DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA747: Measure Theory — End Semester Exam
Instructor: Rajesh Srivastava — May 3, 2016
Time duration: Three hours — Maximum Marks: 35

N.B. Answer without proper justification will attract zero mark.

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1. (a) Let $f_n \in L^1([0,1], M, m)$ be such that $\|f_n\|_1 \to 0$. Does it imply that $\sum_{n=1}^{\infty} f_n$ converges in $L^1([0,1], M, m)$? **[1]**
   (b) Whether $L^1 \cap L^\infty(X, S, \mu)$ is a proper dense subspace of $L^1(X, S, \mu)$ for any non-trivial measure space (X, S, μ)? **[1]**
   (c) Let (X, S, μ) be a finite measure space. Does there exists a measurable function f on X such that $\|f\|_p = 1$, for all $p > 0$? **[1]**
   (d) If $1 \leq p < \infty$. Is it necessary that $L^\infty(X, S, \mu) \subset L^p(X, S, \mu)$ if and only if μ is a finite measure? **[1]**

2. By using the fact that $E \in M(\mathbb{R})$ if there exists a F_σ set F such that $m(E \setminus F) = 0$. Show that every Lebesgue measurable function on $\mathbb{R}$ agrees to a Borel measurable function on $\mathbb{R}$ almost everywhere. **[4]**

3. Let $f \in L^1([0,\infty), M, m)$ be continuous. Prove that the function F defined by $F(x) = \int_{[0,x)} f\, dm$ is differentiable and $F' = f$. **[4]**

4. For $\varphi : (\mathbb{R}, M, m) \to \mathbb{R}$ is continuous, define a measure $\nu(B) = m\{\varphi^{-1}(B)\}$, $\forall\, B \in \mathcal{B}(\mathbb{R})$. If $g \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, then show that $\int_B g\, d\nu = \int_{\varphi^{-1}(B)} g \circ \varphi\, d\mu$. **[5]**

5. Let f be a simple function on a σ-finite measure space (X, S, μ) such that $\|f\|_p \leq \frac{1}{2^p}$, for all $p \in (0, \infty)$. If $\mu(X) < \infty$, then prove that $f = 0$ a.e. Conversely, if any simple function φ satisfies $\|\varphi\|_p \leq \frac{1}{2}$, for all $p \geq 1$. Then show that $\mu(X) = 0$. **[4]**

6. Let (X, S, μ) be a finite measure space. If $f \in L^\infty(X, S, \mu)$, then show that $\lim_{p\to\infty} \left[\mu\left\{x \in X : |f(x)| > 1 - \frac{1}{p}\right\}\right]^{1/p} \leq \|f\|_\infty$. **[4]**

7. Let $f : (\mathbb{R}^2, M \otimes M, m \times m) \to \overline{\mathbb{R}}$ be a measurable function. If either of f^+ or f^- belongs to $L^1(\mathbb{R}^2, M \otimes M, m \times m)$, then show that $\int_{\mathbb{R}} \int_{\mathbb{R}} f\, dm\, dm = \int_{\mathbb{R}^2} f\, d(m \times m)$. **[2]**

8. Let $E = \{(x, y) : x^2 + y^2 \leq 1 \text{ and } x + y \geq 1\}$. Show that E is $M \otimes M$- measurable and find the product measure of E. **[2+2]**

9. Let $f \in L^1(\mathbb{R}, M, m)$. If $\varphi(x, y) = \dfrac{f(x+y)}{1+y^2}$, then show that φ is $M \otimes M$- measurable and $\varphi \in L^1(\mathbb{R}^2, M \otimes M, m \times m)$. **[2+2]**

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