# Surface integral

## A revision of Riemann integral of one variable

Let $f : [a,b] \to \mathbb{R}$ be bounded function and $P = \{x_o, x_1, \ldots, x_n\}$ is a partition of $[a,b]$, where $\{a = x_o < x_1 < \cdots < x_n = b\}$. Let $\Delta x_i = x_i - x_{i-1}$. Define $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$ Write

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \text{ and } U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Since $f$ is bounded, there exist $m, M \ge 0$ such that $m \le f(x) \le M$ for all $x \in [a,b]$. Hence

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a).$$

It is easy to see that if $P_1 \subseteq P_2$, then $U(P_1,f) \ge U(P_2,f)$ and $L(P_1,f) \le L(P_2,f)$. It is clear that $L(P,f)$ is an increasing function over the set of all finer partitions while $U(P,f)$ is a decreasing function of $P$.

**Definition 1.** The function $f$ is said to be Riemann integrable (or $f \in \mathcal{R}[a,b]$) if

$$\inf_P U(P,f) = \sup_P L(P,f).$$

Let $\omega(P,f) = U(P,f) - L(P,f)$. From the definition, it follows that

$$\inf_P \omega(P,f) = \inf_P \{U(P,f) - L(P,f)\} = 0, \tag{1}$$

where $\omega(P,f)$ is known as oscillatory sum of $f$ over the partition $P$. Hence, if $f \in \mathcal{R}[a,b]$, then for each $\epsilon > 0$, there exists a partition $P$ such that $\omega(P,f) < \epsilon$. On the other hand, for $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}$, there exists a partition $P_n$ such that $\omega(P_n,f) < \frac{1}{n}$. Thus, $\lim\limits_{n\to\infty} \omega(P_n,f) = 0$.

**Theorem 2.** *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a,b]$ if and only if there exists a sequence $\{P_n\}$ of partitions of $[a,b]$ such that $\lim\limits_{n\to\infty} \omega(P_n,f) = 0$.*

*Proof.* We have already seen the forward implication. For the other one, if $\lim\limits_{n\to\infty} \omega(P_n,f) = 0$, then for each $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ such that $\omega(P_n,f) < \epsilon$, whenever $n \ge n_o$. But, then $\inf\limits_P \omega(P,f) \le \omega(P_{n_o},f) < \epsilon$ for all $\epsilon > 0$. Since $f$ is bounded, both $\inf\limits_P U(P,f)$ and $\sup\limits_P L(P,f)$ exist, and from (4) it follows that $\inf\limits_P U(P,f) = \sup\limits_P L(P,f)$. Hence $f \in \mathcal{R}[a,b]$. □

**Example 3.** Let $f : [0,1] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f$ is bounded and for $P_n = \{\frac{i}{n} : \ i = 0, 1, \ldots, n\}$, we have

$$\omega(P_n, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq 2.\frac{1}{n} \to 0,$$

since $\frac{1}{2}$ can belong to two consecutive subintervals. Hence $f \in \mathcal{R}[0, 1]$.

Recall that if $P_1 \subseteq P_2$, then $U(P_1, f) \geq U(P_2, f)$ and $L(P_1, f) \leq L(P_2, f)$. Hence $\omega(P_1, f) \geq \omega(P_2, f)$. Using this fact, it is enough to workout $\lim_{n\to\infty} \omega(P_n, f) = 0$, while $\{P_n\}$ is an increasing sequence of partitions.

**Theorem 4.** *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a, b]$ if and only if there exists an increasing sequence of partitions $\{P_n\}$ of $[a, b]$ such that $\lim_{n\to\infty} \omega(P_n, f) = 0$.*

*Proof.* Since $f \in \mathcal{R}[a, b]$, by Theorem 45, there exists a sequence of partition $\{P_n\}$ such that $\lim_{n\to\infty} \omega(P_n, f) = 0$. Let $Q_1 = P_1$ and $Q_n = P_1 \cup P_2 \cup \cdots \cup P_n$. Then $\omega(Q_n, f) \leq \omega(P_n, f) \to 0$. The converse part is obvious from Theorem 45. □

**Remark 5.** From Theorem 4 it follows that $\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = \int_a^b f(x)dx$.

**Theorem 6.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}[a, b]$.*

*Proof.* Since $f$ is continuous on the closed interval $[a, b]$, $f$ is bounded and uniformly continuous. For each $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$. Choose a partition $P$ of $[a, b]$ such that $\Delta x_i < \delta$. Since $f$ attains its infimum and supremum on each subinterval, we get $M_i - m_i \leq \frac{\epsilon}{2(b-a)}$. Hence

$$\omega(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq \sum_{i=1}^{n} \frac{\epsilon}{2(b-a)} \Delta x_i < \epsilon.$$

□

**Example 7.** Every monotone function $f$ on $[a, b]$ is Riemann integrable. Assume $f$ is monotone increasing. Let $P_n = \left\{x_i = a + \frac{(b-a)i}{n} : \ i = 0, 1, \ldots, n\right\}$. Then the oscillatory sum

$$\omega(P_n, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\} \frac{b-a}{n} = \{f(b) - f(a)\} \frac{b-a}{n} \to 0.$$

Hence by Theorem 4 we conclude that $f \in \mathcal{R}[a, b]$.

**Continuity like condition for Riemann integrability on** $[a, b]$.

We know that the oscillatory sum $\omega(P, f)$ decreases over the set of finer partitions. And $f$ is Riemann integrable if and only if there is a sequence of partitions $\{P_n\}$ such that $\omega(P_n, f) \to 0$. Using this fact, we derive a continuity like condition for Riemann integrability of bounded function on $[a, b]$. For a given partition $P = \{x_o, x_1, \ldots, x_n\}$ of $[a, b]$, we define $|P| = \max\limits_{1 \le i \le n} \Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$.

**Theorem 8.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}([a, b])$ if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each partition $P$ with $|P| < \delta$ implies $\omega(P, f) < \epsilon$.*

*Proof.* Since $f$ is Riemann integrable, for each $\epsilon > 0$ there exists a partition $P$ of $[a, b]$ such that $\omega(P, f) < \epsilon$. Let $\delta > 0$ be small enough and $P'$ be a refinement of $P$ such that $|P'| < \delta$. As $P \subseteq P'$, it follows that $\omega(P', f) \le \omega(P, f) < \epsilon$. The other implication is obvious by definition of $\mathcal{R}([a, b])$. $\square$

**Corollary 9.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}([a, b])$ if and only if for each sequence of partitions $\{P_n\}$ with $|P_n| \to 0$ implies $\omega(P_n, f) \to 0$.*

**Question*.** Think about, how far can be a Riemann integrable function from continuous function.

# Double integrals

We know that the Riemann integral of a non-negative function of one variable on a finite interval is the area of the region under the graph of the function. In a similar way, the double integral of a non-negative function $f(x, y)$ defined on a region in the plane is the volume of the region under the graph of $f(x, y)$.

First, we discuss double integral on the rectangular region, and later we consider more general region with curvilinear boundary.

Let $D = [a, b] \times [c, d]$ and $f : D \to \mathbb{R}$ be bounded. Let $P_1 = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and $P_2 = \{y_0, y_1, \ldots, y_m\}$ be a partition of $[c, d]$. Note that the partition $P = P_1 \times P_2$ decomposes $D$ into $mn$ sub-rectangles (or cells). Let $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Let $m_{ij} = \inf\{f(x, y) : (x, y) \in D_{ij}\}$. Define

$$L(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} \Delta x_i \Delta y_j.$$

Similarly, we can define

$$U(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} \Delta x_i \Delta y_j,$$

where $M_{ij} = \sup\{f(x, y) : (x, y) \in D_{ij}\}$. The lower integral of $f$ is defined by $\sup_P L(P, f)$. The upper integral of $f$ is defined by $\inf_P U(P, f)$. Note that both the integrals exist because $f$ is bounded. We say that $f$ is integrable on $D$ (or $f \in \mathcal{R}(D)$) if both lower and upper integrals of $f$ are equal. If the function $f$ is integrable on $D$, then the **double integral** is denoted by

$$\iint\limits_D f(x, y)dxdy \text{ or } \iint\limits_D f(x, y)dA.$$

**Example 10.** Let $f : D = [0, 1] \times [0, 1] \to \mathbb{R}$ is given by

$$f(x, y) = \begin{cases} 1 & \text{if } x, y \in \mathbb{Q} \cap [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Then $f$ is not integrable on $D$, because for any partition $P$ of $D$ defined as above, we get $U(P, f) = 1 \neq 0 = L(P, f)$.

**Theorem 11.** *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}(D)$ if and only if for each $\epsilon > 0$ there exists a partition $P$ of $D$ such that $\omega(P, f) = U(P, f) - L(P, f) < \epsilon$.*

**Theorem 12.** *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}(D)$ if and only if there exists an increasing sequence of partitions $\{P_n\}$ of $D$ such that $\lim_{n\to\infty} \omega(P_n, f) = 0$.*

Since the proof of Theorem 12 is similar to Theorem 4, we omit here.

**Example 13.** Let $f : D = [0, 1] \times [0, 1] \to \mathbb{R}$ is given by

$$f(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$

Then $\iint\limits_D f(x, y)dxdy = 0$. Let $P_n = \{\frac{i}{n} : \ i = 0, 1, \ldots, n\} \times \{\frac{i}{n} : \ i = 0, 1, \ldots, n\}$. In this case, $\Delta x_i = \Delta y_j = \frac{1}{n}$. The oscillatory sum of the function $f$ on $D$ satisfies

$$\omega(P_n, f) = \sum_{i=1}^{n} \sum_{j=1}^{n} (M_{ij} - m_{ij}) \Delta x_i \Delta y_j = \sum_{i=1}^{n} \sum_{j=1}^{n} (M_{ij} - 0) \frac{1}{n^2} = \sum_{i=j,\ j=1}^{n} 1.\frac{1}{n^2} = \frac{1}{n} \to 0.$$

**Theorem 14.** *Let $D = [a, b] \times [c, d]$. If $f : D \to \mathbb{R}$ is continuous, then $f$ is integrable on $D$.*

*Proof.* Since $f$ is continuous on the closed rectangle $D$, it follows that $f$ is bounded and uniformly continuous on $D$. Hence for given $\epsilon > 0$ there exists $\delta >$ such that for $(x, y), \ (x', y') \in$

$D$ with $\sqrt{(x-x')^2+(y-y')^2} < \delta$ implies $|f(x,y)-f(x',y')| < \frac{\epsilon}{2A}$, where $A$ is the area of the rectangle $D$. Let $P = \{D_{ij} : \ i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m\}$, where $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Write $d(D_{ij}) = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{j-1})^2}$. Now, suppose $P$ satisfies $d(D_{ij}) < \delta$ for all $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$. Since $f$ attains its infimum and supremum on each closed cell $D_{ij}$, we get $M_{ij} - m_{ij} \leq \frac{\epsilon}{2A}$. Hence

$$\omega(P,f) = \sum_{i=1}^{n}\sum_{i=1}^{m}(M_{ij} - m_{ij})\Delta x_i \Delta y_i \leq \sum_{i=1}^{n}\sum_{i=1}^{m}\frac{\epsilon}{2A}\Delta x_i \Delta y_j < \epsilon.$$

Hence by Theorem 12, we conclude that $f \in \mathcal{R}(D)$. $\square$

**Continuity like condition for Riemann integrability**

Let $P = \{D_{ij} : \ i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m\}$ be a partition of $D = [a,b] \times [c,d]$, where $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Write $d(D_{ij}) = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{j-1})^2}$. Define $|P| = \max\{d(D_{ij}) : \ i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m\}$.

**Theorem 15.** *Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}(D)$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each partition $P$ of $D$ with $|P| < \delta$ implies $\omega(P,f) < \epsilon$.*

Since the proof of Theorem 15 is similar to Theorem 8, we omit here.

**Corollary 16.** *Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}(D)$ if and only if for each sequence of partitions $\{P_n\}$ of $D$ with $|P_n| \to 0$ implies $\omega(P_n, f) \to 0$.*

**Note that** in order to show $f \notin \mathcal{R}(D)$, it is enough to show that there exists a sequence of partitions $\{P_n\}$ with $|P_n| \to 0$ but $\omega(P_n, f) \not\to 0$.

**Geometric Interpretation**

If $f : D = [a,b] \times [c,d] \to [0,\infty)$ is integrable. Then $\iint\limits_D f(x,y)dxdy$ is the volume of the region bounded by planes $x = a, \ x = b, \ y = c, \ y = d$ and the surface $z = f(x,y)$.

## Repeated Integrals

The next result illustrates that the evaluation of the double integral can be reduced to the repeated integrals. This result is known as Fubini's Theorem. Before we come to the main result let us have a look at the following examples.

**Example 17.** Consider $f : D = [0,1] \times [0,1] \to \mathbb{R}$, defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 2y, & \text{if } x \in \mathbb{Q}^c \cap [0,1] \end{cases}.$$

Then $\int_0^1 \left( \int_0^1 f(x,y)dy \right) dx = 1$. However $f$ is not integrable on $D$. (**Hint:** Use Corollary 16 to deduce that $f \notin \mathcal{R}(D)$.)

**Example 18.** Consider $f : D = [0,1] \times [0,1] \to \mathbb{R}$, defined by

$$f(x,y) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2},\ y \in \mathbb{Q} \cap [0,1] \\ -1 & \text{if } x = \frac{1}{2},\ y \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Note that for $x = \frac{1}{2}$, $\int_0^1 f(x,y)dy$ does not exists. However, $\iint\limits_D f(x,y)dxdy$ exists.

**Theorem 19.** (Fubini's Theorem) Let $f : D = [a,b]\times[c,d] \to [0,\infty)$ be integrable. If for each $y \in [c,d]$, the function $f(\cdot,y) \in \mathcal{R}[a,b]$, then the function $F$ defined by $F(y) = \int\limits_a^b f(x,y)dx$ is integrable on $[c,d]$ and

$$\iint\limits_D f(x,y)dxdy = \int_c^d \left( \int_a^b f(x,y)dx \right) dy.$$

*Proof.* Since $f \in \mathcal{R}(D)$, for each $\epsilon > 0$ there exists a partition

$$P = P_1 \times P_2 = \{D_{ij} :\ i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m\}$$

of $D$ such that $U(P,f) - L(P,f) < \epsilon$. Recall that $m_{ij} = \inf\{f(x,y) :\ (x,y) \in D_{ij}\}$ and $M_{ij} = \sup\{f(x,y) :\ (x,y) \in D_{ij}\}$. Let us define $k_j = \inf\{F(y) :\ y_{j-1} \leq y \leq y_j\}$ and $K_j = \sup\{F(y) :\ y_{j-1} \leq y \leq y_j\}$. Since $m_{ij} \leq f(x,y) \leq M_{ij}$ for each $(x,y) \in D_{ij}$, it follows that

$$\sum_{i=1}^n m_{ij}\Delta x_i \leq L(P_1, f(\cdot,y)) \leq \int_a^b f(x,y)dx = F(y) \leq \sum_{i=1}^n M_{ij}\Delta x_i \tag{2}$$

for each $y \in [y_{j-1}, y_j]$. Note the **first inequality** in (6) follows due to the fact that infimum $m_{ij}$ of $f$ on $D_{ij}$ is smaller than the infimum of $f$ over $[x_{i-1}, x_i] \times \{y\}$.

From the above it follows that

$$L(P,f) = \sum_{i=1}^n \sum_{j=1}^m m_{ij}\Delta x_i \Delta y_j \leq \sum_{j=1}^n k_j \Delta y_j = L(P_2, F) \leq U(P_2, F)$$

and

$$U(P_2, F) = \sum_{j=1}^n K_j \Delta y_j \leq \sum_{i=1}^n \sum_{j=1}^m M_{ij}\Delta x_i \Delta y_j = U(P,f).$$

Hence

$$L(P,f) \le L(P_2,F) \le U(P_2,F) \le U(P_2,F) \le U(P,f). \tag{3}$$

Since $U(P,f) - L(P,f) < \epsilon$, from (5) we get $U(P_2,F) - L(P_2,F) < \epsilon$. That is, $F \in \mathcal{R}[c,d]$, and hence once again from (5) we infer that

$$L(P,f) \le \int_c^d F(y)dy \le U(P,f) \text{ and } L(P,f) \le \iint\limits_D f(x,y)dxdy \le U(P,f).$$

Thus,

$$-\epsilon < \iint\limits_D f(x,y)dxdy - \int_c^d F(y)dy \le \epsilon$$

for each $\epsilon > 0$. Hence

$$\iint\limits_D f(x,y)dxdy = \int_c^d F(y)dy.$$

This completes the proof. $\square$

Note that if we define $G(x) = \int\limits_c^d f(x,y)dy$, then the similar result holds.

**Corollary 20.** (Fubini's Theorem) Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be a continuous function. Then

$$\iint\limits_D f(x,y)dxdy = \int_c^d \left(\int_a^b f(x,y)dx\right)dy = \int_a^b \left(\int_c^d f(x,y)dy\right)dx.$$

**Example 21.** Let $f(x,y) = xe^{xy}$ for $(x,y) \in D = [0,2] \times [0,1]$. Then $f$ is continuous and hence by Fubini's theorem

$$\iint\limits_D f(x,y)dxdy = \int_0^2 \left(\int_0^1 xe^{xy}dy\right)dx = \int_0^2 \left[e^{xy}\right]_0^1 dx = \int_0^2 (e^x - 1)dx = e^2 - 3.$$

## Bounded functions with discontinuities

We know from Theorem 14 that if $f$ is continuous on $D$ then $f$ is integrable. In this section, we discuss that the integral of a function $f$ also exists if the set of discontinuities of $f$ is not too large. In order to measure discontinuities, we introduce the following concept.

**Definition 22.** Let $A$ be a bounded subset of $\mathbb{R}^2$. Then $A$ is said to be of content zero if for each $\epsilon > 0$ there exist finitely many rectangles $\{R_i\}_{i=1}^n$ such that $A \subseteq \bigcup\limits_{i=1}^n R_i$ and $\text{Area}\left(\bigcup\limits_{i=1}^n R_i\right) < \epsilon$.

**Example 23.** (i) Any finite set of points in $\mathbb{R}^2$ has content zero.

(ii) Every subset of a set of content zero has content zero.

(iii) The union of finite numbers of bounded sets of content zero is also of content zero.

(iv) Every line segment has content zero.

**Exercise 24.** Any bounded subset of $\mathbb{R}^2$ having non-empty interior cannot have content zero.

**Theorem 25.** *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be a bounded function. If the set of discontinuities of $f$ in $D$ is a set of content zero, then $f$ is integrable.*

*Proof.* Let $M > 0$ be such that $|f(x, y)| \leq M$ for all $(x, y) \in D$. Suppose $E$ is set of discontinuities of $f$ in $D$. In order to prove this result, we need to reorganize some symbols. Let $P = \{D_i : \ D_i \text{ subrectangles in D}\}$ be a partition of $D$. Let $m_i = \inf_{D_i}(f), \ M_i = \sup_{D_i}(f)$ and $A(D_i) = \text{Area}(D_i)$. Now, choose a partition $P$ of $D$ such that

$$E \subset \bigcup_{i=1}^{m} D_i \text{ and } \sum_{i=1}^{m} A(D_i) < \frac{\epsilon}{4M}.$$

Note that $f$ is uniformity continuous on each closed subrectangle $D_i : \ i = m + 1, \ldots, n$. Hence $f$ attains its infimum and supremum on each $D_i$. Thus, as similar argument used in the proof of Theorem 14, we can have selected the partition $P$ such that $M_i - m_i \leq \frac{\epsilon}{2A(D)}$. Hence

$$\begin{aligned}
\omega(P, f) &= \sum_{i=1}^{n} (M_i - m_i) A(D_i) \\
&= \sum_{i=1}^{m} (M_i - m_i) A(D_i) + \sum_{i=m+1}^{n} (M_i - m_i) A(D_i) \\
&\leq \sum_{i=1}^{m} 2M A(D_i) + \sum_{i=m+1}^{n} \frac{\epsilon}{2A(D)} A(D_i) \\
&< 2M \frac{\epsilon}{4M} + \frac{\epsilon}{2} \frac{A(D)}{A(D)} = \epsilon.
\end{aligned}$$

Thus, for each $\epsilon > 0$ we have constructed a partition $P$ of $D$ such that $\omega(P, f) < \epsilon$. This implies $f \in \mathcal{R}(D)$. $\square$

## Double integral over general bounded regions

Let $D$ be a bounded region in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a bounded function defined on $D$. Let $Q$ be a rectangle such that $D \subseteq Q$. Extend $f$ on $Q$ as $\tilde{f} : Q \to \mathbb{R}$, where

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in Q \setminus D. \end{cases}$$

If $\tilde{f}$ is integrable over $Q$, then we say that $f$ is integrable over $D$ and define

$$\iint\limits_D f(x,y)dxdy = \iint\limits_Q \tilde{f}(x,y)dxdy.$$

**Theorem 26.** (Fubini's Theorem) Let $f$ be a bounded continuous function over a bounded region $D$ in $\mathbb{R}^2$.

(i) If $D = \{(x,y) : a \le x \le b \text{ and } f_1(x) \le y \le f_2(x)\}$ for some continuous functions $f_1, f_2 : [a,b] \to \mathbb{R}$, then

$$\iint\limits_D f(x,y)dxdy = \int_a^b \left( \int_{f_1(x)}^{f_2(x)} f(x,y)dy \right) dx.$$

(ii) If $D = \{(x,y) : c \le y \le d \text{ and } g_1(y) \le x \le g_2(y)\}$ for some continuous functions $g_1, g_2 : [c,d] \to \mathbb{R}$, then

$$\iint\limits_D f(x,y)dxdy = \int_c^d \left( \int_{g_1(y)}^{g_2(y)} f(x,y)dx \right) dy.$$

For a proof of Theorem 26, we refer to Chapter 11, Calculus Vol. II, by Apostol.

**Example 27.** $(i)$ Let $D$ be the region bounded by the lines joining the points $(0,0), (0,1)$ and $(2,2)$. Evaluate the integral $\iint\limits_D (x+y)^2 dxdy$.

$(ii)$ Evaluate the integral $\int\limits_0^2 \left( \int\limits_{\frac{y}{2}}^1 e^{x^2} dx \right) dy$.

Riemann integrable functions on $D$ satisfy the following algebraic relations.

**Theorem 28.** *Let $f$ and $g$ be Riemann integrable functions on the region $D$ in the plane and $c \in \mathbb{R}$. Then*

(i) $cf + g \in R(D)$, $\iint\limits_D \{cf(x,y) + g(x,y)\}dxdy = c\iint\limits_D f(x,y)dxdy + \iint\limits_D g(x,y)dxdy.$

(ii) *If* $f(x,y) \leq g(x,y)$ *for all* $(x,y) \in D$, *then* $\iint\limits_D f(x,y)dxdy \leq \iint\limits_D g(x,y)dxdy$.

(iii) $|f| \in \mathcal{R}(D)$ *and* $\left|\iint\limits_D f(x,y)dxdy\right| \leq \iint\limits_D |f(x,y)|dxdy$.

We very often use a short notion for such function as $f = \chi_{\mathbb{Q}\cap[0,1]}$ and call characteristic (or indicator) function of $\mathbb{Q} \cap [0,1]$.

**Remark 29.** I shall explain a bit in the class that every Riemann integrable function is "eventually" continuous. This result is also true for higher dimensional integral. Proof of the result is beyond the scope of the syllabus.

## Change of variable

Change of variables formula is one of the most important results in multivariable calculus. The reason that many problems have a natural coordinate system, and if we look from the right perspective, the calculation gets considerably simplified. As an effect, making the function and the domain of integration simpler.

## Change of variable in single integral

Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Suppose $g : [c,d] \to [a,b]$ is continuously differentiable function ($C^1$ - function) such that $g'(t) \neq 0$ for all $t \in (c,d)$. Then $g$ is one to one (by Mean Value Theorem) and hence monotone. We also assume that $g$ is surjective. Put $x = g(t)$. Then $dx = g'(t)dt$. If $g$ is monotone increasing, then

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))g'(t)dt = \int_c^d f(g(t))g'(t)dt.$$

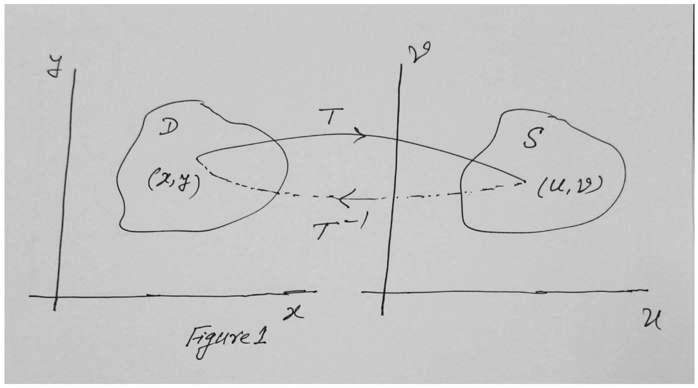
In case, if $g$ is monotone decreasing, then $[c,d] = [g^{-1}(b), g^{-1}(a)]$. Thus, we have the formula

$$\int_a^b f(x)dx = \int_c^d f(g(t))|g'(t)|dt.$$

## Change of variable in double integral

We shall try to imitate the idea developed in one variable to the case of double integral. However, proof of the change of variable formula in higher dimension required some advanced topic and hence we avoid presenting proof of the result here.

Suppose $S$ is a bounded region in the $uv$-plane which is transformed onto the bounded region $D$ in the $xy$-plane by $x = \varphi(u, v)$ and $y = \psi(u, v)$. Now, consider mapping $T$ from $D$ to $S$ which is bijective, continuously differentiable and it's inverse $T^{-1} : S \to D$ is defined by $T^{-1}(u, v) = (\varphi(u, v), \psi(u, v))$. Please see Figure 1.




**Let us assume $T^{-1}$ is also continuously differentiable and it's derivative $(T^{-1})'$ invertible (non-singular) on the interior of $S$.** Hence

$$\det(T^{-1})' = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{vmatrix} = J(u, v) \neq 0 \ ,$$

where function $J$ is known as Jacobi of the transformation. In this way, the function $f(x, y)$ defined on $D$ can be thought of as a function $f(\varphi(u, v), \psi(u, v))$ defined on $S$.

**Theorem 30.** *Let $f(x, y) : D \to \mathbb{R}$ be continuous function and $J$ be as defined above. Then the integral of $f(x, y)$ over $D$ and the integral of $f(\varphi(u, v), \psi(u, v))$ over $S$ are related by*

$$\iint\limits_D f(x, y)dxdy = \iint\limits_S f[\varphi(u, v), \psi(u, v)]\, |J(u, v)| dudv.$$

**Example 31.** Find the area of the region $D$ bounded by the hyperbolas $xy = 1$ and $xy = 2$ and the curves $xy^2 = 3$ and $xy^2 = 4$.

Note that the area of the region is given by $\iint\limits_D dxdy$. Let $u = xy$ and $v = xy^2$. Then $x = \frac{u^2}{v}$ and $y = \frac{v}{u}$. Also $J(u, v) = \frac{1}{v}$. Thus, $\iint\limits_D dxdy = \int\limits_{u=1}^{2} \int\limits_{v=3}^{4} \frac{1}{v} dvdu = \log(\frac{4}{3})$.

**Example 32.** Evaluate the double integral $\iint\limits_D \frac{(x-y)}{(x+y+2)^2} dxdy$ over the region $D$ bounded by the lines $x + y = \pm 1$ and $x - y = \pm 1$.

Let $u = x + y$ and $v = x - y$. Then $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Here $J(u,v) = -\frac{1}{2}$. Also, $u = \pm 1$ and $v = \pm 1$. Hence we have

$$\iint\limits_D \frac{(x-y)}{(x+y+2)^2} dxdy = \frac{1}{2} \int_{u=-1}^{1} \int_{v=-1}^{1} \frac{v}{(u+2)^2} dudv.$$

**Polar coordinates:** In this case the variables $x$ and $y$ are changed to $r$ and $\theta$ by the following two equations

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

We assume that $r > 0$ and $\theta$ lies in $[0, 2\pi)$ so that the mapping $T^{-1}(r,\theta) = (r\cos\theta, r\sin\theta)$ is bijective. Then

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Hence the change of variable formula in this case is

$$\iint\limits_D f(x,y)dxdy = \iint\limits_S f(r\cos\theta, r\sin\theta)\, r\, drd\theta.$$

**Example 33.** Let $D = \{(x,y) : x^2 + y^2 \leq a^2\}$ and $f : D \to \mathbb{R}$ is given by $f(x,y) = 2\sqrt{a^2 - x^2 - y^2}$. Then

$$\begin{aligned} \iint\limits_D f(x,y)dxdy &= 2\int_0^a \int_0^{2\pi} \sqrt{a^2 - r^2} r d\theta dr \\ &= 4\pi \int_0^a r\sqrt{a^2 - r^2} dr \\ &= 4\pi \left. \frac{(a^2 - r^2)^{\frac{3}{2}}}{-3} \right|_0^a = \frac{4\pi a^3}{3}. \end{aligned}$$

# Triple integrals

The concept of double integrals can be extended to functions defined on $D = [a,b] \times [c,d] \times [e,f]$. Consider the partition $P$ of $D$ is of the form $P = P_1 \times P_2 \times P_3$, where $P_1, P_2$ and $P_3$ are partitions of $[a,b], [b,c]$ and $[e,f]$, respectively. For a given partition $P$ and a bounded function $f$ defined on $D$, we can define $\inf\limits_P L(P,f)$ and $\sup\limits_P U(P,f)$, lower integral and upper integral of $f$. If lower and upper integrals are equal, then we say $f$ is integrable and the integral is known as triple integral. It is denoted by

$$\iiint\limits_D f(x,y,z)dxdydz \quad \text{or} \quad \iiint\limits_D f(x,y,z)dV.$$

**Remark 34.** Note that most of the results related to the integrability test of the function of two variables will analogously hold true in the case of the function of three variables. Hence we avoid mentioning those results over here.

**Theorem 35.** (Fubini's Theorem) Let $R$ be a bounded region in $\mathbb{R}^2$ and let $D$ be a bounded domain in $\mathbb{R}^3$ given by $D = \{(x, y, z) : (x, y) \in R \quad \text{and} \quad f_1(x, y) \leq z \leq f_2(x, y)\}$, where $f_1, f_2$ are continuous functions on $R$. If $f$ is continuous on $D$, then

$$\iiint\limits_D f(x, y, z)dV = \iint\limits_R \left( \int_{f_1(x,y)}^{f_2(x,y)} f(x, y, z)dz \right) dA.$$

## Change of variable in a triple integral

The change of variable formula for a double integral can be extended to triple integrals.

$$\iiint\limits_S f(x, y, z)dxdydz = \iiint\limits_T f[\varphi(u, v, w), \psi(u, v, w), \eta(u, v, w)]|J(u, v, w)|dudvdw,$$

where

$$J(u, v, w) = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} & \frac{\partial\varphi}{\partial w} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial w} \\ \frac{\partial\eta}{\partial u} & \frac{\partial\eta}{\partial v} & \frac{\partial\eta}{\partial w} \end{vmatrix}.$$

**Example 36.** Consider the integral $\iiint\limits_D x\, dxdydz$, where $D$ is the region in $\mathbb{R}^3$ bounded by $x = 0, y = 0, z = 2$ and the surface $z = x^2+y^2$. Here $D = \{(x, y, z) : (x, y) \in R, x^2 + y^2 \leq z \leq 2\}$ and $R = \left\{(x, y) : 0 \leq x \leq \sqrt{2}, 0 \leq y \leq \sqrt{2 - x^2}\right\}$. Therefore

$$\begin{aligned} \iiint\limits_D xdxdydz &= \iint\limits_R \left( \int_{x^2+y^2}^{2} x\, dz \right) dA \\ &= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x\, dzdydx \\ &= \frac{8\sqrt{2}}{15}. \end{aligned}$$

**Cylindrical co-ordinates:** In this case the variables $x, y$ and $z$ are changed to $r, \theta$ and $z$ by the following three equations

$$x = r\cos\theta, y = r\sin\theta \quad \text{and} \quad z = z,$$

where $r > 0$ and $\theta \in [0, 2\pi)$. The Jacobian is

$$J(r, \theta, z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Therefore the change of variable formula is

$$\iiint\limits_S f(x, y, z) dxdydz = \iiint\limits_T f(r\cos\theta, r\sin\theta, z)\, r\, drd\theta dz.$$

**Example 37.** Consider $\iiint\limits_D (z^2x^2 + z^2y^2)\, dxdydz$, where $D$ is the region determined by $x^2 + y^2 \leq 1$ and $-1 \leq z \leq 1$. We can describe $D$ in cylindrical coordinates by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ and $-1 \leq z \leq 1$. Therefore,

$$\begin{aligned} \iiint\limits_D \left(z^2x^2 + z^2y^2\right) dxdydz &= \int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1} \left(z^2r^2\right) r\, drd\theta dz \\ &= \int_{-1}^{1}\int_{0}^{2\pi} z^2 \frac{r^4}{4}\bigg|_{r=0}^{1} d\theta dz \\ &= \int_{-1}^{1} \frac{2\pi}{4} z^2 dz = \frac{\pi}{3}. \end{aligned}$$

**Spherical co-ordinates:** In this case the variables $x, y$ and $z$ are changed to $r, \theta$ and $\phi$ by the following three equations

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi.$$

We assume $r > 0, 0 \leq \theta < 2\pi$ and $0 \leq \phi < \pi$ to get mapping of transformation one-one. The Jacobian is

$$J(r, \theta, \phi) = -r^2 \sin\phi.$$

Hence the change of variable formula is

$$\iiint\limits_S f(x, y, z) dxdydz = \iiint\limits_T f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, r^2 \sin\phi\, drd\theta d\phi.$$

# Surface area and surface integrals

A surface is the locus of two points moving in the space ($\mathbb{R}^3$) with two degrees of freedom. A surface can be represented in several ways, however, we discuss some of them.

**Parametric surface:** A parametric surface is the graph of a continuous function of two variables that taking values in $\mathbb{R}^3$. That is, given a continuous function $r : T \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, defined by $r(u, v) = f(u, v)i + g(u, v)j + h(u, v)k$, the set $r(T) = \{r(u, v) : (u, v) \in T\}$ is called parametric surface determined by $r$. We assume that the function $r$ is **one-one in the interior** of $T$ so that the surface does not cross itself. The equations

$$x = f(u, v),\ \ y = g(u, v),\ \ z = h(u, v), \text{where } (u, v) \in T$$

are called parametric equations of the surface $r(T)$.

**Example 38.** $(i)$. Let $T = \{(x, y) : x^2 + y^2 \leq 1\}$. Consider the function $r : T \to \mathbb{R}^3$ given by $r(x, y) = xi + yj + \sqrt{x^2 + y^2}k$. This represents a cone of height 1.

$(ii)$. For fixed $a > 0$ with $0 \leq \theta < 2\pi$ and $0 \leq \phi < \pi$, the equations

$$x = a \sin \phi \cos \theta, \quad y = a \sin \phi \sin \theta, \quad z = a \cos \phi$$

represent a sphere. Here the parameters are $\theta$ and $\phi$.

Note that a parametric surface can degenerate to a point or curve. For instance, if $x = f(u, v),\ y = g(u, v),\ z = h(u, v)$ are constant, then $r(T)$ is a point. On the other hand if $x = u + v, y = (u + v)^2$ and $z = (u + v)^3$, by letting $t = u + v$, $r(T)$ becomes a curve in $\mathbb{R}^3$.

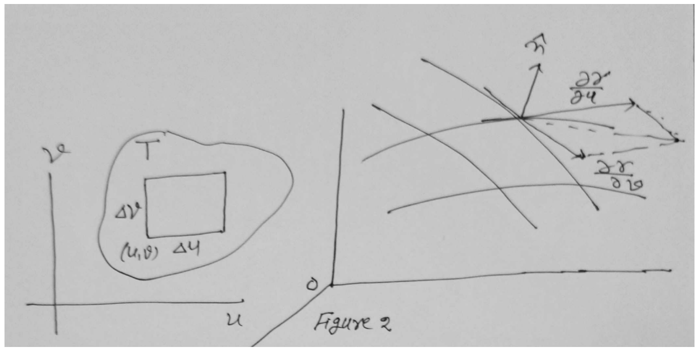
If at $(u, v) \in T$, $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ are continuous and the fundamental product $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \neq 0$, then the point $r(u, v)$ is called a **regular point** of the surface $r(T)$. If one of $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial u}$ is not continuous or $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = 0$ at $(u, v, )$ we say $(u, v)$ is a **singular point**. A surface $r(T)$ is called **smooth** if each point of the surface is regular.

## Area of a parametric surface

Let $S = r(T)$ be be a **smooth** parametric surface defined on the domain $T$. That is, $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ are continuous and $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is never zero on $T$.

If we fix $v$ and allow $u$ to run, then the image of $r$ reduces to a curve in $\mathbb{R}^3$. Hence the distance travel along the curve $r(\cdot, v)$ in a small time interval $\Delta u$ is $\|\frac{\partial r}{\partial u}\|\Delta u$. Similarly, if we fix $u$, then the graph of $r$ is a curve in $\mathbb{R}^3$, and the distance traveled along this curve in a small time interval $\Delta v$ is $\|\frac{\partial r}{\partial v}\|\Delta v$. Please see the Figure 2.

Thus, we see that a small rectangle in $T$ of area $\Delta u \Delta v$ in the $uv$-plane is transferred to a parallelogram on the surface $r(T)$ with area $\left\|\frac{\partial r}{\partial u}\Delta u \times \frac{\partial r}{\partial v}\Delta v\right\| = \left\|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right\| \Delta u \Delta v$.




Note that the point, where $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = 0$, the parallelogram on $r(T)$ will collapse to a curve or a point. Now at each regular point of $r(T)$, the vectors $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ determine a plane having $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ as the normal vector to the surface at the point $(u, v)$. We know that $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \left\|\frac{\partial r}{\partial u}\right\| \left\|\frac{\partial r}{\partial v}\right\| \sin\theta\ \hat{n}$, where $\hat{n}$ is the unit normal to the surface $r(T)$ at $(u, v) \in T$. Hence, the plane determined by $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ is called tangent plane of the surface. Note that the continuity of $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ implies the continuity of $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$. Hence the tangent plane varies continuously on the smooth surface. Thus, the continuity of $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ prevent the occurrence of sharp edges or corners on the surface.

Let us denote the area of small parallelogram obtained by transferring the small rectangle of areas $\Delta u \Delta v$ in the domain $T$ by $d\sigma$. Let $r_u = \frac{\partial r}{\partial u}$ and $r_v = \frac{\partial r}{\partial v}$. Then $d\sigma = \|r_u \times r_v\| \Delta u \Delta v$. Hence the surface area of $r(T)$ denoted by $a(S)$ is given by

$$a(S) = \iint_T \|r_u \times r_v\| \, dudv.$$

**Area of a surface defined by a graph:** Suppose a surface $S$ is given by

$$z = f(x, y), \quad \text{for } (x, y) \in T.$$

That is, $S$ is the graph of the function $f(x, y)$. Then $S$ can be considered as a parametric surface defined by:

$$r(x, y) = xi + yj + f(x, y)k, \quad (x, y) \in T.$$

In this case, $r_x = i + f_x k, \quad r_y = j + f_y k$. Further, $r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x i - f_y j + k.$

Hence the surface area becomes

$$a(S) = \iint_T \sqrt{1 + f_x^2 + f_y^2} \, dxdy.$$

**Example 39.** Let us find the area of the surface of the portion of the sphere $x^2+y^2+z^2 = 4a^2$

that lies inside the cylinder $x^2+y^2=2ax$. Consider $f(x,y)=\sqrt{4a^2-x^2-y^2}$, then

$$f_x=\frac{-x}{\sqrt{4a^2-x^2-y^2}},\quad f_y=\frac{-y}{\sqrt{4a^2-x^2-y^2}}$$

$$\text{and}\quad \sqrt{1+f_x^2+f_y^2}=\sqrt{\frac{4a^2}{4a^2-x^2-y^2}}.$$

Then

$$\begin{aligned}a(S)=&2\iint\limits_T \sqrt{\frac{4a^2}{4a^2-x^2-y^2}}dxdy\\ =&2\times 2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\frac{2ar}{\sqrt{4a^2-r^2}}\,drd\theta.\end{aligned}$$

**Remark 40.** Note that

$$\begin{aligned}\|r_u\times r_v\|^2&=\|r_u\|^2\|r_v\|^2\sin^2\theta\\ &=\|r_u\|^2\|r_v\|^2\left(1-\cos^2\theta\right)\\ &=\|r_u\|^2\|r_v\|^2-(r_u\cdot r_v)^2.\end{aligned}$$

Let $E=r_u\cdot r_u,\ G=r_v\cdot r_v$ and $F=r_u\cdot r_v$, then

$$a(S)=\iint\limits_T\sqrt{EG-F^2}dudv.$$

## Surface integrals

Let $S$ be a parametric surface defined by $r(u,v)$ over $T$. Suppose $r_u$ and $r_v$ are continuous. Let $g:S\to\mathbb{R}$ be bounded. The surface integral of $g$ over $S$, denoted by $\iint\limits_S gd\sigma$, is defined by

$$\iint\limits_S gd\sigma=\iint\limits_T g(r(u,v))\,\|r_u\times r_v\|\,dudv=\iint\limits_T g(r(u,v))\sqrt{EG-F^2}dudv$$

provided double integral in the RHS exists.

**Remark 41.** $(i)$.

$$\iint\limits_S gd\sigma=\iint\limits_T g(r(u,v))\sqrt{EG-F^2}dudv.$$

$(ii)$. If $S$ is defined by $z=f(x,y)$, then

$$\iint\limits_S gd\sigma=\iint\limits_T g[x,y,f(x,y)]\sqrt{1+f_x^2+f_y^2}\,dxdy,$$

where $T$ is the projection of the surface $S$ over the $xy$-plane.

**Example 42.** Let $S$ be the hemispherical surface $z = (a^2 - x^2 - y^2)^{1/2}$. Evaluate

$$\iint\limits_S \frac{d\sigma}{[x^2 + y^2 + (z+a)^2]^{1/2}}.$$

Consider

$$S := r(\theta, \phi) = (a \sin\phi \cos\theta,\ a \sin\phi \sin\theta,\ a\cos\phi),$$

where $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \frac{\pi}{2}$. Note that

$$\sqrt{EG - F^2} = a^2 \sin\phi \text{ and } \left[x^2 + y^2 + (z+a)^2\right]^{1/2} = 2a\cos\frac{\phi}{2}.$$

Hence

$$\iint\limits_S \frac{d\sigma}{[x^2 + y^2 + (z+a)^2]^{1/2}} = \int_0^{2\pi} \int_0^{\pi/2} \frac{a^2 \sin\phi}{2a\cos\frac{\phi}{2}} d\phi d\theta.$$

# Line Integrals

Let $R : [a, b] \to \mathbb{R}^3$ be a differentiable function and the curve $C$ is parameterized by $R(t)$. Suppose $f : C \to \mathbb{R}^3$ is a bounded function. The line integral of $f$ along $C$ is denoted by the symbol $\int\limits_C f \cdot dR$ and is defined by

$$\int_C f \cdot dR = \int_a^b f(R(t)) \cdot R'(t) dt$$

provided the integral in the right-hand side exists. Please see Figure 1.

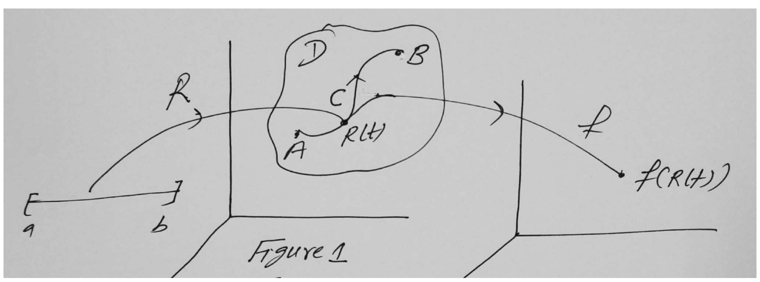


Figure 1

**Remark 43.** Suppose $f = (f_1, f_2, f_3)$ and $R(t) = (x(t), y(t), z(t))$. Then the line integral $\int_C f \cdot dR$ is also written as

$$\int_C f_1 dx + f_2 dy + f_3 dz \quad \text{ or } \quad \int_C f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz.$$

**Example 44.** Let $f = x^2 i + yj +\ (xz - y)k$. Compute the line integral $\int_C f \cdot dR$, along the curve $C$ joining $(0,0,0)$ with $(1,2,4)$.

(i) $C$ is the straight line joining theses points,

(ii) $C$ is the curve given by $R(t) = (t^2, 2t, 4t^2)$.

**The second FTC for line integral:** We know that if $f : [a,b] \to \mathbb{R}$ is continuously differentiable on $[a,b]$, then $\int_a^b f(t)'dt = f(b) - f(a)$. Since $f$ is continuous, $F(x) = \int_\alpha^x f(t)dt$ is differentiable and by FTC it follows that $F(x)' = f(x)$, where $\alpha \in [a,b]$. Hence $\int_a^b F'(x)dx = \int_a^b f(x)dx = \int_\alpha^b f(x)dx - \int_\alpha^a f(x)dx = F(b) - F(a)$. This says that the value of integral of continuously differentiable function depends only on end points and not on the points inside the interval.

We generalize the above second FTC to the line integral.

**Theorem 45.** *Let $D$ be a solid domain in $\mathbb{R}^3$, and $f : D \to \mathbb{R}$ be continuously differentiable. Suppose $A, B$ are two points in $D$. Let $C = \{R(t) : t \in [a,b]\}$ be a curve lying in $D$ and joining the points $A$ and $B$. If $R(t)$ is continuously differentiable on $[a,b]$, then*

$$\int_C \nabla f \cdot dR = f(B) - f(A).$$

*Proof.* Let $h(t) = f(R(t))$. Then by chain rule, we get $h'(t) = (f \circ R)' = \nabla f(R(t)).R'(t)$. Hence

$$\int_C \nabla f \cdot dR = \int_a^b \nabla f(R(t)).R'(t)dt = \int_a^b h'(t)dt = h(b) - h(a) = f(B) - f(A).$$

$\square$

**Remark 46.** Line integral of gradient of a function is independent of the choice of path joining the points $A$ and $B$ in the domain $D$.

**Definition 47.** Let $R : [a,b] \to \mathbb{R}^3$ be a continuous function that represents a curve $C$. The curve $C$ is said to be

(i) **simple** if $R$ is one-one on $(a,b]$.

(ii) **Closed** if $R(a) = R(b)$.

(iii) **Smooth** if $R'$ exists and continuous.

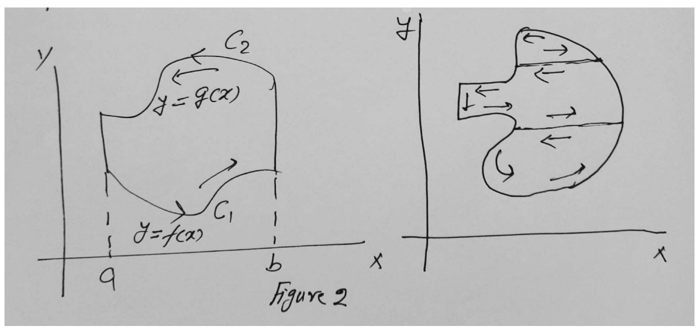
(iv) **Piecewise smooth** if the interval $[a,b]$ can be partitioned into a finite number of subintervals and in each of which the curve is smooth.

**Theorem 48.** *(Green's Theorem) Let $C$ be a piecewise smooth simple closed curve in the $xy$-plane and let $D$ denote the closed region enclosed by $C$. Suppose $M, N, \frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are real valued continuous functions in an open set containing $D$. Then*

$$\iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C (Mi + Nj)dR = \oint_C Mdx + Ndy, \tag{4}$$

*where the line integral is taken around $C$ in the counterclockwise direction.*

Since the identity (4) holds true for every choice of $M$ and $N$ ( satisfying the assumption of Green's Theorem), by letting $M = 0$ and $N$ arbitrary and vice-versa, the identity (4) is equivalent to two identities $\iint\limits_D \frac{\partial N}{\partial x} dxdy = \oint\limits_C Ndy$ and $-\iint\limits_D \frac{\partial M}{\partial y} dxdy = \oint\limits_C Mdy$. We shall present the proof of Green's Theorem for two special cases I and II as shown in Figure 2.




*Proof.* $(i)$ Let $D = \{(x, y) : \ a \leq x \leq b$ and $f(x) \leq y \leq g(x)\}$, where $f$ and $g$ are continuous functions on $[a, b]$. Since $\frac{\partial M}{\partial y}$ is continuous, by Fubini's Theorem, the double integral

$$-\iint\limits_D \frac{\partial M}{\partial y} dxdy = \int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial M}{\partial y} dy\right] dx = \int_a^b M[x, f(x)]dx - \int_a^b M[x, g(x)]dx. \tag{5}$$

On the other hand, we can write

$$\int_C Mdx = \int_{C_1} Mdx + \int_{C_2} Mdx, \tag{6}$$

since the line integral along each of vertical segment is zero. Note that $C_1$ and $C_2$ can be represented by $r_1(t) = ti + f(t)j$ and $r_1(t) = ti + g(t)j$ respectively. Hence

$$\int\limits_{C_1} Mdx = \int_a^b M[t, f(t)]dt \text{ and } \int\limits_{C_2} Mdx = -\int_a^b M[t, f(t)]dt. \tag{7}$$

Negative sign appeared in the second equation since the carve $C_2$ traverses in the reverse direction. Thus, from (5-7) we conclude that the identity (4) holds for the type I region. Similarly, we can obtain the result for the type II region. Further, we can obtain the result for any region which an be decomposed into finitely many regions of the above two types. □

**Area expressed as a line integral:** Let $C$ be a simple (piecewise smooth) closed curve and $D$ be the region enclosed by $C$. Let $M(x,y) = -\frac{y}{2}$ and $N(x,y) = \frac{x}{2}$. Then by Green's Theorem the area of $D$ is

$$a(D) = \iint\limits_D dxdy = \int_D (N_x - M_y)\, dxdy = \int_a^b Mdx + Ndy = \frac{1}{2}\int_C -ydx + xdy.$$

**Example 49.** Note that the integral

$$\int_C xy^2dx + \left(x^2y + x\right)dy = \iint\limits_D dxdy = \text{Area(D)}$$

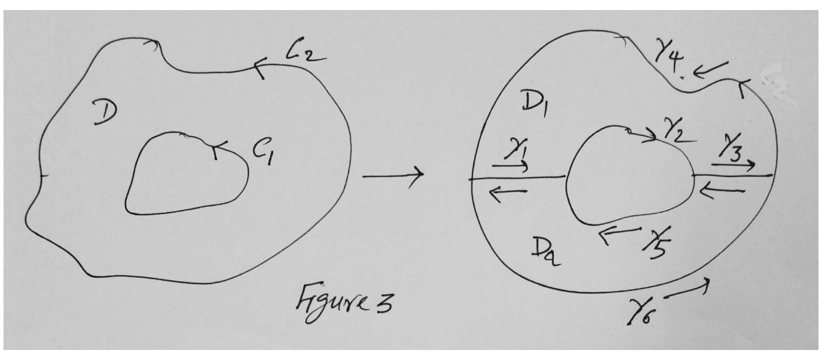
(By Green's Theorem), where $D$ is the region enclosed by $C$. Hence the integral is depending only on the region enclosed by $C$ but not its location.

**Example 50.** Find the area bounded by the ellipse $C = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$.

Consider the parametric form of $C = \{(a\cos t, b\sin t) :\ 0 \le t < 2\pi.\}$ Then the area is

$$\frac{1}{2}\int_C -ydx + xdy = \frac{1}{2}\int_0^{2\pi} -(b\sin t)(-a\sin t)dt + (a\cos t)(b\cos t)dt = \frac{1}{2}\int_0^{2\pi} abdt = ab\pi.$$

**Example 51.** Let $C_1$ and $C_2$ be two simple (piecewise smooth) closed curves as shown in Figure 3.



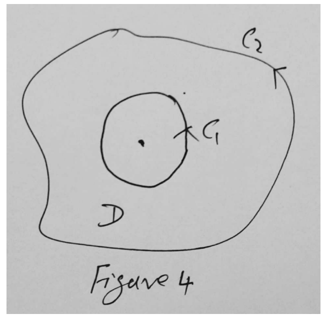

Consider the region $D$ bounded by the curves $C_1$ and $C_2$. Note that $D = D_1 \cup D_2$ and $D_1$ is enclosed by the curves $\gamma_i;\ i = 1,2,3,4$ and $D_2$ is enclosed by curves $\gamma_j;\ i = 1,3,5,6$.

$$\iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint\limits_{D_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy + \iint\limits_{D_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \left(\int_{\gamma_1} \alpha + \int_{\gamma_2} \alpha + \int_{\gamma_3} \alpha + \int_{\gamma_4} \alpha\right) + \left(\int_{\gamma_6} \alpha - \int_{\gamma_3} \alpha + \int_{\gamma_5} \alpha - \int_{\gamma_1} \alpha\right) = \oint_{C_2} \alpha - \oint_{C_1} \alpha,$$

where $\alpha = Mdx + Ndy$.

**Example 52.** Let $C_1$ be unit circle and $C_2$ be any simple closed curve as shown in Figure 4.




Find $\int_{C_2} \frac{xdy - ydx}{x^2+y^2}$. Let $D$ be the domain lies between $C_1$ and $C_2$. A simple calculation shows that $N_x - M_y = 0$ on $D$. By applying Green's Theorem for multiply-connected domain $D$, we get

$$\oint_{C_2} (Mdx + Ndy) - \oint_{C_1} (Mdx + Ndy) \iint_D (N_x - M_y)\, dxdy = 0.$$

Since $C_1 = \{(\cos t, \sin t) : 0 \le t \le 2\pi\}$, we get

$$\int_{C_1} \frac{-ydx + xdy}{x^2 + y^2} = \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\sin^2 t + \cos^2 t} dt = 2\pi.$$

Hence,

$$\oint_{C_2} (Mdx + Ndy) = 2\pi.$$

## Exactness of the line integral

Let $Q$ be a cube in $\mathbb{R}^3$. Suppose $C$ is a curve in $Q$ which is parameterized by $R(t) = (x(t), y(t), z(t))$, where $R : [a, b] \to \mathbb{R}^3$ is continuously differentiable. Now, does there exist a function $F : Q \subset \mathbb{R}^3 \to \mathbb{R}$ such that $\int\limits_C f \cdot dR = \int\limits_C dF$ for every curve $C$ in $Q$? Suppose there exists $F$ such that $\int\limits_C f \cdot dR = \int\limits_C dF$ for every curve $C$ in $Q$. Then by Theorem 45 (second FTC for line integral), it follows that

$$\int_C f \cdot dR = F(R(b)) - F(R(a)) = F(B) - F(A) = \int_C \nabla F \cdot dR.$$

That is,

$$\int_C (f - \nabla F) \cdot dR = 0$$

for all curves $C$ in $Q$. It is easy to see that $f = \nabla F$ on $Q$.

**Remark 53.** Note that it is difficult to prove that $f = \nabla F$ on a general domain $D$. However, the following exercise can be done with small effort.

**Exercise 54.** Let $D = \{(x, y) : \ x^2 + y^2 < 1\}$. If $f : D \to R^2$ is a continuously differentiable function such that $\int\limits_{\Gamma} f \cdot dR = 0$ for every curve $\Gamma$ in $D$, then $f$ constant.

**Example 55.** Show that the line integral

$$\int_C 2x \sin y \ dx + (x^2 \cos y - 3y^2)dy$$

is path independent joining the points $(-1, 0)$ and $(5, 1)$.

# Curl and Divergence

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field given by $F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$.

**Definition 56.** (Curl of $F$) The curl of $F$ is another vector field denoted by curl $F$ and defined by the vector

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times f,$$

where $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$.

**Definition 57.** (Divergence of $F$) The divergence of $F$ is a scalar valued function denoted by div $F$ and is defined by div $F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. We can rewrite the div $F$ as div $F = \nabla \cdot F$.

Now, we recall Green's Theorem to get a motivation for Stoke's Theorem. Let $C$ be the piece-wise smooth curve which encloses the domain $D$ in $\mathbb{R}^2$. Let $F : D \to \mathbb{R}^2$ be a vector field in the plane given by $F(x, y) = M(x, y)i + N(x, y)j + 0k$. By Green's Theorem

$$\iint\limits_D (N_x - M_y) \, dxdy = \oint_C Mdx + Ndy,$$

where $C = \{R(t) : t \in [a, b]\}$. The above identity can be represented as

$$\iint\limits_D \text{curl } F \cdot k \ dxdy = \oint_C F \cdot dR, \tag{8}$$

where curl $F = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k$. Stoke's Theorem is a generalization of the identic (8) in $\mathbb{R}^3$. Before we make a formal statement for Stoke's Theorem, we discuss unit normal vector on some special surfaces.

(i) Suppose the surface $S$ is given by $f(x,y,z)=c$, where $f$ is differentiable function on some domain $D$ in $\mathbb{R}^3$. Please see Figure 5.

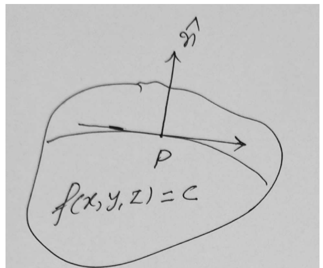


Figure 5

Consider a smooth curve $C$ given by $R:[a,b]\to\mathbb{R}^3$ which lies on the surface $S$ and passes through a point $P$ on $S$. Then $f(R(t))=c$. By the chain rule, we get $f'(R(t)).R'(t)=0$. That is, $\nabla f(R(t)).R'(t)=0$. Since $R'(t)$ is the tangent vector at point $P$, the vector $\nabla f(R(t))$ is the normal vector at $P$. Hence the unit normal vector $\hat{n}$ is given by $\hat{n}=\frac{\nabla f}{\|\nabla f\|}$. Note that $\hat{n}:D\subset\mathbb{R}^3\to\mathbb{R}^3$. If $\hat{n}$ is continuous and never vanishes on $D$, then the surface $S$ is called **orientable.**

(ii) Let $D$ be a domain in $\mathbb{R}^2$. Let $F:D\to\mathbb{R}^3$ given by $F(s,t)=x(s,t)i+y(s,y)j+z(s,t)k$ is a parametrization of surface $S$, where $F$ is smooth (continuously differentiable). Let $P=F(s_o,t_o)$ be a point on the surface $S$. Then $F(s,t_o)$ and $F(s_o,t)$ are curves on $S$ passing through $P$ as shown in Figure 6.

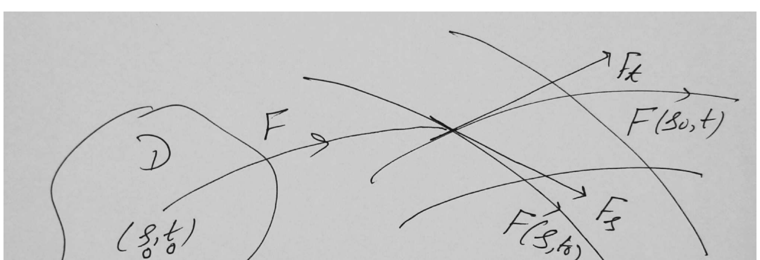


Figure 6

Recall that the fundamental product $F_s\times F_t$ is the normal to the surface $S$ at $P$. Hence unit normal vector to the surface $S$, in this case, is given by $\hat{n}=\frac{F_s\times F_t}{\|F_s\times F_t\|}$. Note that $\hat{n}:D\subset\mathbb{R}^2\to\mathbb{R}^3$.

(iii) If the surface $S$ is given by the graph of smooth function $f : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. That is, $F(x, y) = xi + yj + f(x, y)k$. Then unit normal vector is given by

$$\hat{n} = \frac{F_x \times F_y}{\|F_x \times F_y\|} = \frac{-f_x i - f_y j + k}{\| - f_x i - f_y j + k\|} = \frac{-f_x i - f_y j + k}{\sqrt{1 + f_x^2 + f_y^2}}.$$

**Definition 58.** A surface $S$ is called orientable if unit normal vector to the surface $S$ is continuous and never vanishes.

Hence **orientable surface** is a two-sided surface. Möbius strip is not an orientable surface.

**Theorem 59.** (Stokes' Theorem) Let $S$ be a piecewise smooth orientable surface and $C$ be the piecewise smooth boundary of $S$. Let $F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$ be a vector field such that $P, Q$ and $R$ are continuously differentiable on an open set containing $S$. If $\hat{n}$ is a unit normal vector to $S$, then

$$\iint\limits_S \text{curl } F \cdot \hat{n}\, d\sigma = \oint_C F \cdot dR, \tag{9}$$

where the line integral is evaluated around $C$ in the direction of the orientation of $C$ with respect to $\hat{n}$.

Please see Figure 7.

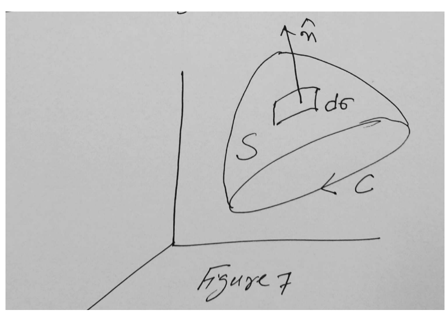


Figure 7

(i) Note that the value of surface integral in (9) depends only on the boundary $C$ and not to the shape of the surface $S$.

(ii) If $S$ is a plane surface, then identity (9) reduces to the identity (8). Thus, Stoke's Theorem can be considered as a direct extension of Green's Theorem.

(iii) For the closed smooth surface like sphere and donut, there is no boundary and in this case $\iint\limits_S \text{curl } F \cdot \hat{n}\, d\sigma = 0$.

(iv) Stoke's Theorem can be extended to a smooth surface whose boundary contains more than one simple smooth closed curve.

**Remark 60.** If a surface $S$ is given by the graph of smooth function $f$ defined on the domain $D \subset \mathbb{R}^2$, then

$$\oint_C F \cdot dR = \iint\limits_D (-f_x i - f_y j + k) \cdot \text{curl } F \; dxdy.$$

**Example 61.** Let $S$ be the part of the cylinder $z = 1 - x^2,\ 0 \le x \le 1,\ -2 \le y \le 2$. Let $C$ be the boundary of the surface $S$ and $F(x, y, z) = yi + yj + k$. Use Stoke's Theorem to find the line integral $\int\limits_C F \cdot dR$.

Here curl $F = -\vec{k}$. Let $z = f(x, y) = 1 - x^2$. The unit normal to the surface $S$ will be given by

$$\hat{n} = \frac{-f_x\vec{i} - f_y\vec{j} + \vec{k}}{\sqrt{1 + f_x + f_y^2}} = \frac{2x\vec{i} + \vec{k}}{\sqrt{1 + 4x^2}}.$$

Surface element $d\sigma(x, y) = \sqrt{1 + f_x + f_y^2}\ dxdy$. Please refer to Figure 8.

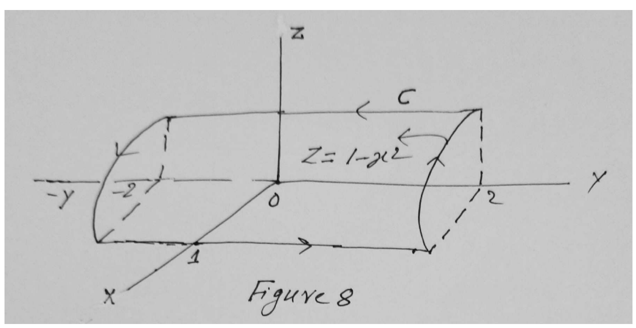


Figure 8

By stoke's Theorem as mentioned in Remark 60,

$$\oint_C F \cdot dR = \iint\limits_D (-f_x i - f_y j + k) \cdot \text{curl } F \; dxdy = \int_{y=-2}^{2} \int_{x=0}^{1} (-1) dxdy = -4.$$

Once again let us look at the Green's Theorem in the plane. Let $F(x, y) = M(x, y)i + N(x, y)j$ be smooth on the domain $D \subset \mathbb{R}^2$, where $D$ is enclosed by the simple and smooth curve $C = \{R(t) : \; t \in [a,b]\}$. Then $R'(t) = x'(t)i + y'(t)j$ is the tangent vector to the curve. Hence $n = y'(t)i - x'(t)j$ is a normal vector to the curve $C$. By Green's Theorem

$$\oint_C (F \cdot n) dt = \oint_C M dy - N dx = \iint\limits_D \left( \frac{\partial M}{\partial x} - \left( -\frac{\partial N}{\partial y} \right) \right) dxdy = \iint\limits_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy.$$

Hence

$$\iint\limits_D \text{div } F \; dxdy = \oint_C (F \cdot n) ds. \tag{10}$$

The generalization of the identity (10) is known as the divergence theorem.

**Theorem 62.** (Divergence Theorem) Let $D$ be a solid domain in $\mathbb{R}^3$ bounded by piecewise smooth and orientable surface $S$. Let $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$ be vector filed which is continuously differentiable on an open set that contains $D$. Let $\hat{n}$ be the unit outward normal to the surface $S$. Then

$$\iiint\limits_D \text{div}\, F \; dV = \iint_S F \cdot \hat{n} \; d\sigma.$$

Refer to Figure 9.

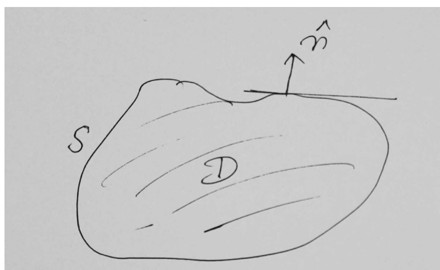
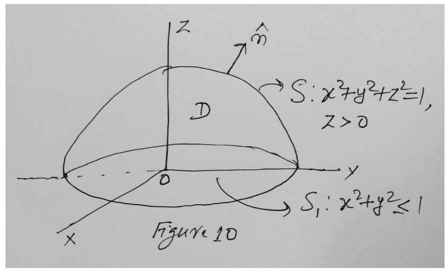


Figure 9

**Example 63.** Let $F(x,y,z) = (x+y)i + z^2 j + x^2 k$. Let $\hat{n}$ be the unit outward normal to the hemisphere $S = \{(x,y,z) : x^2 + y^2 = z^2 = 1 \text{ and } z > 0\}$. Find the surface integral $\iint\limits_S F \cdot \hat{n} \; d\sigma$ using divergence theorem.

Let $F(x,y,z) = (x+y, z^2, x^2)$. Then div $F = 1$. Note that $S$ is not a closed surface. Let $S_1 = \{(x,y) : \; x^2 + y^2 \le 1\}$. Then $S \cup S_1$ is a closed surface and we can apply divergence theorem for it. Please refer to Figure 10.




By divergence theorem,

$$\iint_S F \cdot \hat{n}\ d\sigma + \iint_{S_1} F \cdot \hat{n_1}\ d\sigma_1 = \iiint_D \text{div}\ F\ dV = \frac{2\pi}{3}.$$

Here

$$\iint_{S_1} F \cdot \hat{n_1}\ d\sigma_1 = \iint_{S_1} (x+y, z^2, x^2).(-k) dxdy = \iint_{x^2+y^2\leq 1} x^2 dxdy.$$