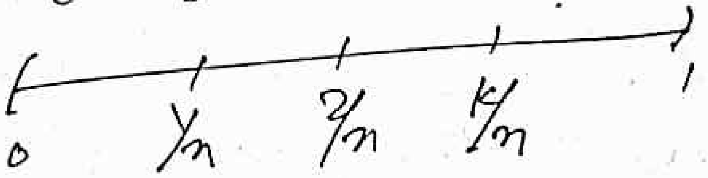
Totally bounded set:

Suppose A be a bounded set in $\mathbb{R}$, and (w.l.g) $A \subset (0,1)$. Then for $\epsilon = 1/n > 0$,

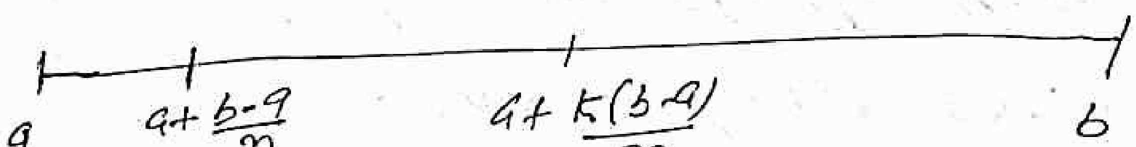


$$A \subset \bigcup_{k=1}^{n} \left(\frac{k-1}{n}, \frac{k}{n} \right].$$

That is, A can be covered by finitely many intervals of arbitrarily small length.

Similar argument can be produced for bounded set $A \subset \mathbb{R}^n$ (or in finite dim. spaces).

Notice that if A is bounded in $\mathbb{R}$, then $A \subseteq [a,b]$ ($a = \inf A$, $b = \sup A$, $b-a<\infty$).



Hence, $A \subseteq \bigcup_{k=1}^{n} \left[a + \frac{(k-1)(b-a)}{n},\ a + \frac{k(b-a)}{n} \right]$.

Notice that with small perturbation of the intervals, A can be covered by open intervals of arbitrarily small length $\epsilon>0$. (123)

However, if the dim of the space X is infinite, then the above property need not be inherited for arbitrary bounded set.

E.g. $X = \ell^1$, $e_n = (0,0,\dots,\underset{n^{th}}{1},0,\dots)$,

$\|e_n - e_m\|_1 = 2$, if $n \neq m$.

$\Rightarrow A = \{e_n : n \geq 1\} \subset B_1[0] \subset B_2(0)$.

That is A is bounded.

Notice that for any ϵ, $0<\epsilon<1$, if

$A \subset \bigcup_{n=1}^{\infty} B_\epsilon(e_n)$.

But A cannot be covered by finitely many balls of arbitrarily small radius

i.e. if $A \subset \bigcup_{i=1}^{n} B_\epsilon(f_i)$; $f_i \in \ell^1$,

then for $\epsilon < 1$, each ball $B_\epsilon(x_i)$ can contain exactly one point of A. (124)
$(\because \|e_n - e_m\|_1 = 2)$.

Also, notice that A has no convergent subsequence. Since ℓ^1 is complete, it is equivalent to say that A has no Cauchy subsequence.

Defn: $A \subseteq (X, d)$ is said to be totally bounded (TB) if $\forall \epsilon > 0$, $\exists x_1, \dots x_n \in X$ s.t. $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$.

We can show that centers of these balls can be taken some points of A. Since

$$A \subseteq \bigcup_{i=1}^{n} B_{\epsilon/2}(x_i).$$

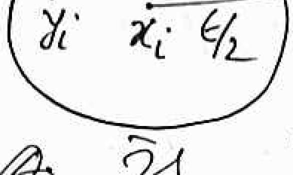
Also, we can assume that $A \cap B_{\epsilon/2}(x_i) \neq \emptyset$ $\forall i = 1, \dots n$. Then $\exists y_i \in A \cap B_{\epsilon/2}(x_i)$.

Rem. It is easy to see that

$$A \subset \bigcup_{i=1}^{n} B_\epsilon(y_i)$$

(Hint: $x \in A \Rightarrow d(x, x_i) < \epsilon/2$ for some i)

& $\exists\, y_i \in A \cap B_{\epsilon/2}(x_i) \Rightarrow d(x, y_i) < d(x, x_i) + d(x_i, y_i) < \epsilon$)



Moreover, if A is T.B, then we can replace balls with sets in A with ~~small~~ arbitrarily small diameter.

Result: A in (X, d) is T.B. iff $\forall \epsilon > 0$, $\exists$ sets $A_1, \ldots, A_n \subset A$ with $\delta(A_i) < \delta$ s.t. $A \subseteq \bigcup_{i=1}^{n} A_i$.

Pf: Let A be T.B. Then $\forall \epsilon > 0$, $\exists$ $x_1, \ldots, x_n \in A$ s.t.

$$A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i).$$

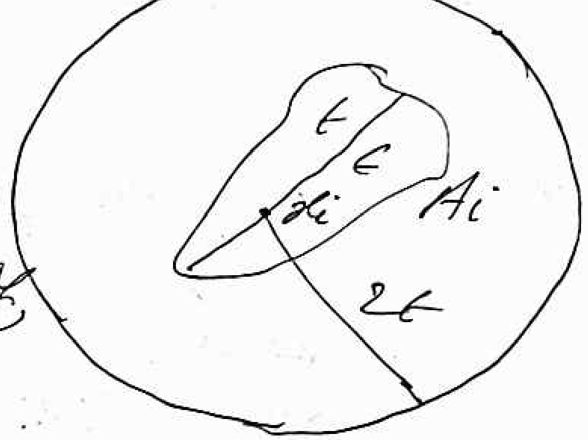
Set $A_i = A \cap B_\epsilon(x_i) \subseteq A$ & $\delta(A_i) \leq 2\epsilon$.

i.e. $A = \bigcup_{i=1}^{n} A_i$, $\delta(A_i) < 2\epsilon$.

Conversely, suppose $\forall \epsilon > 0$, $\exists A_i \subset A$ s.t. $A \subseteq \bigcup_{i=1}^{n} A_i$, with $\delta(A_i) < \epsilon$.

Let $x_i \in A$, then

$A_i \subseteq B_{2\epsilon}(x_i)$.



Since $\epsilon > 0$ is arbitrary we get

$$A \subseteq \bigcup_{i=1}^{n} B_{2\epsilon}(x_i).$$

Notice that if $A \subseteq \bigcup_{i=1}^{n} B_i$; $B_i \subseteq X$, with $\delta(B_i) < \epsilon$, then for $A_i = A \cap B_i \subset A$.

(*) $$A = \bigcup_{i=1}^{n} A_i ; \quad \delta(A_i) < \delta.$$

It is easy to see that if A is T.B in (X,d), then A is bounded.

Also, every finite set $A = \{x_1, \dots, x_n\}$ is T.B. b.c. $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$.

Notice that total boundedness of a set is solely depends upon metric. (127)

In fact, in discrete metric space (X, d_0), $A \subset X$ is T.B. iff A is finite.

(Hint: If $A \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)$; $x_i \in A$, then for $0<\epsilon<1/2$, each $B_\epsilon(x_i) = \{x_i\}$.).

However, if $X = l_1$, $\|e_n - e_m\|_1 = 2$, $n \neq m$, $A = \{e_n : n \geq 1\}$ cannot be covered by finitely balls of radius < 2.

In fact, $A = \{e_n : n \geq 1\}$ with

$$d(e_n, e_m) = \begin{cases} 2 & \text{if } n \neq m \\ 0 & \text{o.w.} \end{cases}$$

(in its own discrete metric) is not totally bounded.

Ex. Every subset of T.B set is T.B.

Ex. $A \subseteq \mathbb{R}$ is T.B. iff A is bounded.

Ex. A is T.B. iff A is covered by finitely many closed sets of arbitrarily small diameters. (128)

(Hint: $A \subset \bigcup_{i=1}^{n} A_i$; $\delta(A_i) < \epsilon$, but $\delta(\bar{A_i}) = \delta(A_i) < \epsilon$ & $A \subset \bigcup_{i=1}^{n} \bar{A_i}$)

Ex. A is T.B. iff $\bar{A}$ is T.B.

If A is T.B., then $A \subseteq \bigcup_{i=1}^{n} A_i$, $\delta(A_i) < \epsilon$.

$\Rightarrow \bar{A} \subseteq \bigcup_{i=1}^{n} \bar{A_i}$; $\delta(\bar{A_i}) < \epsilon$.

$\Rightarrow \bar{A}$ is T.B.

On the other hand, if $\bar{A}$ is T.B., then for $\epsilon > 0$, $\exists$ $x_1 \dots x_n \in X$ s.t

$A \subseteq \bar{A} \subseteq \bigcup_{i=1}^{n} B_i$; $\delta(B_i) < \epsilon$.

Ex. If $A \subset \mathbb{R}^n$ is bounded, then A is T.B.

Result: Let (x_n) be a seq^n in (X, d), & let $A = \{x_n : n \geq 1\}$ (range of (x_n)).

(i) If (x_n) is C.S., then A is T.B.

(ii) If A is T.B, then (a_n) has a Cauchy subsequence.

Proof (i) Since (x_n) a C-S, for $\epsilon>0$, $\exists\, N\in\mathbb{N}$ s.t. $d(x_n, x_N) < \epsilon \quad \forall\, n \geq N$

$\Rightarrow \delta\{x_n : n \geq N\} \leq \epsilon$

$A = \{x_i\}_{i=1}^{N-1} \cup \{x_n : n \geq N\}$

$A \subseteq \bigcup_{i=1}^{N-1} B_\epsilon(x_i) \cup B_\epsilon(x_N)$

$\Rightarrow$ A is totally bounded.

(ii) If A is finite, then trivial.

Suppose A is an infinite set & T.B. Then A can be covered by finitely many sets of diameter < 1. And one of them, say A_1 will contain infinitely many points of A.

But A_1 is also T.B. & hence

covered by finitely many sets of diameter $< 1/2$. Let A_2 be one of them having infinitely many points from A. Thus,

$$A_1 \supset A_2 \supset \cdots \supset A_K \supset A_{K+1} \supset \cdots,$$

where each A_K is an infinite set with $\delta(A_K) < \frac{1}{K}$.

Choose $x_{n_K} \in A_K$. Then

$$\delta\{x_{n_K} : j \geq K\} \leq \delta(A_K) < \frac{1}{K}$$

($\because A_K$ are decreasing).

Thus x_{n_K} is a Cauchy sequence.

Ex. $x_n = (-1)^n$ has Cauchy subsequences, as it is T.B.

Ex. Let $e_n \in l^2$; $e_n = (0, 0, \overset{n\text{th}}{1}, 0, \cdots)$. Then (e_n) has no Cauchy subsequence.

<u>Theorem!</u> A set $A \subset (X, d)$ is T.B. iff every sequence in A has Cauchy subseq^n.

Proof: Let A be a T.B. set in X, and (x_n) be a seq^n in A. Then (x_n) is T.B, & by previous result (x_n) has Cauchy Subsequence. (131)

For the other implication, suppose A is not T.B. Then $\exists\ \epsilon > 0$, s.t.

$$A \neq \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

for every choice of finite set $\{x_1, \cdots x_n\}$.

Thus, for each $n \geq 1$, $\exists\ y_n \in A$ s.t

$$d(y_n, x_i) \geq \epsilon \quad \forall\ i = 1, 2, \cdots, n.$$

Notice that y_n's must be distinct (or an infinite set), else A will be covered by finitely many balls of radius ϵ.

Also, notice that (y_n) cannot be a C-S, else y_n will be covered by finitely many ϵ-balls & hence A is covered by ϵ-balls.

This implies that (x_n) has no Cauchy subsequence (as x_n's are distinct). (132)
Therefore, A must be totally bounded.

Cor (The Bolzano-Weierstrass theorem):

Every bounded infinite subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$.

Pf: Let A be an infinite bounded set in $\mathbb{R}$. Then $\exists$ a distinct seq^n $x_n \in A$. Since A is T.B, x_n has ~~conv. S.~~ Cauchy subsequence x_{n_K}. But $\mathbb{R}$ is complete, implies $x_{n_K} \to x \in \mathbb{R}$. Thus, x is a limit point of A.

We know that a metric space X is complete iff every C.C. in X has limit in X, and every closed set in X is complete. In fact, if X is complete, then $A \subseteq X$ is complete iff A is closed.

We can see that complete metric space has some common properties like $\mathbb{R}$. (133)

<u>Theorem</u>: Let (X,d) be a metric space. Then TFAE:

(i) (X,d) is complete

(ii) (<u>Nested set theorem</u>): Let F_n be a decreasing seq^n of closed sets in X with $\delta(F_n) \to 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \Phi$. (exactly one pt)

(iii) (<u>Bolzano-Weierstrass thm</u>): Every infinite, totally bounded subset of X has a limit point in X

<u>Proof</u>: (i) $\Rightarrow$ (ii):

Let $F_n \supset F_{n+1} \supset \cdots$ & $\delta(F_n) \to 0$. Choose $x_n \in F_n$, then $\delta\{x_k : k \geq n\} \leq \delta(F_n) \to 0$.

Hence, (x_n) is a c.s in X, and by (i), $x_n \to x \in X$.

Since F_n's are closed, $x \in F_n$ for each n.

$\Rightarrow x \in \bigcap_{n=1}^{\infty} F_n \Rightarrow \bigcap_{n=1}^{\infty} F_n \neq \Phi$.

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(In fact $\bigcap_{n=1}^{\infty} F_n = \{x\}$, exactly one pt).

(ii) $\Rightarrow$ (iii): Let A be an infinite, T.B. set in X. Notice that A contains ~~an~~ a distinct Cauchy seqn x_n ($x_n \neq x_m$ if $n \neq m$), because A is T.B. Set

$$A_n = \{x_k : k \geq n\}.$$

Then $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$

and $\delta(A_n) \to 0$ ($\because x_n$ is a C.C.).

But then $\overline{A_n} \supseteq \overline{A_{n+1}} \cdots$ &

$$\delta(\overline{A_n}) = \delta(A_n) \to 0.$$

By (ii), $\exists\ x \in \bigcap_{n=1}^{\infty} \overline{A_n} \neq \Phi$.

Now, $x_n \in A$, and $d(x_n, x) \leq \delta(\overline{A_n}) \to 0$.

Hence, $x_n \to x$. So x is a limit point of A.

(iii) $\Rightarrow$ (i): Let x_n be a C.C. in X. We only need to show that

(x_n) has conv. subsequence.

Note that $A=\{x_n : n\geq 1\}$ is T.B. because (x_n) is a C.S. If A is finite, then trivial, otherwise (iii) implies A has a limit point. That is, $\exists\ x_{n_k}\to x\in X$. Hence, $x_n\to x\in X$.

Ex. Suppose that every countable, closed set in X is complete. Show that X is complete.

Ex. Show that X is complete iff every closed ball in X is complete.

<u>Remark</u>: The total boundedness of a set is all about; an infinite set cannot be too scattered. That is, the substantial portion of the set can be put into (or lies in) into a set of arbitrarily "small" size by a continuous dissection process by leaving finitely many.

Def^n: A metric space (X, d) is said to be compact if X is complete & totally bounded.

Theorem: (X, d) is compact iff every seq^n in X has conv. subsequence.

Pf: Suppose X is compact (complete & TB). Let $x_n \in X$. Then $A = \{x_n : n \geq 1\}$ is T.B, and hence has Cauchy subsequence, say x_{n_k}. But X is complete, implies

$$x_{n_k} \to x \in X.$$

On the other hand, if every seq^n (x_n) in X has conv subsequence (x_{n_k}), which is then Cauchy, and it implies that X is T.B.
Also, let x_n be a C.C. in X. Then, again $A = \{x_n : n \geq 1\}$ is T.B, and has conv. sub-sequence, say $x_{n_k} \to x \in X$. Thus, $x_n \to x$.

$$\begin{bmatrix} \text{T.B} \\ + \\ \text{complete} \end{bmatrix} \iff \begin{bmatrix} \forall \text{ seq}^n \text{ has Cauchy } \cancel{\text{conv}} \text{ subseq}^n \\ + \\ \text{C.C is conv} \end{bmatrix}$$

Cor: (i) Let $A \subset X$. If A is cpt, then A is closed.

(ii) If X is compact & A is closed, then A is compact.

i.e. Compact subsets of a compact metric spaces are closed sets.

(Hint (i): if A is cpt, then for $x_n \in A$ & $x_n \to x$
$\Rightarrow x_{n_k} \to y \in A \Rightarrow x_n \to x = y$:

(ii) if A is closed, & $x_n \in A$, then $x_n \in X$
$\Rightarrow \exists\, x_{n_k} \to x \in X \Rightarrow x \in A$, since A is closed)

Ex. If K is compact subset of $(\mathbb{R}, \mathcal{U})$, then $\inf K$ & $\sup K \in K$.

By defn of infimum, $\exists\, x_n \in K$ st. $x_n \to \inf K$.
But, since K is compact, $\exists\, x_{n_k} \to x \in K$.
$\Rightarrow \inf K = x$, etc.

Ex. Let $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$. Show that E is closed & bounded in $(\mathbb{Q}, \mathcal{U})$, but not cpt.

(Hint: $\mathbb{Q}$ is not complete)

Ex. If $f:(X,d)\to(Y,\rho)$ is continuous, then for $K\subset X$ to be compact, $f(K)$ is compact in Y.

Let $y_n\in f(K)$, then $y_n=f(x_n)$, for some $x_n\in K \Rightarrow \exists\, x_{n_k}\to x\in K \Rightarrow f(x_{n_k})\to f(x)\in f(K)$.

Ex. If $A\subset X$ is compact, then show that $\delta(A)<\infty$. If $A\neq\Phi$, then $\exists\, x,y\in A$ s.t. $\delta(A)=d(x,y)$.

Note that $\delta(A)=\sup\underbrace{\{d(x,y):(x,y)\in A\times A\}}_{S}$

And $d: A\times A\to\mathbb{R}$ is (jointly) continuous.

As $A\times A$ is compact, S is compact in $\mathbb{R}$.

Hence, $\exists\,(x_0,y_0)\in A\times A$ s.t. $\delta(A)=d(x_0,y_0)$.

Ex. Show that $S_1[0]=\{x\in l^2: \|x\|_2\leq 1\}$ is not compact.

(Hint: $\{e_n: n\geq 1\}$ is not T.B.)

Ex. Show that $A = \{x \in l^2 : |x_n| \leq \frac{1}{n}, n = 1, 2, \cdots\}$ is compact. (139)

(Hint: A is closed, hence complete. A is T.B, since for $\epsilon = 1$, okay. for $\epsilon < 1$, only finitely many co-ordinates left un-patched (uncovered), hence for each $\epsilon < 1$, $A = A_\epsilon \cup B_\epsilon$, $A_\epsilon \in \mathbb{R}^n$ for some n.)

Cor: Let (X, d) be compact. If $f : X \to \mathbb{R}$ is continuous, then f is bounded. Moreover, f ~~attains~~ its maxi & ~~sup.~~ mini.

Pf: $f(X)$ is cpt in $\mathbb{R}$, implies $f(X)$ is closed & bounded & hence

$$\sup_{x \in X} f(x) \ \& \ \inf_{x \in X} f(x) \in \mathbb{R}$$

i.e. $\exists x_0, y_0 \in X$ st $f(y_0) = \sup_{x \in X} f(x)$, $f(x_0) = \inf_{x \in X} f(x)$.

Hence $f(x_0) \leq f(t) \leq f(y_0)$.

Cor: If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is compact & $f([a, b]) = [c, d]$.

Cor: If (X,d) is a compact metric space and $C(X) = \{ f : X \xrightarrow{\text{cont}} \mathbb{R} \text{ or } \mathbb{C} \}$, define

$$\|f\|_\infty = \sup_{x \in X} |f(x)| < \infty.$$ (140)

Then $(C(X), \|\cdot\|_\infty)$ is complete n.l.s.

Lemma: Let (X,d) be a metric space. Then TFAE:

(a) If $\mathcal{G}$ is a ~~finite~~ arbitrary collection of open sets in X with $\bigcup\{G : G \in \mathcal{G}\} \supseteq X$, then $\exists\ G_1, \dots, G_n$ (finitely many) s.t.

$$\bigcup_{i=1}^{n} G_i \supseteq X$$

(every open cover has finite subcover)

(b) If $\mathcal{F}$ is a collection of closed sets in X with $\bigcap_{i=1}^{n} F_i \neq \emptyset$ for every choice of finitely many F_i's in $\mathcal{F}$, then

$$\bigcap\{F : F \in \mathcal{F}\} \neq \emptyset.$$

(finite intersection property)

Notice that (a) $\Rightarrow$ X is T.B., since

$$X \subseteq \bigcup_{x \in X} B_\varepsilon(x) \Rightarrow X \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i).$$

(b) $\Rightarrow$ X is complete, since every decreasing seq^n of closed sets has non-empty intersection.

Proof of lemma:

(a) $\Rightarrow$ (b): let $\mathcal{F}$ be a collection of closed sets in X s.t. $\bigcap_{i=1}^{n} F_i \neq \Phi$ for every choice of finitely many F_i's in $\mathcal{F}$. On contrary, suppose

$$\bigcap \{F : F \in \mathcal{F}\} = \emptyset.$$

Then $X = \bigcup \{F^c : F \in \mathcal{F}\}$ is a open cover of X, hence $X = \bigcup_{i=1}^{n} \{F_i^c : F_i \in \mathcal{F}\}$

$\Rightarrow \bigcap_{i=1}^{n} F_i = \Phi$, an absurd.

(b) $\Rightarrow$ (a): Suppose $X = \bigcup \{G \in \mathcal{G}\}$, but $X \neq \bigcup_{i=1}^{n} G_i$ for any choice of finitely many G_i's in $\mathcal{G}$. Then $X \setminus \bigcup_{i=1}^{n} G_i \neq \Phi$

$\Rightarrow \bigcap_{i=1}^{n} G_i^c \neq \Phi$, for every choice of

$$\bigcap_{G \in \mathcal{G}} G^c \neq \Phi \Rightarrow \bigcup\{G : G \in \mathcal{G}\} \neq X.$$

Theorem: X is compact iff either ~~(hence both)~~ (a) or (b) (hence both) of the previous lemma is satisfied.

Proof: Notice that (a) & (b) imply that X is T.B & Complete. Hence X is Cpt.

Now, Suppose X is compact, and $\mathcal{G}$ is an open cover that admits no finite subcover.

Since X is T.B, it can be covered by finitely closed sets of diameter ≤ 1.

But, then it implies that one of these, say A_1, will not be covered by finitely many open sets in $\mathcal{G}$.

It follows that $A_1 \neq \Phi$, it must be an infinite set

(Else covered by finitely many G's).

Next, A_1 is T-B, so A_1 is covered by finitely many closed sets of diameter $\leq 1/2$.

Choose one of them, say A_2 such that A_2 cannot be covered by finitely many G's from $\mathcal{G}$. (143)

Thus, $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$,

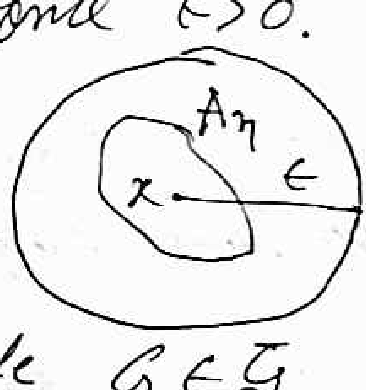
where A_n is closed, infinite, $\operatorname{diam}(A_n) \leq \frac{1}{n}$, and cannot be covered by finitely many G's from $\mathcal{G}$.

Notice that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ ($\because X$ is complete)

Let $x \in \bigcap_{n=1}^{\infty} A_n$, then $x \in A_n$. Then $x \in G$ for some $G \in \mathcal{G}$. But G is open, hence $x \in B_\epsilon(x) \subset G$ for some $\epsilon > 0$.

For $1/n < \epsilon$, we get

$$x \in A_n \subset B_\epsilon(x) \subset G.$$



Hence, A_n is covered by a single $G \in \mathcal{G}$, which is a contradiction.

Cor: X is compact iff every decreasing seqn of non-empty closed sets has non-empty intersection. (144)
(ie $F_1 \supset F_2 \supset \cdots F_n \supset F_{n+1} \supset \cdots$, implies $\bigcap_{n=1}^{\infty} F_n \neq \phi$.)

proof: The forward implication is followed by the previous theorem.
On the other hand, suppose every nested (decreasing) seqn of closed ~~intervals~~ sets in X has non-empty intersection.
We prove compactness of X in the sense of BWT. ie let $x_n \in X$. write
$A_n = \{x_k : k \geq n\}$. Then $\bigcap_{n=1}^{\infty} \overline{A_n} \neq \phi$.
Let $x \in \bigcap_{n=1}^{\infty} \overline{A_n} = A$ (say). Then A is closed
hence $\exists\ x_{n_k}$ s.t. $x_{n_k} \to x$.
(Notice that x_n has been taken distinct, ie an infinite set)

Remark: Note that, as long as compactness is concerned we donot require diameter of F_n tends to 0, hence $\bigcap_{n=1}^{\infty} F_n$ can contain more than one point. This makes a sharp contrast with the condition for completeness.

Cor: X is compact iff every countable open cover admits a finite subcover.

pf: $\Rightarrow$: Compact $\Rightarrow$ lemma (a) holds

$\Rightarrow$ countable cover has finite subcover

$\Leftarrow$: Suppose every countable open cover has finite subcover.

$\Leftrightarrow$ every countable family of closed sets has finite intersection property.
(can be proved similar to the previous lemma).

Let $(x_n) \in X$ be a seqn of distinct terms. Write $A_n = \{x_k : k \geq n\}$. Then $x \in \bigcap_{n=1}^{\infty} \overline{A_n} \neq \emptyset$. $\Rightarrow \exists\, x_{n_k} \in X$ st $x_{n_k} \to x$.
Hence, X is compact

Separable metric spaces:

If a space admit a countable dense set, we say that the space is separable. Eventually, it helps determine the size of the space, certainly not in terms of cardinally only, rather dimensions or in more general aspect of size. Evidently, every totally bounded space is separable.

Defn: A metric space (X,d) is said to be separable if $\exists$ a countable set $A \subset X$ s.t. $\bar{A} = X$.

For example, $\mathbb{Q}$ (the set of rationals) is a countable dense subset of $\mathbb{R}$.

Likewise, $\mathbb{Q}^n$ and $\mathbb{Q}^n + i\mathbb{Q}^n$ are countable dense subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively.

It is easy to see that $(\mathbb{R}^n, \|\cdot\|_p)$ is separable for $1 \le p \le \infty$. However, $(\ell^p, \|\cdot\|_p)$

is Separable for $1 \le p < \infty$, and not Separable for $p = \infty$.

(147)

We know that $\overline{C_{00}} = \ell^p$, C_{00} - the space of finite seqns. Let $x \in \ell^p$. then

$x = (x_1, x_2, \dots x_n, x_{n+1}, \dots)$. Write

$X_n = (x_1, \dots x_n, 0, 0, \dots)$. Then

$\|X - X_n\|_p \to 0$ as $n \to \infty$ —— (1)

Since $x_i \in \mathbb{C}$, $\exists$ $x_i^k \in \mathbb{Q} + i\mathbb{Q}$ s.t.

$|x_i^k - x_i|^p \to 0$; $i = 1, 2, \dots, n$.

$\Rightarrow \left(\sum_{i=1}^{n} |x_i^k - x_i|^p \right)^{1/p} \to 0$

i.e $\|X_n^k - X_n\|_p \to 0$, —— (2)

where $X_n^k = (x_1^k, \dots x_n^k) \in \mathbb{Q}^n + i\mathbb{Q}^n$

From (1) & (2),

$\|X - X_n^k\|_p \le \|X_n^k - X_n\|_p + \|X_n - X\|_p \to 0$.

That is, $\overline{C_{00}(\mathbb{N}, \mathbb{Q} + i\mathbb{Q})} = \ell^p(\mathbb{N}, \mathbb{C})$.

Next, we shall show that $\ell^\infty(\mathbb{N}, \mathbb{C})$ is not separable, by proving that ℓ^∞ cannot be the union of countably many balls of arbitrarily small radius. (148)

Let $A = \{\tilde{x}_1, \tilde{x}_2, \cdots\}$ be any countable set in ℓ^∞. Consider

$$S = \{x = (x_1, \cdots x_n, \cdots) \in \ell^\infty ; \ x_i \in \{0,1\}\}$$

Then S is an uncountable set. For this,

$$x \in S \Rightarrow y = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots, \quad x_i \in \{0,1\}.$$

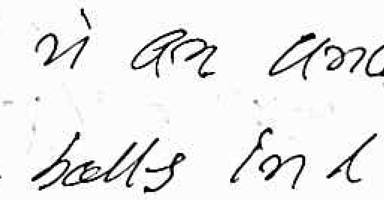
Then the map from S to $[0,1]$ is surjective, and hence S is uncountable.

Let $x, y \in S$ be such that $x \neq y$.

Then $\|x - y\|_\infty = 1$.

Hence, $\{B_{1/2}(x) : x \in S\}$ is an uncountable disjoint collection of open balls in ℓ^∞.

Since A is countable, A ~~cannot~~ intersect only countably many balls $B_{1/2}(x)$'s. Hence, A cannot be dense.



Ex. Show that $\overline{C_{00}} = C_0$, and hence deduce that C_0 is separable. (149)

Ex. Let $B[0,1]$ be the space of all bounded functions on $[0,1]$. Show that $(B[0,1], \|\cdot\|_\infty)$ is not separable.

For $t \in (0,1)$, write $f_t = \chi_{[0,t)}$. Then for $s \neq t$, $s,t \in (0,1)$, we get

$$\|f_s - f_t\|_\infty = 1.$$

Then $S = \{ B_{1/2}(f_t) : t \in (0,1)\}$ is an uncountable collection of disjoint open balls for $B[0,1]$.

If A is any countable set, say

$A = \{g_1, g_2, \ldots\} \subset B[0,1]$, then

$\exists\ t_0 \in (0,1)$ s.t. $B_{1/2}(f_{t_0}) \cap A = \emptyset$.

That is, except countably many, all the balls in S left un-intersected by A.

Ex. The space $(C[0,1], \|\cdot\|_\infty)$ is separable.

(Hint: proof of this will be done by Weierstrass approxi. thm, which we do later.)

Ex. Every totally bounded metric space is separable. (150)

Let (X,d) be T.B. For $\epsilon = 1/n$, $\exists$ $x_{n_1}, \dots, x_{n_K}$ s.t. $X = \bigcup_{j=1}^{n_K} B_{1/n}(x_{n_j})$.

Let $D_{nK} = \{x_{n_1}, \dots x_{n_K}\}$. Then $D = \bigcup D_{nK}$ is a countable dense set in X.

Next, we consider the compact subsets of the space of continuous function $C(X)$, when X is a compact metric space.

Notice that $\dim C(X) < \infty$ iff (151)

X is a finite set. Hence, ~~$C(X)$ is~~ closed & bounded subset of $C(X)$ are compact if X is finite.

But the question of compact subsets of $C(X)$, X cpt, is same as when a subset of $C(X)$ is T.B?

In terms of BWT, we can rephrase when (unif) bounded seq^n in $C(X)$ has uniformly conv. subseq^n?

We will see later that this question relates to the ~~eivt~~ earlier question of asking, when p.w. conv. seq^n is uniformly conv.

ie p.w. conv + $\boxed{}$ $\Rightarrow$ uniform conv.

Ex. If $f_n \in C(X)$, X compact, $f_n \xrightarrow{unif} f$, then $\{f\} \cup \{f_n : n \geq 1\}$ is compact. (152)
($\because$ every C.C. is T.B.).

Defn: A collection $F \subset C(X)$ is said to be unif bounded if

$$\sup_{f \in F} \sup_{x \in X} |f(x)| = \sup_{f \in F} \|f\|_\infty < \infty.$$

Ex. Any unif conv. seqn f_n in $B(X)$ (or $C(X)$) is unif bounded.
(Hint: $\|f_n\|_\infty \leq \|f\|_\infty + 1$ (for $\epsilon = 1$), $n \geq N$.)

Defn: A collection F $C(X)$ is said to be p.w. bounded, if for each $x \in X$,

$$\sup_{f \in F} |f(x)| < \infty$$

~~Defn~~ Ex. If $f_n \to f$ p.w, then f_n is p.w. bdd.

<u>Theorem</u>: Let (X, d) be a compact metric space, & $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is continuous, then f is <u>unif.</u> cont.

Proof: Let $x \in X$ (cpt), & $\epsilon > 0$, then $\exists\, \delta_x > 0$ s.t. $d(x,y) < \delta_x \Rightarrow |f(x) - f(y)| < \epsilon$

i.e. $y \in B_{\delta_x}(x) \Rightarrow |f(x) - f(y)| < \epsilon$

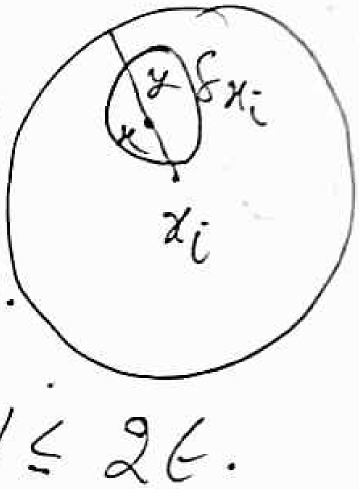
Notice that $X = \bigcup_{x \in X} B_{\delta_x}(x)$. Since X is compact, $X = \bigcup_{i=1}^{n} B_{\delta_{x_i}}(x_i)$.

Let $\delta = \frac{1}{2} \min_{1 \le i \le n} \{\delta_{x_i}\}$. Then $\delta > 0$.

Let $x, y \in X$ & ~~and~~ they closed enough.

$\exists\, x_i$ s.t. $x \in B_{\delta_{x_i}}(x_i)$

Choose $\delta' > 0$ st $\delta' < \delta$ & $d(x,y) < \delta'$, we'll $y \in B_{\delta_{x_i}}(x_i)$.



Then $d(x,y) < \delta' \Rightarrow |f(x) - f(y)| \le 2\epsilon$.

Thus, for $\epsilon > 0$, $\exists\, \delta' > 0$ st whenever $d(x,y) < \delta' \Rightarrow |f(x) - f(y)| < 2\epsilon$

Next, we shall discuss the missing ~~ingredient~~ ingredient of p.w. conv to be unif. conv.

Def^n: A collection $\mathcal{F} \subseteq C(X)$ is said to be (unif) equi-continuous if $\forall \epsilon>0$, $\exists\ \delta>0$ s.t.

$$d(x,y)<\delta \Rightarrow |f(x)-f(y)|<\epsilon, \quad \forall f\in\mathcal{F}.$$

Example (i) finite subset of $C(X)$ is (unif) equicontinuous, & every sub-collection of (unif) equicontinuous collection is equicontinuous.

(ii) let $0<\alpha\leq 1$, & $K>0$. Then

$$\mathrm{Lip}_K^{\alpha} = \{ f\in C[0,1] : |f(x)-f(y)|\leq K|x-y|^{\alpha} \}$$

is equi-continuous, but not T.B, since all constant functions are satisfying this condition.

Lemma: If $\mathcal{F} \subset C(X)$ is T.B, then $\mathcal{F}$ is uniformly bounded & (unif) equi-continuous.

Pf: Since a T.B set is (unif) bounded, we only need to show that $\mathcal{F}$ is equi-continuous.

Since $\mathcal{F}$ is T.B., for $\epsilon>0$, $\exists\ f_1, \ldots f_n \in \mathcal{F}$

~~s.t. $\|f - f_i\|_\infty < \epsilon$,~~
such that for $f \in F$, $\exists$ f_i with
$$\|f - f_i\|_\infty < \epsilon.$$
But $\{f_1, \dots f_n\}$ is equi-continuous, for $\epsilon > 0$, $\exists$ $\delta > 0$ st
$$d(x,y) < \delta \Rightarrow |f_i(x) - f_i(y)| < \epsilon. \quad \forall i = 1,2,\dots,n.$$
now,
$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)|$$
$$\leq \epsilon + \epsilon + \epsilon.$$

<u>Cor</u>: If f_n is unif conv. in $C(X)$, then f_n is unif. bounded + (unif) equi-cont.

Pf: Notice that $\{f\} \cup \{f_n : n \geq 1\}$ is compact, hence $\{f_n : n \geq 1\}$ is T.B $\Rightarrow$ (unif) equi-cont.

<u>Arzila-Ascoli thm</u>:
Let X be a compact metric space, and $F \subset C(X)$. Then F is compact iff F is closed, unif. bounded, and unif equi-conti

Proof: The forward implication follows from the previous lemma. 156

On the other hand, let $f_n \in F$ be a seq. Claim f_n has a (unif.) conv subseq.

Note that (f_n) is equi-cont. For $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$, $\forall\, n \geq 1$.

Since X is T.B, $\exists$ a finite set $x_1 \ldots x_k \in X$ s.t.

$$X = \bigcup_{i=1}^{k} B_\delta(x_i).$$

Let $x \in X$, then $\exists\, x_i$ s.t. $d(x, x_i) < \delta$. Also, (f_n) is unif bounded, hence

$$\{f_n(x_i)\}_{n=1}^{\infty} \text{ is bounded (in } \mathbb{R})$$

for $i = 1, 2, \ldots, k$.

So w.l.g, we may assume that $\{f_n(x_i)\}_{n=1}^{\infty}$ is conv. for each $i = 1, 2, \ldots, k$

In particular, for $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$$|f_m(x_i) - f_n(x_i)| < \epsilon \quad \text{for } m, n \geq N$$

for each $i = 1, 2, \ldots, k$.

Now, for $x \in X$, $\exists\, x_i$ st $d(x, x_i) < \delta$. Hence

$$|f_m(x) - f_n(x)| \le |f_m(x) - f_m(x_i)| + |f_m(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)|$$

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$$< \epsilon + \epsilon + \epsilon = 3\epsilon.$$

i.e. $\|f_m - f_n\|_\infty \le 3\epsilon$ for $m, n \ge N$.

$\Rightarrow$ f_n is (unif) Cauchy seq^n, hence f_n is conv. (because $C(X)$ is complete).

<u>Cor</u>: Let X be cpt. If f_n is unif bdd, and (unif) equi-cont. on $C(X)$, then f_n has conv. subseq^n.

(Hint: $A = \overline{\{f_n : n \ge 1\}}$ is closed)

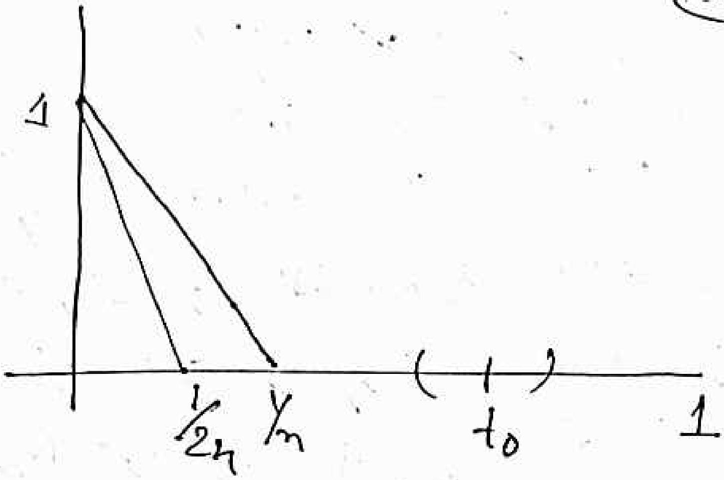
Ex. Let $X = (0,1)$, and define

$$f_n(t) = \begin{cases} 1 - nt & t < \frac{1}{n}; \\ 0 & t \ge 1/n. \end{cases}$$

Show that $(f_n)_{n=1}^{\infty}$ is point-wise equi-cont. but not uniformly equi-cont on $(0,1)$.

Notice that for any point $t \in (0,1)$, $\exists$ $n_0 \in \mathbb{N}$ s.t for each $n \geq n_0$,

$f_n(t) = 0$ in a small nbd of t.



Hence, $(f_n)_{n=1}^{\infty}$ is point-wise equi-cont on $(0,1)$. However,

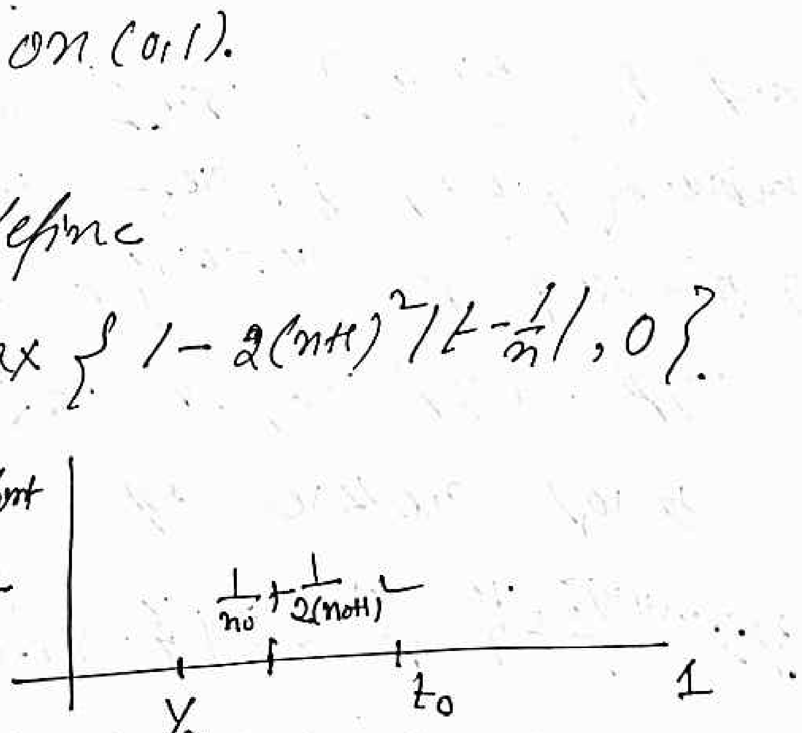
$$\left| f_n\left(\frac{1}{2n}\right) - f_n\left(\frac{1}{n}\right) \right| = \left| 1 - n \cdot \frac{1}{2n} - 0 \right| = \frac{1}{2}$$

where $\left|\frac{1}{2n} - \frac{1}{n}\right| = \frac{1}{2n} \to 0$. Hence, $(f_n)_{n=1}^{\infty}$ is not uniformly equi-cont on $(0,1)$.

Ex. For $X = [0,1]$, define

$$f_n(t) = \max\left\{ 1 - 2(n+1)^2 \left| t - \frac{1}{n} \right|, 0 \right\}.$$

Then $(f_n)_{n=1}^{\infty}$ is equi-cont at each pt $t > 0$, but not at $t = 0$.



For $t_0 > 0$, it follows from the fact that (159)

$$1 - 2(n+1)^2 |t_0 - \tfrac{1}{n}| \le 0 \text{ iff } \frac{1}{n} + \frac{1}{2(n+1)^2} \le t_0.$$

And hence, $f_n(t) = 0$ for $n \ge n_0$ in a small nbd of $t_0 > 0$. Notice that, the above means that $\{f_n\}_{n=n_0}^{\infty}$ is p.w. equi-cont at $t_0 > 0$. Since $\{f_1, \dots f_{n_0 - 1}\}$ (finitely many) is always equi-cont. Thus, $(f_n)_{n=1}^{\infty}$ is p.w. equi-cont for $t > 0$.

However, for $t = 0$; $f_n(0) = 0$, $f_n(\frac{1}{n}) = 1$, but $|0 - \frac{1}{n}| \to 0$, and $|f_n(0) - f_n(\frac{1}{n})| = 1$.

Thus, $(f_n)_{n=1}^{\infty}$ is not point-wise equi-cont at $t = 0$.

<u>Remark</u>: We end this section with a remark on structural property of sets in real line. Any set can be inscribed into countably many disjoint open intervals, however, a bounded (T.B) set can be covered by <u>finitely many</u> <u>almost disjoint</u> intervals of <u>arbitrarily small</u> length.

Remark 2:

A closed observation of totally bounded set reveals that most of the properties, which are true for finitely many points (centers) in a T.B. metric space, can easily be extrapolated to the full space, since any point of the space is in a small (arbitrarily) ball.

Ex (Dini's theorem):

Let X be a compact metric space, and $f, f_n \in C(X)$ s.t. $f_n \downarrow f$ pointwise on X. Then $f_n \downarrow f$ uniformly on X.

Proof: Let $g_n = f_n - f$. Then $g_n \downarrow 0$ p.w. on X. Notice that for each $\epsilon > 0$, $|g_n(x)| < \epsilon$ for large n, depends upon x.

Let $E_n = \{x \in X : g_n(x) < \epsilon\}$. Then

$E_n = g_n^{-1}(-\infty, \epsilon)$, hence open.

Also, $E_n \subset E_{n+1} \subset \cdots$

Since $g_n \downarrow 0$ at each point, it follows that $X = \bigcup_{n\ge 1}^{\infty} E_n$

(If $x \in X$ & $x \notin E_n \; \forall n \ge 1 \Rightarrow g_n(x) \ge \epsilon$ $\forall n \ge 1$, which is a contradiction)

But X is cpt, hence $\exists\, N \in \mathbb{N}$ st

$$X = \bigcup_{n\ge 1}^{N} E_n = E_N.$$

Thus, for $x \in X$, & $n \ge N$,

$$g_n(x) \le f_N(x) < \epsilon.$$

i.e. $|g_n(x)| < \epsilon \quad \forall\, n \ge K, \; \forall\, x \in X.$

Hence $g_n \downarrow 0$ uniformly on X.

Cr: Suppose $f, f_n \in C(X)$ & $f_n \uparrow f$ p.w, then $f_n \uparrow f$ uniformly.

(Hint: $g_n = f - f_n \downarrow 0$ p.w etc).

1. Notice that the limit function f must be continuous, else $f_n(x) = x^n$ with

Contradict the above theorem.

2. If X is not compact, then the conclusion of the theorem might not be true.

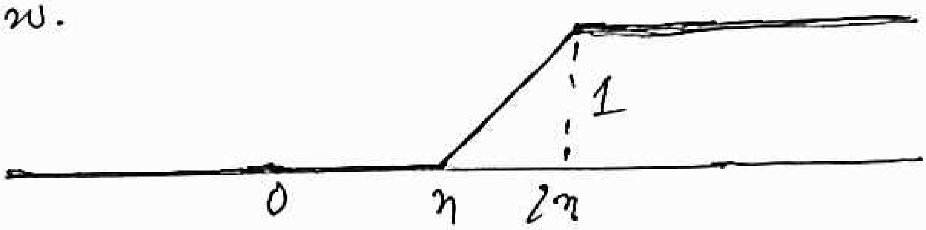
For $X = \mathbb{R}$,

$$f_n(t) = \begin{cases} 0 & -\infty < t \leq n \\ \frac{x}{n} - 1 & n < t \leq 2n \\ 1 & t > 2n \end{cases}$$

$f_n \downarrow 0$ p.w.

but

$\|f_n\|_\infty = 1$.



Remark: However, a point-wise conv seq^n is differs with uniform conv. on arbitrarily small set (Egoroff's theorem).

<u>Upper Semi-Continuity</u>

Let $f : (X, d) \longrightarrow \mathbb{R}$. Then f is said to be Upper Semi-cont on X if for each $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) < \alpha\}$ is open.

<u>Result</u>: $f : X \to \mathbb{R}$ is USC iff for any $x \in X$, and each seq^n $x_n \to x$ implies

$$\limsup_{n \to \infty} f(x_n) \leq f(x).$$

Proof: Let $x_0 \in X$ & $\epsilon > 0$. Then

$$x_0 \in \{x : f(x) < f(x_0) + \epsilon\} \text{ - open.}$$

$\Rightarrow \exists$ a nbd $B_\delta(x_0)$ s.t

$$f(x) < f(x_0) + \epsilon, \quad \forall x \in B_\delta(x_0).$$

$\left\{ \begin{array}{l} \text{Let } \frac{1}{n} < \delta, \\ \text{\& } x_n \to x \end{array} \right\}$, then $\exists x_n \in B_{1/n}(x_0)$ s.t

$$f(x_n) < f(x_0) + \epsilon$$

Hence, $x_n \to x_0 \Rightarrow \limsup_{n \to \infty} f(x_n) \leq f(x_0) + \epsilon \quad \forall \epsilon > 0.$

Conversely, suppose (on contrary) that f is not USC. on X. Then

$\exists\ \alpha \in \mathbb{R}$ s.t

$$A_\alpha = \{x \in X : f(x) < \alpha\} \quad (164)$$

is not open. That is, $\exists\ x_0 \in A_\alpha$ s.t for any nbd $B_\delta(x_0)$, $\exists\ x_\delta \in B_\delta(x_0)$ with $x_\delta \notin A_\alpha \Rightarrow f(x_\delta) \geq \alpha$

For $\delta = 1/n$. $x_n \in B_{1/n}(x_0) \Rightarrow x_n \to x_0$ but

$$f(x_n) \geq \alpha > f(x_0)$$

$$\Rightarrow \limsup f(x_n) \geq \alpha > f(x_0), \quad \text{a}$$

contradiction.

Ex. If X is compact, & $f : X \to \mathbb{R}$ is USC, then f attains its maximum.

note that $X = \bigcup_{\alpha \in \mathbb{R}} \{x \in X : f(x) < \alpha\}$, but X is compact, hence,

$$X = \bigcup_{i=1}^{K} \{x \in X : f(x) < \alpha_i\}$$

for $x \in X$, $f(x) < \alpha_i < \max\{\alpha_i\} = \alpha < \infty$.

Hence, f is bounded above.

Next, f attains its supremum on X.

If not, then $f(x) < \sup f$, $\forall\, x \in X$.
For $n \in \mathbb{N}$, $\exists\, x_n \in X$ s.t.

$$\sup f - \frac{1}{n} < f(x_n).$$

Now, $x_n \in X$, and X is compact, hence $\exists\, x_{n_k}$ st $x_{n_k} \to x \in X$. But, then

$$\sup f \leq \limsup_{k\to\infty} f(x_{n_k}) \leq f(x).$$

i.e. $\sup f \leq f(x)$, which is not possible, as it contradicts the ~~hypothesis~~ assumption.

Note that, similar way we can define lower semi-continuity; i.e. if $\{x \in X : f(x) > \alpha\}$ is open for each $\alpha \in \mathbb{R}$. Again, it follows that f is LCS iff $\forall\, x_n \to x$,

$$f(x) \leq \liminf_{x\to\infty} f(x_n).$$

Thus, f is continuous iff f is LSC & USC.

(Hint: $\limsup_{n\to\infty} f(x_n) \leq f(x) \leq \liminf_{x\to\infty} f(x_n)$ whenever $x_n \to x$).

Remark 1: Note that if $f: X \longrightarrow \mathbb{R}$ is USC, then $f^{-1}(-\infty, \alpha)$ is open, and hence (166)
$f^{-1}\{[\beta, \alpha)\}$ is open if $\beta < \alpha$, but it does not imply that $f^{-1}\{(\beta, \alpha)\}$ is open, for each $\alpha, \beta \in \mathbb{R}$, else f is continuous. (However, f is Lebesgue measurable!). But if f is both LSC & USC, then
$$f^{-1}\{(\alpha, \beta)\} = f^{-1}\{(-\infty, \beta) \cap (\alpha, \infty)\} \text{ is open,}$$
hence f is continuous.

Remark 2: There is no relation between LSC & USC with left limit & right limit.

Ep. $f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

is upper semi-continuous, but none of left limit and right limit ~~exits~~ exists.

Ex. Check for LSC & USC for $f(x) = [x]$, the greatest integer function.

Weierstrass approximation theorem

We shall see that polys are dense in $(C[a,b], \|\cdot\|_\infty)$ if $b-a<\infty$. And as a consequence, $C[a,b]$ is a separable space. The question of density of polys in $C[a,b]$ can be transferred to $C[0,1]$ with help of the map $\gamma(t) = t-a/b-a$.

For $f \in C[0,1]$ & $n = 0,1,2,\dots$, define

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Then $B_n(f_n)$ is a poly. of degree $\le n$. Here, $B_n(f_n)$ is known as Bernstein polys. In fact, we have

$$B_n(f)(0) = f(0) \text{ and } B_n(f)(1) = f(1).$$

Let us denote $f_n(x) = x^n$; $n = 0,1,2,\dots$

The following lemmas which is bit involved with combinatorics, is crucial in proving the density of $B_n(f)$ in $C[0,1]$.

Lemma:

(i) $B_n(f_0) = f_0$ & $B_n(f_1) = f_1$

(ii) $B_n(f_2) = (1-\frac{1}{n})f_2 + \frac{1}{n}f_1$ &

hence $B_n(f_2) \longrightarrow f_2$ uniformly.

(iii) $$\sum_{k=0}^{n} \left(\frac{k}{n}-x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n} \leq \frac{1}{4n}$$

(iv) Given $\delta > 0$, & $0 \leq x \leq 1$, let F denote set of k in $\{0, 1, 2, \ldots n\}$ for which $|k/n - x| \geq \delta$.

Then $$\sum_{k \in F} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{4n\delta^2}$$

Proof: (i) is trivial, as follows from simple binomial expansions.

(Hint: $$\sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k} = x \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k}$$
$$= x \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{(n-1)-j}$$
$$= x[x + (1-x)]^{n-1} = x.)$$

$\Rightarrow B_n(f_1) = f_1$)

(ii) To compute $B_n(f_2)$, we break the sum into two parts: for k

$$\left(\frac{k}{n}\right)^2\binom{n}{k} = \frac{k}{n}\binom{n-1}{k-1} = \left(1-\frac{1}{n}\right)\binom{n-2}{k-2} + \frac{1}{n}\binom{n-1}{k-1},$$

if $k \geq 2$. (169)

Thus, $$B_n(f_2) = \left(1-\frac{1}{n}\right)\sum_{k=2}^{n}\binom{n-2}{k-2}x^k(1-x)^{n-k}$$
$$+ \frac{1}{n}\sum_{k=1}^{n}\binom{n-1}{k-1}x^k(1-x)^{n-k}$$
$$= \left(1-\frac{1}{n}\right)x^2 + \frac{1}{n}x \to f_2 \text{ uniformly.}$$

(iii) Note that
$$(k/n - x)^2 = (k/n)^2 - 2x(k/n) + x^2, \text{ hence}$$
$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \left(1-\frac{1}{n}\right)x^2 + \frac{1}{n}x - 2x^2 + x^2$$
$$= \frac{1}{n}x(1-x) \leq \frac{1}{4n}.$$
(by (ii)).

(iv) We have $1 \leq (k/n - x)^2/\delta^2$ for $k \in F$.

Hence,
$$\sum_{x\in F}\binom{n}{k}x^k(1-x)^{n-k} \leq \frac{1}{\delta^2}\sum_{k\in F}\left(\frac{k}{n}-x\right)^2\binom{n}{k}(1-x)^{n-k}x^k$$
$$\leq \frac{1}{\delta^2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}$$
$$\leq \frac{1}{4n\delta^2}.$$

Theorem (Bernstein):

Let $f \in C[0,1]$, then $B_n(f) \to f$ uniformly.

Proof: Since f is unif. cont; for $\epsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon/2$.

Now,

$$|f(x) - B_n(f)| = \left| \sum_{k=0}^{n} \left(f(x) - f(k/n)\right) \binom{n}{k} x^k (1-x)^{n-k} \right|$$

$$\left(\because \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1 \right)$$

$$\leq \sum_{k=0}^{n} |f(x) - f(k/n)| \binom{n}{k} x^k (1-x)^{n-k}.$$

~~Fix~~ Let us fix n (will specified soon). Let F denote the set of k in $\{0, 1, \ldots n\}$ s.t. $|k/n - x| \geq \delta$. Then $|f(x) - f(k/n)| < \epsilon/2$ for $k \notin F$, and $|f(x) - f(k/n)| \leq 2\|f\|_\infty$ of $k \in F$.

Thus,

$$|f(x) - B_n(f)| \leq \epsilon/2 \cdot \sum_{k \notin F} \binom{n}{k} x^k (1-x)^{n-k} + 2\|f\|_\infty \sum_{k \in F} \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq \epsilon/2 \cdot 1 + 2\|f\|_\infty \cdot \frac{1}{4n\delta^2} < \epsilon$$

if $n > \frac{\|f\|_\infty}{\epsilon\delta^2}$.

i.e $\|B_n(f)-f\|_\infty \leq \epsilon$ if $n \geq \frac{\|f\|_\infty}{\epsilon\delta^2}$.

Ex. If $f \in C[0,1]$ & $\int_0^1 x^n f(x)\,dx = 0$, $\forall n \geq 0$, then $f = 0$.