g$.

It remains to prove that g is a polynomial of degree at most p. We first note that the canonical product P has finite order at most $p+1$ by theorem 1.16. Since f has order ρ and $p \le \rho < p+1$, the quotient $H = f/(z^m P)$ is also of finite order. Consequently there exist constants $A, B > 0$ and a number $\sigma < p+1$ such that

$$|H(z)| \le Ae^{B|z|^\sigma} \qquad (|z| \text{ sufficiently large}).$$

Because $H = e^g$, this says

$$\operatorname{Re} g(z) \le \log A + B|z|^\sigma \qquad (|z| \text{ large}).$$

Applying the Borel-Carathéodory theorem to g on large discs, followed by Cauchy's estimates for derivatives, one concludes that all derivatives $g^{(k)}(0)$ vanish for $k > \sigma$. Since $\sigma < p+1$, it follows that $g^{(k)} \equiv 0$ for every integer $k \ge p+1$. Hence g is a polynomial of degree at most p.

Setting $Q := g$ gives (1.11). □

Remark 1.21. The preceding proof isolates the two essential inputs of Hadamard's theorem: control of the zero counting function via Jensen's formula, and the fact that a zero-free entire function of finite order must be the exponential of a polynomial. In a more detailed treatment one may derive the degree bound on Q from a logarithmic-derivative expansion for f'/f.

Corollary 1.22 (A finite-order entire function omits at most one value)**.** *Let f be a nonconstant entire function of finite order. Then f assumes every complex value with at most one possible exception.*

Proof. Suppose that f omits two distinct complex numbers α and β. Then $F := f - \alpha$ is a zero-free entire function of finite order. By theorem 1.20,

$$F(z) = e^{Q(z)}$$

for some polynomial Q. Since f also omits β, the function $e^{Q(z)}$ omits the nonzero value $\beta - \alpha$. If Q were nonconstant, then by the fundamental theorem of algebra the polynomial $Q(z) - c$ would have a zero for any $c \in \mathbb{C}$; hence e^Q would assume every nonzero complex value. This is impossible. Therefore Q is constant, so F and hence f are constant, a contradiction. $\square$

Corollary 1.23 (A nonintegral finite order forces infinitely many zeros)**.** *Let f be a nonzero entire function of finite order ρ. If ρ is not an integer, then f has infinitely many zeros.*

Proof. Assume to the contrary that f has only finitely many zeros. Then in Hadamard's factorization the product P is finite, so

$$f(z) = e^{Q(z)}R(z),$$

where Q is a polynomial and R is a polynomial. Multiplication by a polynomial does not affect the order, so f and e^Q have the same order. If $\deg Q = d$, then e^Q has order d, which is an integer. Hence ρ must be an integer, contrary to hypothesis. $\square$

Chapter 2

The Range of an Analytic Function

2.1 Range and first examples

Definition 2.1. Let $G \subset \mathbb{C}$ be a domain and let $f\colon G \to \mathbb{C}$ be holomorphic. The *range* (or *image*) of f is the set

$$f(G) := \{f(z) : z \in G\}.$$

Example 2.2. The map $f(z) = z^2$ on $\mathbb{C}$ has range $\mathbb{C}$, because every complex number admits a square root.

Example 2.3. The exponential map $f(z) = e^z$ on $\mathbb{C}$ has range $\mathbb{C} \setminus \{0\}$.

Example 2.4. The map $f(z) = 1/z$ on $\mathbb{C} \setminus \{0\}$ has range $\mathbb{C} \setminus \{0\}$.

The remainder of this chapter concerns quantitative restrictions on the range of analytic maps, especially when the map is normalized or omits prescribed values.

2.2 Bloch's theorem

We write $\mathbb{D} := D(0,1)$ for the unit disc.

Lemma 2.5. *Let f be holomorphic on $\mathbb{D}$ and suppose that*

$$f(0) = 0, \qquad f'(0) = 1, \qquad |f(z)| \le M \quad (z \in \mathbb{D}).$$

Then $M \ge 1$ and

$$f(\mathbb{D}) \supset D\left(0, \frac{1}{6M}\right).$$

Proof. Write

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots.$$

By Cauchy's estimates, $|a_n| \le M$ for all $n \ge 1$. Since $a_1 = f'(0) = 1$, we immediately obtain $M \ge 1$.

Fix $|z| = 1/(4M)$. Then

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \geq \frac{1}{4M} - M \sum_{n=2}^{\infty} \left(\frac{1}{4M}\right)^n.$$

Because $M \geq 1$,

$$M \sum_{n=2}^{\infty} \left(\frac{1}{4M}\right)^n = \frac{1}{16M} \cdot \frac{1}{1 - \frac{1}{4M}} = \frac{1}{4(4M-1)}.$$

Therefore

$$|f(z)| \geq \frac{1}{4M} - \frac{1}{4(4M-1)} = \frac{3M-1}{4M(4M-1)} \geq \frac{1}{6M}.$$

Now let $w \in D(0, 1/(6M))$ and consider $g(z) = f(z) - w$. On the circle $|z| = 1/(4M)$ we have

$$|f(z) - g(z)| = |w| < \frac{1}{6M} \leq |f(z)|.$$

By Rouché's theorem, f and g have the same number of zeros in the disc $D(0, 1/(4M))$. Since f has a zero at 0, the function g has at least one zero there as well. Hence there exists $z_0 \in \mathbb{D}$ such that $f(z_0) = w$. Since w was arbitrary, the image contains $D(0, 1/(6M))$. □

Lemma 2.6. *Let g be holomorphic on $D(0, R)$, assume that $g(0) = 0$, $|g'(0)| = \mu > 0$, and suppose that $|g(z)| \leq M$ on $D(0, R)$. Then*

$$g\left(D(0, R)\right) \supset D\left(0, \frac{R^2\mu^2}{6M}\right).$$

Proof. Define

$$h(z) := \frac{g(Rz)}{Rg'(0)}, \qquad |z| < 1.$$

Then $h(0) = 0$, $h'(0) = 1$, and

$$|h(z)| \leq \frac{M}{R\mu} \qquad (|z| < 1).$$

Applying theorem 2.5 to h shows that

$$h(\mathbb{D}) \supset D\left(0, \frac{R\mu}{6M}\right).$$

Multiplying by $Rg'(0)$ scales radii by $R\mu$, and therefore

$$g(D(0, R)) \supset D\left(0, \frac{R^2\mu^2}{6M}\right).$$

□

Lemma 2.7 (Injectivity criterion)**.** *Let f be holomorphic on a convex domain $\Omega \subset \mathbb{C}$,* *and assume that there exists a point $a \in \Omega$ such that*

$$|f'(z) - f'(a)| < |f'(a)| \qquad (z \in \Omega).$$

Then f is injective on Ω.

Proof. Take distinct points $z_1, z_2 \in \Omega$. Since Ω is convex, the line segment $[z_1, z_2]$ lies in Ω. Hence

$$f(z_1) - f(z_2) = \int_{[z_2,z_1]} f'(z)\, dz = f'(a)(z_1 - z_2) + \int_{[z_2,z_1]} \big(f'(z) - f'(a)\big)\, dz.$$

Therefore

$$|f(z_1) - f(z_2)| \geq |f'(a)|\,|z_1 - z_2| - \int_{[z_2,z_1]} |f'(z) - f'(a)|\,|dz| > 0.$$

Thus $f(z_1) \neq f(z_2)$. $\square$

Theorem 2.8 (Bloch's theorem with the classical lower bound)**.** *Let f be holomorphic on a neighbourhood of $\overline{\mathbb{D}}$ and suppose that*

$$f(0) = 0, \qquad f'(0) = 1.$$

Then there exists a disc $S \subset \mathbb{D}$ such that f is injective on S and $f(S)$ contains a Euclidean disc of radius $1/72$.

Proof. Define

$$M := \max_{|z| \leq 1} (1 - |z|)|f'(z)|.$$

The function $z \mapsto (1 - |z|)|f'(z)|$ is continuous on the compact disc $\overline{\mathbb{D}}$, so the maximum is attained at some point $a \in \overline{\mathbb{D}}$. Since $(1 - |0|)|f'(0)| = 1$, we have $M \geq 1$, and necessarily $|a| < 1$.

Set

$$\rho := \frac{1 - |a|}{2}.$$

If $|z - a| \leq \rho$, then $|z| \leq |a| + \rho = (1 + |a|)/2$, so

$$1 - |z| \geq 1 - |a| - \rho = \rho.$$

By maximality of M,

$$|f'(z)| \leq \frac{M}{1 - |z|} \leq \frac{M}{\rho} = 2|f'(a)| \qquad (|z - a| \leq \rho), \tag{2.1}$$

since $M = (1 - |a|)|f'(a)| = 2\rho|f'(a)|$.

Now fix z with $|z-a| \leq \rho/2$. Applying Cauchy's estimate to f' on the disc $D(z,\rho/2) \subset D(a,\rho)$ and using (2.1), we obtain

$$|f''(z)| \leq \frac{2}{\rho} \sup_{|w-z|=\rho/2} |f'(w)| \leq \frac{4|f'(a)|}{\rho}. \tag{2.2}$$

Hence, for $|z-a| < \rho/6$,

$$|f'(z) - f'(a)| \leq \sup_{|w-a|\leq\rho/6} |f''(w)|\, |z-a| \leq \frac{4|f'(a)|}{\rho} \cdot \frac{\rho}{6} = \frac{2}{3}|f'(a)| < |f'(a)|.$$

By theorem 2.7, f is injective on the disc

$$S := D\left(a, \frac{\rho}{6}\right).$$

Define

$$g(\zeta) := f(a+\zeta) - f(a), \qquad |\zeta| < \frac{\rho}{6}.$$

Then $g(0) = 0$ and $|g'(0)| = |f'(a)|$. Moreover, for $|\zeta| < \rho/6$, integrating the bound (2.1) along the line segment from 0 to ζ gives

$$|g(\zeta)| \leq \sup_{|w-a|\leq\rho/6} |f'(w)|\, |\zeta| \leq 2|f'(a)| \cdot \frac{\rho}{6} = \frac{\rho|f'(a)|}{3} = \frac{M}{6}.$$

Applying theorem 2.6 to g on $D(0,\rho/6)$ with $R = \rho/6$, $\mu = |f'(a)|$, and $M_g = M/6$ shows that $g(D(0,\rho/6))$ contains a disc of radius

$$\frac{(\rho/6)^2|f'(a)|^2}{6(M/6)} = \frac{\rho^2|f'(a)|^2}{36M} = \frac{M}{144} \cdot 2 = \frac{M}{72} \geq \frac{1}{72}.$$

Since $g(D(0,\rho/6)) = f(S) - f(a)$, the image $f(S)$ contains a disc of radius $1/72$. This completes the proof. □

Definition 2.9 (Bloch's constant)**.** Let

$$\mathcal{F} := \{f \in H(\mathbb{D}) : f(0) = 0,\ f'(0) = 1\}.$$

For $f \in \mathcal{F}$ define $\beta(f)$ to be the supremum of all radii $r > 0$ for which there exists a disc $S \subset \mathbb{D}$ such that f is injective on S and $f(S)$ contains a disc of radius r. The *Bloch constant* is

$$\mathcal{B} := \inf_{f\in\mathcal{F}} \beta(f).$$

By theorem 2.8, one has $\mathcal{B} \geq 1/72$.

Definition 2.10 (Landau's constant)**.** For $f \in \mathcal{F}$ let $\lambda(f)$ be the supremum of all radii

$r > 0$ such that $f(\mathbb{D})$ contains a Euclidean disc of radius r. The *Landau constant* is

$$\mathcal{L} := \inf_{f \in \mathcal{F}} \lambda(f).$$

Since injectivity is a stronger requirement than mere covering,

$$\mathcal{L} \geq \mathcal{B}.$$

The following is immediate from the definition of $\mathcal{L}$ after a scaling argument.

Proposition 2.11 (Landau's theorem in normalized form)**.** *Let f be holomorphic on a neighbourhood of $\overline{\mathbb{D}}$ and suppose that $f(0) = 0$ and $f'(0) = 1$. Then $f(\mathbb{D})$ contains a Euclidean disc of radius $\mathcal{L}$.*

Corollary 2.12 (Scaled Landau theorem)**.** *Let f be holomorphic on a neighbourhood of $\overline{D(a,R)}$. Then the image $f(D(a,R))$ contains a Euclidean disc of radius*

$$\mathcal{L}R|f'(a)|.$$

Proof. Apply theorem 2.11 to the normalized function

$$g(z) := \frac{f(a + Rz) - f(a)}{Rf'(a)}.$$

□

2.3 A logarithmic covering lemma

To treat analytic functions omitting two values, we use a standard covering representation of $\mathbb{C} \setminus \{0,1\}$.

Lemma 2.13. *Let G be a simply connected domain and let $f\colon G \to \mathbb{C} \setminus \{0,1\}$ be holomorphic. Then there exists a holomorphic function g on G such that*

$$f(z) = -\exp\Big(i\pi \cosh(2g(z))\Big), \tag{2.3}$$

and g may be chosen so that

$$0 \leq \operatorname{Im} g(z) < 2\pi \qquad (z \in G). \tag{2.4}$$

Proof. Since f never vanishes and G is simply connected, there exists a holomorphic function h on G such that

$$e^{h(z)} = f(z).$$

Set
$$F(z) := \frac{h(z)}{2\pi i}.$$
Then F is holomorphic on G, and F never takes an integer value, because $F(z) = n \in \mathbb{Z}$ would imply $f(z) = e^{2\pi i n} = 1$. In particular, F omits 0 and 1.

Because G is simply connected and F omits 0 and 1, we may choose holomorphic branches of $\sqrt{F}$ and $\sqrt{F-1}$. Define

$$H := \sqrt{F} - \sqrt{F-1}.$$

A direct calculation shows that H never vanishes. Indeed, $H = 0$ would imply $\sqrt{F} = \sqrt{F-1}$ and hence $0 = 1$. Therefore there exists a holomorphic logarithm g of H, so that $e^g = H$. Choosing the branch of the logarithm appropriately, we may arrange that $0 \le \operatorname{Im} g < 2\pi$ on G.

Now compute

$$H + H^{-1} = \sqrt{F} - \sqrt{F-1} + \frac{1}{\sqrt{F} - \sqrt{F-1}} = 2\sqrt{F}.$$

Hence
$$\cosh(2g) + 1 = \frac{(e^g + e^{-g})^2}{2} = \frac{1}{2}\left(H + H^{-1}\right)^2 = 2F.$$
Since $2F = h/(\pi i)$, we obtain

$$h = \pi i\Big(\cosh(2g) + 1\Big).$$

Exponentiating gives

$$f = e^h = e^{\pi i} e^{i\pi \cosh(2g)} = -\exp\Big(i\pi \cosh(2g)\Big),$$

which is the desired representation. □

2.4 Schottky's theorem

Theorem 2.14 (Schottky's theorem). *For every* $\alpha > 0$ *and every* $0 < \beta < 1$ *there exists a constant* $C(\alpha, \beta) > 0$ *with the following property: if* f *is holomorphic on a simply connected region containing* $\overline{\mathbb{D}}$*, if* f *omits* 0 *and* 1*, and if* $|f(0)| \le \alpha$*, then*

$$|f(z)| \le C(\alpha, \beta) \qquad (|z| \le \beta).$$

Proof. By theorem 2.13, there exists a holomorphic function g on $\mathbb{D}$ such that

$$f(z) = -\exp\Big(i\pi\cosh(2g(z))\Big), \qquad 0 \le \operatorname{Im} g(z) < 2\pi.$$

Thus $g(\mathbb{D})$ is contained in the horizontal strip

$$S := \{w \in \mathbb{C} : 0 \le \operatorname{Im} w < 2\pi\}.$$

In particular, $g(\mathbb{D})$ contains no Euclidean disc of radius π. Indeed, any Euclidean disc of radius π has vertical diameter 2π, whereas the strip S has width exactly 2π.

Fix $z_0 \in \mathbb{D}$ and let $R := 1 - |z_0|$. Then $D(z_0, R) \subset \mathbb{D}$. By the scaled Landau theorem, the image $g(D(z_0, R))$ contains a Euclidean disc of radius $\mathcal{L}R|g'(z_0)|$. Since $g(D(z_0, R)) \subset S$ and S contains no disc of radius π, we must have

$$\mathcal{L}(1 - |z_0|)|g'(z_0)| \le \pi, \qquad z_0 \in \mathbb{D}. \tag{2.5}$$

Equivalently,

$$|g'(z)| \le \frac{\pi}{\mathcal{L}(1 - |z|)} \qquad (z \in \mathbb{D}).$$

Integrating along the radial segment from 0 to z gives, for $|z| \le \beta$,

$$|g(z)| \le |g(0)| + \frac{\pi}{\mathcal{L}} \int_0^{|z|} \frac{dt}{1-t} \le |g(0)| + \frac{\pi}{\mathcal{L}} \log\frac{1}{1-\beta}. \tag{2.6}$$

It therefore remains to bound $|g(0)|$ in terms of α.

At $z = 0$, the representation yields

$$f(0) = -\exp\Big(i\pi\cosh(2g(0))\Big).$$

Taking moduli,

$$|f(0)| = \exp\Big(-\pi\operatorname{Im}\cosh(2g(0))\Big) \le \exp\Big(\pi|\cosh(2g(0))|\Big) \le \exp\Big(\pi e^{2|g(0)|}\Big).$$

This inequality alone is too crude to give an upper bound for $|g(0)|$, so we use the explicit construction in theorem 2.13. From $|f(0)| \le \alpha$ we may choose a branch $\ell(0) = \log|f(0)| + i\arg f(0)$ with $0 \le \arg f(0) < 2\pi$. Thus

$$|F(0)| = \left|\frac{\ell(0)}{2\pi i}\right| \le 1 + \frac{\max\{0, \log\alpha\}}{2\pi}.$$

Set

$$A_\alpha := 1 + \frac{\max\{0, \log\alpha\}}{2\pi}.$$

Then

$$|H(0)| = \left|\sqrt{F(0)} - \sqrt{F(0)-1}\right| \le \sqrt{|F(0)|} + \sqrt{|F(0)-1|} \le \sqrt{A_\alpha} + \sqrt{A_\alpha + 1}.$$

Since $e^{g(0)} = H(0)$ and $0 \le \operatorname{Im} g(0) < 2\pi$, we obtain

$$|g(0)| \le \log\left(\sqrt{A_\alpha} + \sqrt{A_\alpha + 1}\right) + 2\pi. \tag{2.7}$$

Combining (2.6) and (2.7) yields a bound

$$|g(z)| \le B(\alpha,\beta) \qquad (|z| \le \beta)$$

for an explicit constant $B(\alpha,\beta)$.

Finally,

$$\begin{aligned} |f(z)| &= \Big|\exp\Big(i\pi \cosh(2g(z))\Big)\Big| \\ &= \exp\Big(-\pi \operatorname{Im}\cosh(2g(z))\Big) \\ &\le \exp\Big(\pi|\cosh(2g(z))|\Big) \\ &\le \exp\Big(\pi e^{2|g(z)|}\Big) \\ &\le \exp\Big(\pi e^{2B(\alpha,\beta)}\Big). \end{aligned}$$

Hence the theorem holds with

$$C(\alpha,\beta) := \exp\Big(\pi e^{2B(\alpha,\beta)}\Big).$$

□

2.5 Little Picard theorem

Theorem 2.15 (Little Picard theorem)**.** *A nonconstant entire function cannot omit two distinct complex values.*

Proof. Suppose that f is entire and omits two distinct values a and b. Define

$$F(z) := \frac{f(z) - a}{b - a}.$$

Then F is entire and omits 0 and 1.

Fix $0 < \beta < 1$, say $\beta = \frac{1}{2}$. For each $R > 0$, consider the rescaled function

$$F_R(\zeta) := F(R\zeta), \qquad \zeta \in \mathbb{D}.$$

Each F_R is holomorphic on $\mathbb{D}$, omits 0 and 1, and satisfies $|F_R(0)| = |F(0)|$. By Schottky's theorem there exists a constant $C = C(|F(0)|, \frac{1}{2})$, independent of R, such that

$$|F_R(\zeta)| \le C \qquad (|\zeta| \le \tfrac{1}{2}).$$

Equivalently,

$$|F(w)| \le C \qquad (|w| \le R/2).$$

Since $R > 0$ is arbitrary, F is bounded on all of $\mathbb{C}$. By Liouville's theorem, F is constant, and therefore so is f. $\square$

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