Connected Sets:

The structure of real line has been invaded in several ways to know the peculiar hidden properties. We have already seen that an open set $O \subseteq \mathbb{R}$ can be expressed as disjoint union of countably many open intervals. That is,

$$O = \bigcup_{n=1}^{\infty} I_n \; ; \; I_n = (a_n, b_n).$$

Hence any set A in $\mathbb{R}$, we get an open set $O \supset A$ & thus

$$A \subset O \subset \bigcup I_n.$$

Hence, any set can be embedded into countably many open intervals.

The "Connected set" has its natural meaning, and we can extract its definition from the intervals.

We know that an interval cannot be

break into two relatively open parts.

On contrary, suppose that (173)

$$[a,b] = A \dot{\cup} B, \text{ where}$$

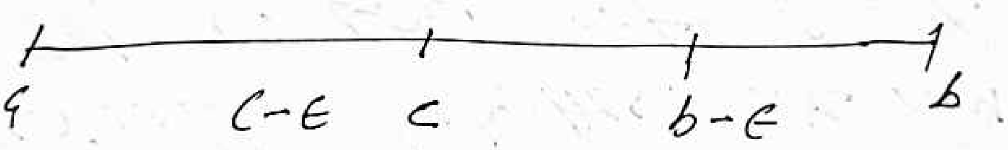
A & B are non-empty, disjoint, and relatively open sets in $[a,b]$.

This implies that A & B are disjoint closed sets too, as

$$A = [a,b] \setminus B \ \& \ B = [a,b] \setminus A.$$

Thus, A & B are disjoint non-empty open & closed sets both (called clopen).

To start with, let $b \in B$. Since B is open, $(b-\epsilon, b] \subset B$ for some $\epsilon > 0$.



Now, let $c = \sup A$. Then $a < c < b$:

If $a = c \Rightarrow A = \{a\}$ - not open, and if

$c = b \Rightarrow A \cap B \neq \Phi$).

By def^n of supremum, $(c-\epsilon, c) \cap A \neq \Phi$,

and $(c, c+\epsilon) \cap B \neq \Phi$. 174

(since c is the deadend of A).

That is, $c \in \bar{A} = A$ and $c \in \bar{B} = B$,

which is a contradiction that $A \cap B = \Phi$.

Hence based on the above observation, we can define connected / disconnected sets.

Def^n: A metric space X is said to be disconnected (not connected) if $\exists$ two non-empty open sets A & B s.t $X = A \cup B$.

The sets A & B are called disconnection of X.

We say that X is connected if X can not be expressed as disjoint union of two non-empty open sets in X.

Thus, the interval $[a,b]$ is connected.

Note that, when $X = A \dot{\cup} B$, where A & B are disjoint, non-empty open sets, it follows that A & B are closed sets too, (as $A = B^c$ & $B = A^c$). Thus, A & B are disjoint non-empty clopen sets. (175)

Thus, X is connected iff X has no nontrivial clopen sets.

(Hint: if A - clopen $\Rightarrow X = A \dot{\cup} A^c$, and A^c is also open)

Def^n: A subset E of a metric space X is called disconnected in E if $\exists$ non-empty disjoint open sets U & V in E such that $E = U \dot{\cup} V$.

Note that $\exists$ open sets A & B in X s.t. $U = A \cap E$ & $V = B \cap E$.

$\Rightarrow E = (A \cap E) \dot{\cup} (B \cap E) = (A \cup B) \cap E$.

$\Rightarrow E \subset A \cup B$.

It is clear that A & B need not be disjoint, However, we can filter them further to make them disjoint, and still cover E.

<u>Lemma:</u> let $E \subset X$. If U & V are disjoint open sets in E, then $\exists$ disjoint open sets A & B in X s.t

$$U = A \cap E \quad \& \quad V = B \cap E.$$

Proof: For $x \in U$, $\exists$ $\epsilon_x > 0$ s.t

$$E \cap B_{\epsilon_x}(x) \subset U. \quad (\because U\text{-open in } E)$$

And for $y \in V$, $\exists$ $\epsilon_y > 0$ s.t

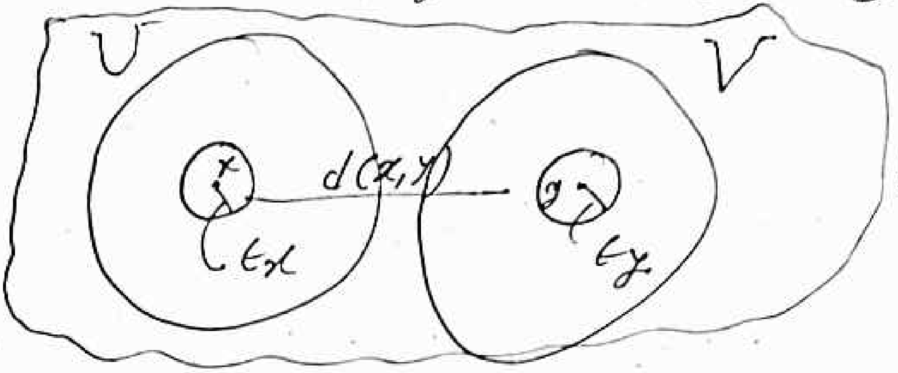
$$E \cap B_{\epsilon_y}(y) \subset V.$$

Now, $U \cap V = \Phi \Rightarrow E \cap (B_{\epsilon_x}(x) \cap B_{\epsilon_y}(y)) = \Phi$.

claim $B_{\epsilon_x/2}(x) \cap B_{\epsilon_y/2}(y) = \Phi$.

Note that if $d(z,x) < \epsilon_x/2$ & $d(z,y) < \epsilon_y/2$, then $d(x,y) < \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2}$.

Choose ϵ_x & ϵ_y s.t. $d(x,y) > \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2}$, then the claim will ~~follows~~. be satisfied.



(177)

write $A = \cup\{B_{\epsilon_x/2}(x) : x \in U\}$

and $B = \cup\{B_{\epsilon_y/2}(y) : y \in V\}$.

Then $A \cap B = \emptyset$, and A & B are open in X, and $E \subset A \cup B$.

Thus, we say, $E \subset X$ is disconnected if $\exists$ disjoint open sets A, B in X s.t $A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$ & $E \subset A \cup B$.

Next, we see that connected subsets of $\mathbb{R}$ are precisely singletons or intervals.

<u>Theorem</u>: A subset E of $\mathbb{R}$ (containing more than one pt) is connected iff for every $x, y \in E$ with $x < y$ implies $[x, y] \subset E$.

Proof: If for some $x, y \in E$, with $x < z < y \Rightarrow z \notin E$, then

$$E \subset (-\infty, z) \dot{\cup} (z, \infty),$$

a disconnection of E. This proves the assertion.

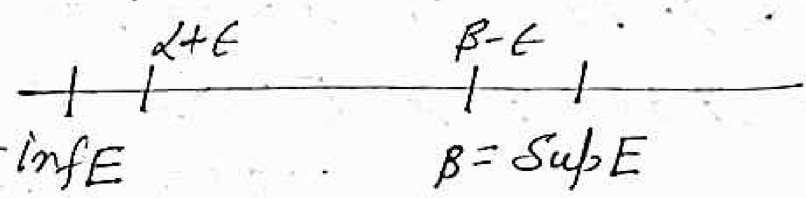
On the other hand, suppose for each pair of $x, y \in E$, we have $[x, y] \subset E$, but E is disconnected. Then $\exists$ of non-empty open sets A & B in $\mathbb{R}$ s.t. $A \cap E \neq \Phi$, $B \cap E \neq \Phi$, & $E \subset A \cup B$.

Let $a \in E \cap A$ and $b \in E \cap B$, and assume $a < b$. Then

$$[a, b] \subset E \subset A \cup B.$$

$\Rightarrow$ $[a, b]$ is disconnected, which is absurd. $(\because [a,b] = (A \cap [a,b]) \dot{\cup} (B \cap [a,b]))$

Finally, suppose for each $x, y \in E$, implies $[x, y] \subset E$. We claim E is an interval.



Note that $(\inf E, \sup E) \subset E$, where we include the possibilities that $\inf E = -\infty$ and $\sup E = +\infty$.

(Hint: for $\epsilon > 0$, $\exists$ $x \in E$, $y \in E$ s.t. $\inf E + \epsilon > x$ & $\sup E - \epsilon < y$)

$\Rightarrow [x, y] \subset E$ $\forall$ $\epsilon > 0$.

Thus, E must be an interval, and it depends upon disposition of $\inf E$ & $\sup E$ as finite or infinite as element, or not, of E.

Ex. Show that the connected subsets of Cantor's set are only singletons (i.e. Cantor set is totally disconnected)

Now, we simplify our study of connected sets with the help of continuous functions.

Notice that discrete metric space (containing more than one pt) is always disconnected. We use this fact to identify disconnected sets through a comparision via conti map.

Theorem: A space X is disconnected iff $\exists$ a continuous onto map $f : X \longrightarrow \{0,1\}$ (two pts discrete space). 180

Proof: If $f : X \longrightarrow \{0,1\}$ is cont & onto, then $A = f^{-1}\{0\}$ & $B = f^{-1}\{1\}$ are non-empty disjoint open sets & $A \cup B = X$. Since, f is cont, A & B are closed. Thus, X has a disconnection.

Conversely, if $X = A \cup B$, where A & B are non-empty disjoint open sets in X, then by setting $f(A) = \{0\}$ & $f(B) = \{1\}$, we can define a continuous onto map

$$f : X \rightarrow \{0,1\}.$$

This result gives a perfect replacement of def^n of connected sets.

Thus, we conclude that X is connected iff every continuous map from X <u>into</u> a discrete space is constant.

Theorem: Let $f:(X,d) \longrightarrow (Y,\rho)$ be continuous, and let $E \subseteq X$. If E is connected, then $f(E)$ is connected. 181

Pf: Suppose $f(E)$ is not connected. Then $\exists\ g: f(E) \xrightarrow[\text{onto}]{\text{cont}} \{0,1\}$.

Thus, $g \circ f: E \xrightarrow[\text{onto}]{\text{cont}} \{0,1\}$

$\Rightarrow E$ is disconnected.

Remark: Non-constant continuous image of and interval is again an interval. This is nothing but the intermediate value ~~property~~ theorem.

Cor: Let I be an interval in $\mathbb{R}$, and $f: I \longrightarrow \mathbb{R}$ be a non-constant continuous function. Then $f(I)$ is an interval.

In particular, if $a, b \in I$, & $f(a) \neq f(b)$, then f assumes all values between $f(a)$ and $f(b)$.

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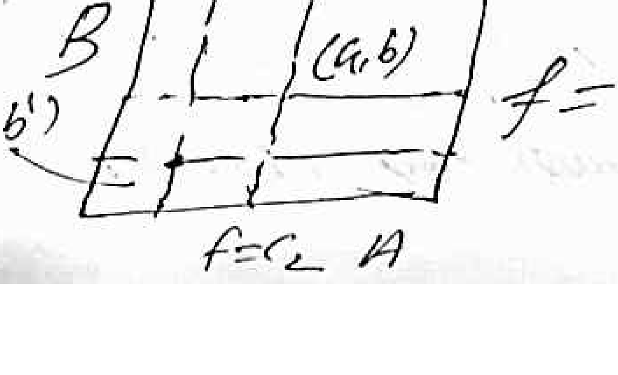
Ex. If A, B are connected subset of a metric space X, then $A \times B$ is connected in $(X \times X, d \times d)$, where

$$(d \times d)(x', y') = d^2(x', y')$$
$$= d(x_1, x_2) + d(y_1, y_2),$$
$$x' = (x_1, y_1),\ y' = (x_2, y_2).$$

Suppose $f: A \times B \longrightarrow \{0, 1\}$ is continuous. We claim f is constant. For $a \in A$ & $b \in B$, $f(a, \cdot)$ & $f(\cdot, b)$ are const function on A & B resp. Since A & B are connected, implies $f(a, \cdot)$ & $f(\cdot, b)$ both are const. for that is, f is constant on every vertical & horizonal lines. Hence f is constant.

$f(a, b) = c_1 = c_2$.



& $f(a', b') = c_1 = c_2$.

Ex. Show that $(0,1)\times(0,1)$ cannot be written as disjoint union of countabely many open balls.

(Hint: $(0,1)\times(0,1)$ is connected) (183)

Ex. Let $D \subset \mathbb{R}$ & $f: D \to \mathbb{R}$ be continuous. Show that D is connected iff the graph of f $G_f = \{(x, f(x)) : x \in D\}$ is connected in $\mathbb{R}^2$.

(Hint: $g: X \to X \times X$, $g(x) = (x, f(x))$ is cont. & G_f is conn ($\because X$ is conn). On the other hand, projection $p_1: G_f \to X \Rightarrow p_1(x, f(x)) = x$ is cont $\Rightarrow X$ is conn.)

Ex. If $A \subset X$ is connected then for $A \subseteq B \subseteq \overline{A}$, it implies that B is connected. In particular, $\overline{A}$ is connected.

Suppose $f: B \xrightarrow[\text{onto}]{\text{cont}} \{0,1\} \Rightarrow f: A \xrightarrow{\text{cont}} \{0,1\}$,
$\Rightarrow f$ is const on $B \Rightarrow f$ is const on B.

Ex. Let $A \subset B \subset X$. If A & X are conn, does it imply B is conn? ($(0,1) \subset (0,1) \cup (1,2) \subset \mathbb{R}$).

Ex. Let $f : [0,1] \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$ [Topologist's Sine Curve]

Show that: f is not cont, but G_f is conn.

(Hint: $g : (0,1] \to [-1,1]$ by $g(x) = \sin\frac{\pi}{x}$. Then is continuous, and hence $g\{(0,1]\}$ is conn $\Rightarrow$ g is onto. Also, G_g is conn. Since

$G_g \subset G_f \subset \overline{G_g} \Rightarrow G_f$ is connected.)

Ex. If $f : X \xrightarrow[\text{onto}]{\text{cont}} Y$ (not conn), then X is not conn.

(Hint: $Y = C \cup D \Rightarrow X = f^{-1}(C) \cup f^{-1}(D)$)

Ex. $L_n(\mathbb{R}) = \{$ Space of all $n \times n$ real matrices$\}$ & $GL_n(\mathbb{R}) = \{A = (x_{ij}) \in L_n(\mathbb{R}) : \det A \neq 0\}$. Then $GL_n(\mathbb{R})$ is disconnected in usual metric on $L_n(\mathbb{R})$.

(Hint: $\det(A) = \sum_{i=1}^{n} x_{ii} \Rightarrow \det$ is conti,

$\Rightarrow GL_n(\mathbb{R}) = (\det)^{-1}(\mathbb{R} \setminus \{0\})$ — is open

Now, $\det : GL_n(\mathbb{R}) \xrightarrow[\text{onto}]{\text{cont}} \mathbb{R}\setminus\{0\}$ (185)

$\Rightarrow GL_n(\mathbb{R}) = GL_n^+(\mathbb{R}) \cup GL_n^-(\mathbb{R})$

is disconnected, where

$\det^{-1}\{(-\infty,0)\} = GL_n^-(\mathbb{R})$ & $(\det)^{-1}\{(0,\infty)\} = GL_n^+(\mathbb{R})$.

(Hint: An easiest metric on $L_n(\mathbb{R})$ is $d(A,B) = \max_{i,j} |a_{ij} - b_{ij}|$)

Path Connected:

A set $E \subset X$ is said to be path connected if for every $x, y \in E$, $\exists$ a conti function $\gamma : [0,1] \to E$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$.

Ex. Show that continuous image of path connected set is path connected.

Let $E \subset X$ be path connected, &

$f : E \to \mathbb{C}$

be continuous. Then for $f(x), f(y) \in f(E)$, $\exists$ path $\gamma : [0,1] \to E$ ($\because x, y \in E$)

s.t. $\gamma(0) = x$ & $\gamma(1) = y$

$\Rightarrow f\circ\gamma(0) = f(x)$, $f\circ\gamma(1) = f(y)$. So

$\gamma' = f\circ\gamma$ is the required path

Connecting $f(x)$ & $f(y)$.

Ex. Let p be a poly. on $\mathbb{C}^n$. Then $\mathbb{C}^n \setminus p^{-1}(0)$ is path connected. (186)

Let $z, w \in \mathbb{C}^n \setminus p^{-1}(0)$. Define

$$\gamma : \mathbb{C} \longrightarrow \mathbb{C}^n \text{ by}$$

$$\gamma(t) = (1-t)z + tw, \quad t \in \mathbb{C}.$$

Then $\{t \in \mathbb{C} : \gamma(t) \in p^{-1}(0)\} = (p \circ \gamma)^{-1}(0)$.

Since $p \circ \gamma$ is a poly on $\mathbb{C}$, it implies that $(p \circ \gamma)^{-1}(0)$ is a finite set. Hence,

$\mathbb{C} \setminus (p \circ \gamma)^{-1}(0)$ is path conn. in $\mathbb{C}$.

Hence $\gamma(\mathbb{C} \setminus (p \circ \gamma)^{-1}(0))$ is path conn. in $\mathbb{C}^n \setminus p^{-1}(0)$, (since $\gamma(\mathbb{C} \setminus (p\circ\gamma)^{-1}(0))$ is contained in $\mathbb{C}^n \setminus p^{-1}(0)$,) containing z & w. Hence, $\mathbb{C}^n \setminus p^{-1}(0)$ is path conn.

(Note that γ is not onto unless $n=1$, hence $\gamma(\mathbb{C} \setminus (p \circ \gamma)^{-1}(0)) \subsetneq \mathbb{C}^n \setminus p^{-1}(0)$.)

One again Topologist's Sine Curve:

$f : [0,1] \longrightarrow [-1,1]$ by

$$f(x) = \begin{cases} \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

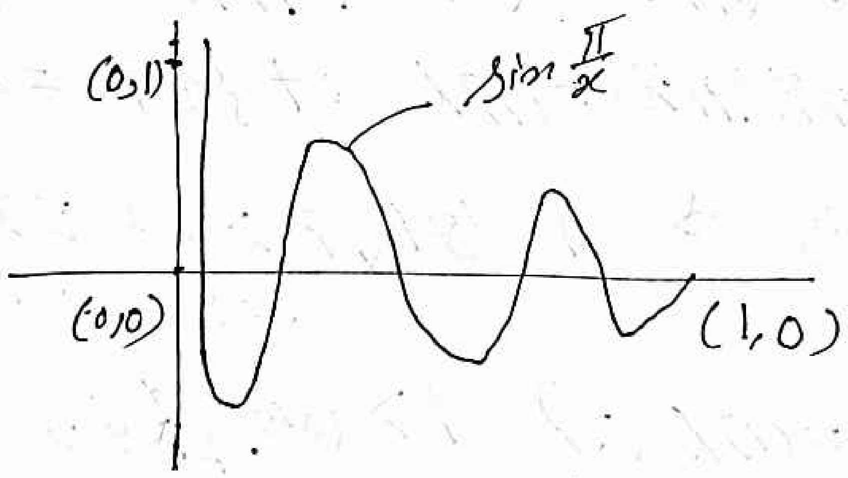
not open.

Then $G_f = \{ (x, \sin\frac{\pi}{x}) : x \in (0,1] \} \cup \{(0,0)\}$.

G_f is not path connected.

(The hope comes from the fact that f is not cont at "0").

On Contrary, Suppose $\exists$ a conti path



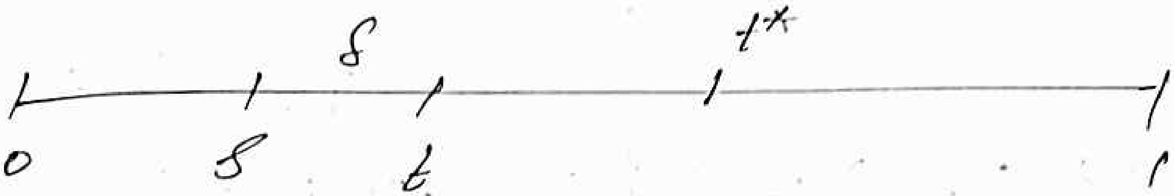
$\gamma : [0,1] \longrightarrow G_f = \{(x, \sin\frac{\pi}{x}) : x \neq 0\} \cup \{(0,0)\}$,

where $\gamma(0) = (0,0)$ & $\gamma(1) = (1,0)$, and $\gamma = (\gamma_1, \gamma_2)$.

Since γ is cont, γ becomes unif cont.

For $\epsilon = 1 > 0$, $\exists$ $\delta > 0$ s.t.

$|s-t| < \delta \Rightarrow |\gamma_2(s) - \gamma_2(t)| < 1$



Since $0 \in \gamma^{-1}\{(0,0)\}$, let

$t^* = \sup \gamma^{-1}(0,0) < 1 \quad (\because \gamma(1) = (1,0))$

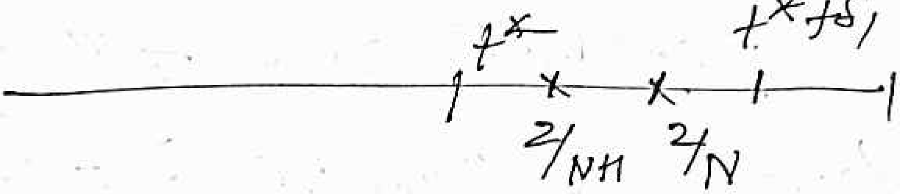
Choose $\delta_1 > 0$ s.t.

$0 \leq t^* < t^* + \delta_1 < 1$ & $\delta_1 < \delta$.

Note that $t^* = \sup\{t : \gamma(t) = (\gamma_1(t), \gamma_2(t)) = (0,0)\}$

$\exists\ t_n \to t^*$ & $\gamma_1(t_n) = 0 \Rightarrow \gamma_1(t^*) = 0$,

but $\gamma_1(t^* + \delta_1) > 0$.



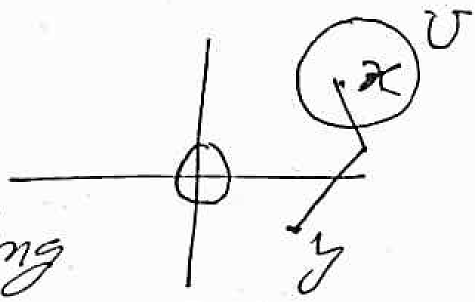
For large N, $\exists\ s, t \in (t^*, t^* + \delta_1)$

s.t. $\gamma_1(t) = 2/N+1$ & $\gamma_1(s) = 2/N$.

$\Rightarrow \gamma_2(t) = \sin\left(\frac{N+1}{2}\right)\pi$, and $\gamma_2(s) = \sin N\pi/2$.

$\Rightarrow |\gamma_2(t) - \gamma_2(s)| = 1$, a contradiction.

Ex. $\mathbb{R}^n \setminus \{0\}$, $(n \geq 2)$ is connected. If not, let U be an open & closed set in $\mathbb{R}^n \setminus \{0\}$. For $x \in U$ & $y \in \mathbb{R}^n \setminus \{0\} \setminus U$, we get a line segments path connecting x & y, say L. Then $L \cap U$ is the ~~non~~ finite union of open & closed sets in $\mathbb{R}$, but $\mathbb{R}$ is connected. Hence, our assumption is wrong and $\mathbb{R}^n \setminus \{0\}$ is connected, infact path connected.



Ex. let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Then S^{n-1} connected. Define

$$\varphi : \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1} \text{ by}$$

$$\varphi(x) = \frac{x}{\|x\|}.$$

Then φ is cont & onto, hence S^{n-1} is connected. Infact, S^{n-1} is continuous image of a path conn. set $\mathbb{R}^n \setminus \{0\}$, hence path conn.

Ex. (Alternative): $I \subseteq \mathbb{R}$ is connected, then I is an interval. (190)

Suppose $\exists\, x, y \in I$. s.t $x < z < y$, but $z \notin I$. Then $f(s) = \begin{cases} 1 & s < z \\ -1 & s > z \end{cases}$

implies $f : I \setminus \{z\} = I \xrightarrow[\text{onto}]{\text{cont}} \{\pm 1\}$.

$\Rightarrow I$ is not connected.

Ex. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $G_f = \{(x, f(x)) : x \in \mathbb{R}\}$ is closed and connected in $\mathbb{R}^2$, then f is continuous.

Let $x_n \to x$. Assume $f(x_n) \to y$. Then $(x_n, f(x_n))$ is a C.C. in $\mathbb{R}^2$ and hence $(x_n, f(x_n)) \to (x, y)$. But G_f is closed, implies $y = f(x)$. Thus, f is continuous.

Notice that $f(x_n) \to y$ can be acheived by considering f is bounded, where $x_n \to x$.

If f is bounded, then $f(x_n)$ is bounded in $\mathbb{R}$, and by B-W-T, $f(x_{n_k}) \to y \in \mathbb{R}$.

$\Rightarrow (x_{n_k}, f(x_{n_k}))$ is a C.S in $\mathbb{R}^2$, and hence conv. Say, $(x_{n_k}, f(x_{n_k})) \to (x, y)$. 191

But G_f is closed, implies $y = f(x)$.

Notice that there is no other limit pt for $(x_n, f(x_n))$ than $(x, f(x))$, else f will not be well-defined. Thus,

$(x_n, f(x_n)) \to (x, f(x))$. Hence, f is continuous.

Notice that so far we have not used the fact that G_f is connected.

Next case is when $|f(x_n)| \to \infty$, where $x_n \to x$.

In this case, we reach to a contradiction that G_f is disconnected in a nbd of x.

We claim that $\exists\, \delta > 0$ st

$|x-y| < \delta \Rightarrow$ either $|f(x)-f(y)| < 1$ or $|f(x)-f(y)| > 2$

(bounded below & above in a nbd of x):
If it is false, then $\exists$ seq^n u_n with
$|u_n - x| < \frac{1}{n}$ s.t. $1 \leq |f(x) - f(u_n)| \leq 2$. (192)

$\Rightarrow \exists$ a subsequence $f(u_{n_k})$ of $f(u_n)$
s.t. $f(u_{n_k}) \to w$. Then
$(u_{n_k}, f(u_{n_k})) \to (x, w)$ & the
graph G_f is closed, implying

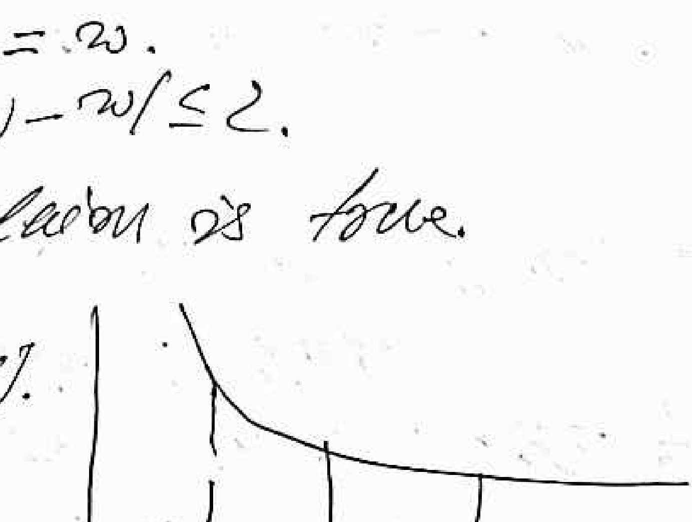
$$f(x) = w.$$

But $1 \leq |f(x) - w| \leq 2$.

Thus, our claim is true.

Let $[a,b] = [x-\delta, x+\delta]$.

We claim that
$G_f \cap \{[a,b] \times R\}$
is connected.



On the other hand,
$G_f \cap ([a,b] \times R) = (G_f \cap \{[a,b] \times R\}) \cap \{(t,s) : |f(x) - s| < 1\}$
$\cup (G_f \cap [a,b] \times R) \cap \{(t,s) : |f(x) - s| > 1\}$.

i.e. $G_f \cap ([a,b]\times\mathbb{R}) = A \cup B$. ——— (*)

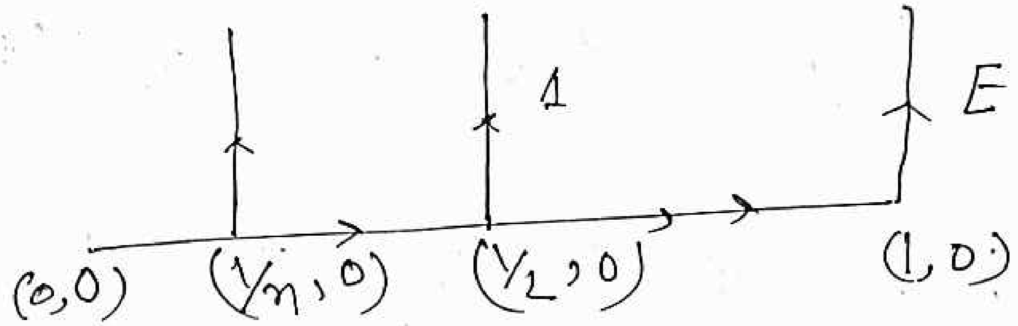
$\Rightarrow G_f \cap ([a,b]\times\mathbb{R})$ is disconnected as (193)
$(x, f(x)) \in A$ & $(x_n, f(x_n)) \in B$, for large n.

This implies, $x_n \to x \Rightarrow f(x_n)$ is bounded.
Hence, from the previous case, it follows that $f(x_n) \to f(x)$.

To show G_f is connected, let
$$g : G_f \cap ([a,b]\times\mathbb{R}) \to \{0,1\}$$
be continuous. Then g can be extended continuously outside $G \cap ([a,b]\times\mathbb{R})$ by constant. Hence,
$$g : G \to \{0,1\} \text{ is cont.}$$
But G is conn, hence g is constant.
Thus, $G_f \cap ([a,b]\times\mathbb{R})$ is connected.

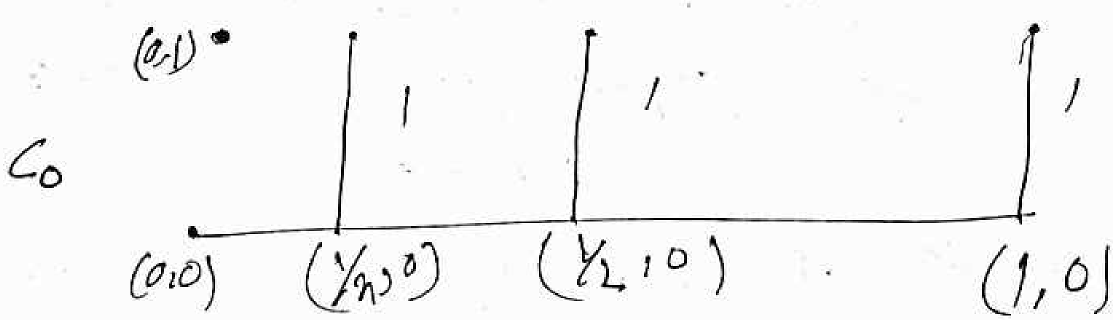
Ex. Let $K = \{\frac{1}{n} : n \geq 1\}$ and
$$E = ([0,1]\times\{0\}) \cup (K\times[0,1]).$$

Then E is path connected. (why?)



let $C = E \times (\{0\} \times [0,1])$, known as Comb space is path connected. The deleted comb space $C_0 = E \cup \{(0,1)\}$ is connected, since $E \subset C_0 \subseteq \overline{E}$, and E is connected.

But C_0 is not path-connected,



because, there is no path connecting $(0,1)$ & $(1,0)$.

On contrary, suppose

$\gamma : [0,1] \longrightarrow C_0$ be cont path s.t. $\gamma(0) = (0,1)$ & $\gamma(1) = (1,0)$.

Then $\gamma^{-1}((0,1))$ is a closed set, and

let $t_0 = \sup \gamma^{-1}((0,1))$

$= \sup\{t \in [0,1] : \gamma(t) = (0,1)\}$.

We claim that $\exists\ t_1 \in (t_0, 1]$ s.t

$(P_1 \circ \gamma)\{(t_0, t_1)\} \subseteq K$, where

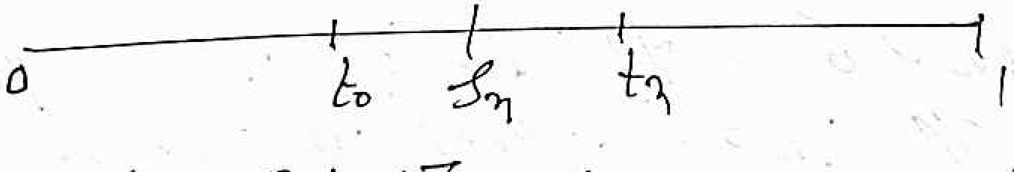
$P_1 : \mathbb{R}^2 \to$ x-axis is the projection.

Suppose the claim is false, then $\exists\ t_n \in (t_0, 1]$ s.t $t_n \to t_0$. By assumption, $\exists\ s_n \in (t_0, t_n)$ s.t $\gamma(s_n) = (x_n, 0)$ for some $x_n \in [0,1] \setminus K$.

Note that $s_n \to t_0$. By continuity

$(x_n, 0) = \gamma(s_n) \to \gamma(t_0) = (0,1)$,

which is absurd.



Thus, $t_1 \in [t_0, 1]$ s.t $(P_1 \circ \gamma)\{(t_0, t_1)\} \subseteq K$.

$\Rightarrow$ $1 \in (P_1 \circ \gamma)(t_0, t_1)$ is connected subset of K. Hence $(P_1 \circ \gamma)[t_0, t_1) = \{1\}$ (by continuity), but $(P_1 \circ \gamma)(t_0) = 0$, an absurd.

Ex. Let U be an open set in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then U path conn. iff conn.

Let A be the collection of all path connecting a point $p \in U$. Then A is open. Let $q \in A$, then $q \in U$ (196)

$\Rightarrow B_r(q) \subset U$ for some $r > 0$.

Let $s \in B_r(q)$, then s is connected by a path to q, say straight line, and q is conn. to p. Hence,

$$B_r(q) \subset U.$$

Let $B = A \setminus U$. Then B is also open. Since for $t \in B$, $\nexists$ path connecting p. Then we can draw a ~~for~~ ball ~~[illegible]~~ surrounding t, which is not conn. to p. Thus,

$$U = A \dot{\cup} B.$$

Since U is connected, implies $B = \Phi$.

Thus, U is path connected.