DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA747: Measure Theory — Mid Semester Exam
Instructor: Rajesh Srivastava — March 2, 2016
Time duration: Two hours — Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) Let $E = \bigcap_{x \in C} (x + [0,1])$, where C is the Cantor set. Whether E is a Lebesgue measurable set? **[1]**

   (b) Let $E \in \mathcal{M}(\mathbb{R})$ and $m(E) < \infty$. Is it possible that $m(E \cap (n, n+1)) \geq \frac{1}{2}$ for all $n \in \mathbb{Z}$? **[1]**

   (c) Does there exist an F_σ-set in $[0, \infty)$ which is not a G_δ-set? **[1]**

2. For $A \times B \in \mathbb{R}^2$, define $A - B = \{x - y : (x, y) \in A \times B\}$. If $m^*(A - B) > 0$. Show that there exists $(x, y) \in A \times B$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$. **[2]**

3. If $E \subset \mathbb{R}$ is Lebesgue measurable, then show that $F = \bigcup_{x \in E} [x - 1, x + 1]$ is Lebesgue measurable. **[3]**

4. Let (X, S, μ) be complete measure space. Let $f : (X, S, \mu) :\to \mathbb{R}$. If for each $\epsilon > 0$ there exists a set $E \in S$ such that $\mu(E) < \epsilon$ and f is constant on $X \setminus E$, then show that f is measurable. **[3]**

5. Let (X, S, μ) be σ-finite measure space. Define an outer measure $\mu^* : P(X) \to [0, \infty]$ by $\mu^*(A) = \inf\{\mu(E) : A \subset E \text{ and } E \in S\}$. Show that $A \in \mathcal{M}_{\mu^*}$ if and only if for each $\epsilon > 0$ there exists $E \in S$ with $A \subset E$ such that $\mu^*(E \setminus A) < \epsilon$. **[3]**

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sup\{|x + y| : y \in [0, 1]\}$. Show that f is Borel measurable. **[3]**

7. Let $f : (X, S) \to \mathbb{R}$ be measurable and $g : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Show that the function $g \circ f$ is measurable. Does the above conclusion hold if g is Lebesgue measurable ? **[1+1]**

8. Let $f : (X, S, \mu) \to \mathbb{R}$ be measurable and $\mathcal{B}(\mathbb{R})$ denotes the Borel sigma algebra on $\mathbb{R}$. Define a set function $\mu_f : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ by $\mu_f(B) = \mu\left(f^{-1}(B)\right)$. Show that μ_f is a measure on $\mathcal{B}(\mathbb{R})$. **[3]**

9. Let $E \subset \mathbb{R}$ is Lebesgue measurable be such that $m(E) = \alpha$. Show that there exists a Lebesgue measurable set $A \subset E$ such that $m(A) = \frac{\alpha}{2}$. **[4]**

10. If $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that the function g defined by $g(x) = \inf\{|f(t)| : x < t < x + 1\}$ is Lebesgue measurable. Does the conclusion hold if f is bounded Lebesgure measurable? **[3+1]**

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