Notice that in the previous exercises, we have seen that $(C[0,1], \|\cdot\|_\infty)$ is complete. i.e if $\|f_n - f_m\|_\infty \to 0$, then $\exists f \in C[0,1]$ s.t $\|f_n - f\|_\infty \to 0$. But then (90)

$$|f_n(t) - f(t)| < \|f_n - f\|_\infty \to 0, \forall t \in [0,1].$$

i.e $f_n(t) \to f(t)$ for each pt $t \in [0,1]$.

We say that $f_n \to f$ uniformly if

$$\sup_t |f_n(t) - f(t)| \to 0.$$

But there are seq^n of functions, which converges point wise (p.w) but not uniformly.

Ex. let $f_n(t) = t^n$; $t \in [0,1]$.

Then $f(t) = \lim f_n(t) = \begin{cases} 0 & 0 \le t < 1, \\ 1 & t = 1 \end{cases}$

So $\sup_t |f_n(t) - f(t)| = 1 \not\to 0$.

Ex. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by
$$f_n(t) = e^{-nt^2}, \quad n \in \mathbb{N}.$$
Then $f(t) = \lim f_n(t) = \begin{cases} 1 & t = 0 \\ 0 & |t| > 0. \end{cases}$

Notice that for $t = 0$,
$$|f_n(0) - f(0)| = |1 - 1| = 0 < \epsilon, \ \forall\ n \geq 1.$$
If $|t_0| > 0$; $t_0^2 > 0$. Then for
$$|f_n(t_0) - 0| < \epsilon, \text{ we get}$$
$$e^{-nt_0^2} < \epsilon \Rightarrow n > \frac{\log 1/\epsilon}{t_0^2}.$$
Let $n_0 = \left[\frac{\log 1/\epsilon}{t_0^2}\right] + 1$. Then ~~for t~~
$$|f_n(t_0) - f(t_0)| < \epsilon \quad \text{for } n \geq n_0 \cancel{= n_0(\epsilon, t_0)}.$$
Notice that $n_0 = n_0(\epsilon, t_0)$ & n_0 is large for $|t_0|$ is close to "0"! Thus, n_0 cannot be free from t_0. ~~Thus~~ Therefore,
$f_n \to f$ p.w. but not uniformly.
(Also, $\|f_n - f\|_\infty = \sup_{t \in \mathbb{R}} e^{-nt^2} = 1 \not\to 0$)

If $f_n(t) = e^{-nt}$ for $t \in [1, \infty)$, then
$$\sup |f_n(t) - 0| = e^{-n} \to 0 \Rightarrow e^{-nt} \xrightarrow[{[1,\infty)}]{\text{unif.}} 0.$$

Ex. Let $f_n, f: A \subseteq \mathbb{R} \to \mathbb{R}$ be such that $f_n \to f$ uniformly on A. Then for $|f_n(t)| \le M_n$'s (i.e. f_n's are bounded), implies f is bounded.

(Hint: $|f(t)| \le |f_{n_0}(t) - f(t)| + |f_{n_0}(t)|$

$< \epsilon + M_{n_0} < \infty \quad \forall\, t \in A$).

We shall see later that uniform conv. seqn is a good carrier for many under-line properties.

Ex. Let $f, f_n: A \subseteq \mathbb{R} \to \mathbb{R}$ be such that $f_n \to f$ unif. Then

~~(i)~~ f is cont. for f_n's are continuous

(iii) (i.e. uniform limit of ~~cont~~ seqn of continuous function is continuous)

For $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$\sup_{t \in A} |f_{n_0}(t) - f(t)| < \epsilon$$

$$\Rightarrow |f_{n_0}(t) - f(t)| < \epsilon \quad \forall\, t \in A.$$

Since f_{n_0} is cont. on A, for fixed t, for $\epsilon > 0$, $\exists\, \delta > 0$

s.t. $|t - s| < \delta \Rightarrow |f_{n_0}(t) - f_{n_0}(s)| < \epsilon$.

Thus, $|f(s)-f(t)| \le |f(s)-f_{n_0}(s)| + |f_{n_0}(s)-f_{n_0}(t)|$ (93)
$+ |f_{n_0}(t)-f(t)| < 3\epsilon.$

Result: Let $R[a,b]$ denote the space of all Riemann integrable functions on $[a,b]$. Let $f, f_n \in R[a,b]$ and $f_n \to f$ uniformly. Then $\int f_n \to \int f$ (or $\lim \int f_n = \int \lim f_n$)

Pf: $\left|\int_a^b (f_n - f)\right| \le \int_a^b |f_n - f| \le \|f_n - f\|_\infty (b-a) \to 0.$

Cor: If $f_n \in R[a,b]$ s.t. $S_n = f_1 + \cdots + f_n \to S$ unif., then $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.

(Obvious from the previous result)

Result: Let $f_n \in C^1[a,b]$ be such that $f_n' \to g$ uniformly. If $\exists\, x_0 \in [a,b]$ such that $f_n(x_0)$ is conv, then $\exists\, f \in C^1[a,b]$ such that $f_n \to f$ unif. & $f' = g$.

Proof: Since $f_n' \to g$ unif, & f_n' is conts, g will be continuous. Define $f : [a,b] \to \mathbb{R}$ by $f(x_0) = \lim f_n(x_0)$,

and $f(x) = \begin{cases} f(x_0) + \int_{x_0}^{x} g(t)\,dt, & \text{if } x > x_0. \\ f(x_0) - \int_{x}^{x_0} g(t)\,dt, & \text{if } x \leq x_0. \end{cases}$

Then $f'(x) = g(x)$, $\forall x \in [a,b]$. Hence, $f \in C^1[a,b]$. Now,

$$f_n(x) - f_m(x) = f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0)) + (f_n(x_0) - f_m(x_0))$$

$$= (x - x_0)(f_n'(t) - f_m'(t)) + (f_n(x_0) - f_m(x_0))$$

$\Rightarrow \|f_n - f_m\|_\infty \leq (b-a)\|f_n' - f_m'\|_\infty + |f_n(x_0) - f_m(x_0)| \to 0.$

$\Rightarrow (f_n)$ is a C.S. in $(C[a,b], \|\cdot\|_\infty)$.

Hence f_n ~~$\Rightarrow f$~~ converges uniformly.

Again, $f_n' \to g = f'$ uniformly, it follows that

$$\int_{x_0}^{x} f_n'(t)\,dt \to \int_{x_0}^{x} f'(t)\,dt$$

$$\lim \left(f_n(x) - f_n(x_0) \right) = f(x) - f(x_0)$$

$$\lim f_n(x) = f(x). \quad \{ \because \lim f_n(x_0) = f(x_0) \}$$

<u>Remark:</u> Conv. of $(f_n(x_0))$ for some pt is necessary. Consider,

$$f_n(t) = \sqrt{t+n}, \quad t \in [0,1]$$

Then f_n does not converge at any pt of $[0,1]$, but

$$f_n'(t) = \frac{1}{2\sqrt{t+n}} \xrightarrow{\text{unif}} 0.$$

Since $\sup |f_n'(t) - 0| = \sup\limits_{0 \le t \le 1} \frac{1}{2\sqrt{t+n}} = \frac{1}{2\sqrt{n}} \to 0.$

Ex. Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$. Check for uniform conv. of f_n to some f.

(i) $f_n(t) = \frac{\sin nt}{\sqrt{n}}$, (ii) $f_n(t) = n^2 t (1-t^2)^n$

(iii) $f_n(t) = t e^{-nt}$

Also, verify for term by term integration/differentiation for each of the above.

Theorem: Let $E \subseteq \mathbb{R}$, and $f_n \to f$ uniformly on E. For a limit pt. x of E, suppose

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(*) $\lim_{t \to x} f_n(t) = A_n$ (finite).

Then A_n is conv., ~~In other words,~~ and

$$\lim_{t \to x} f(t) = \lim A_n$$

That is,
$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t).$$

Proof: Since $f_n \to f$ uniformly on E. For $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

(*) $|f_n(t) - f_m(t)| < \epsilon, \ \forall\, n, m \geq n_0, \ \forall\, t \in E$

By (*), it implies that

$$|A_n - A_m| \leq \epsilon, \ \forall\, n, m \geq n_0.$$

$\Rightarrow A_n \to A$ (say).

Now, $|f(t) - A| = |f(t) - f_n(t) + f_n(t) - A_n + A_n - A|$

$\leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$

$< \epsilon + \epsilon + \epsilon$ for $t \in (x-\delta, x+\delta) \setminus \{x\}$

for $n \geq n_0'$ (free of t)

i.e $\lim_{t \to x} f(t) = A = \lim_{n \to \infty} A_n$

$\Rightarrow \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$.

Theorem: Let $f_n : [a,b] \to \mathbb{R}$ be s.t. (f_n') converges uniformly. If $\exists\, x_0 \in [a,b]$ s.t. $(f_n(x_0))$ is conv, then (f_n) is uniformly conv and $\lim f_n'(x) = (\lim f_n(x))'$.
(i.e limit & derivative commute)

Proof: The 1st part of the proof is as earlier. By MVT, it follows that

$$|f_n(x) - f_m(x)| \leq (b-a) \|f_n' - f_m'\| + |f_n(x_0) - f_m(x_0)|.$$

Since f_n' conv. uniformly, & $f_n(x_0)$ is conv, it follows that $f_n \to f$ (say) uniformly.

Claim $\lim f_n'(x) = f'(x)$.

Notice that f_n' need not be continuous, hence FTC cannot be applied.

Therefore, we need to exploit the differentiability of f. (98)

For $x \in [a,b]$, define

$$\varphi_n(t) = \frac{f_n(x) - f_n(t)}{x - t}, \quad t \in [a,b] \setminus \{x\}.$$

Then

$$\lim \varphi_n(t) = \frac{f(x) - f(t)}{x - t} = \varphi(t) \text{ (say)}$$

Notice that $\lim\limits_{t \to x} \varphi_n(t) = f_n'(x)$ (finite).

Also, $|\varphi_n(t) - \varphi_m(t)| = |f_n'(\alpha) - f_m'(\alpha)|$ (by MVT)

$< \epsilon$ for $n, m \geq n_0$

$\forall\, t \in [a,b] \setminus \{x\}$.

$\Rightarrow \varphi_n \to \varphi$ uniformly on $[a,b] \setminus \{x\}$.

Apply previous theorem with $E = [a,b]$.

Then

$$\begin{aligned}
\lim_{n \to \infty} f_n'(x) &= \lim_{n \to \infty} \lim_{t \to x} \varphi_n(t) \\
&= \lim_{t \to x} \lim_{n \to \infty} \varphi_n(t) \\
&= \lim_{t \to x} \varphi(t) \text{ (exists)} \\
&= f'(x).
\end{aligned}$$

Thus $\lim f_n'(x) = (\lim f_n(x))'$.

Term by term differentiation:

let $S_n = f_1 + \cdots + f_n$, $f_i : [a,b] \to \mathbb{R}$.
s.t $S_n' \xrightarrow{\text{unif}} S$ & $S_n(x_0) \to L$. Then

$$\lim (S_n') = (\lim S_n)'$$

i.e $f_1' + \cdots + f_n' + \cdots \infty = (f_1 + \cdots + f_n + \cdots \infty)'$

This sparks a very fundamental question: when

$$\left(\int_a^x f(t)\,dt\right)' \overset{?}{=} \int_a^x f'(t)\,dt \quad (**)$$

Notice that if f' is cont, then for
$F(x) = \int_a^x f'(t)\,dt$, by FTC, $F' = f'$
$(F-f)' = 0$, by MVT $\Rightarrow$ ~~$F - f = 0$~~
$F - f = \text{const} \Rightarrow$ ~~$F = f - f(a)$ ($\because F(a) = 0$)~~
$F(x) = f(x) - f(a)$ ($\because F(a) = 0$).

However, if f' is not cont, e.g $f' \in R[a,b]$
then $(**)$ need not be true.

Uniform Continuity:

Def^n: A function $f: A \subset (X,d) \to \mathbb{R}$ is said to be uniformly continuous on A if for each $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Notice that δ is free of choice of locations of points $x, y \in A$ (rather depends only upon their separation).

Ex. For $x_0 \in X$, let $f(x) = d(x, x_0)$. Then f is uniformly cont on X.

(Hint: $d(x, x_0) \leq d(x,y) + d(y, x_0)$

$f(x) - f(y) \leq d(x,y)$

& replacing x with y, it implies that

Ex. For $x \in X$, $\phi \neq A \subset X$, we define $d(x,A) = \inf_{a \in A} d(x,a)$, called distance of A from x, is uniformly

(Hint: $d(x,a) \leq d(x,y) + d(y,a)$)

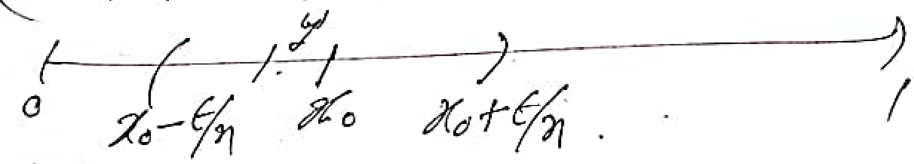
$\Rightarrow d(x,A) \le d(x,y) + d(x,A)$

$\Rightarrow |f(x)-f(y)| \le d(x,y) \quad (\because x \leftrightarrow y)$

Ex. $f:(0,1)\to\mathbb{R}$, $f(x)=\frac{1}{x}$ is continuous on $(0,1)$ but not uniformly continuous.

<u>Pointwise continuity of f:</u>

Let $x_0\in(0,1)$, then for $\epsilon>0$, $\exists\, n\in\mathbb{N}$ s.t. $(x_0-\epsilon/n,\ x_0+\epsilon/n)\subset(0,1)$



Suppose $\left|\frac{1}{x_0}-\frac{1}{y}\right|<\epsilon$ for $y\in\underbrace{(x_0-\epsilon/n,\ x_0+\epsilon/n)}_{=I_{x_0}}$.

Then $|x_0-y|<\epsilon x_0 y$.

Let $\delta=\min\limits_{y\in I_{x_0}}\{\epsilon x_0 y\}=\epsilon x_0(x_0-\epsilon/n)>0$.

Let $|x_0-y|<\delta$. Then

$$\left|\frac{1}{x_0}-\frac{1}{y}\right|=\frac{|x_0-y|}{x_0 y}<\frac{\delta}{x_0 y}=\frac{\epsilon x_0(x_0-\epsilon/n)}{x_0 y}<\epsilon$$

Hence, f is cont. at each $x_0\in(0,1)$.

<u>f is not uniformly conti:</u>

Let $\epsilon=1/2$, $x=\frac{1}{n}$, $y=\frac{1}{n+1}$; $n\in\mathbb{N}$.

Then for any $\delta>0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.

$|x-y| = \left|\frac{1}{n}-\frac{1}{n+1}\right| < \delta$, but $|f(x)-f(y)| = 1 \not< \frac{1}{2}$.

Hence, f is not uniformly cont. on $(0,1)$.

From the above argument, we can prove the following result.

<u>Theorem</u>: Let $f: A \subset (X,d) \to \mathbb{R}$. Then f is uniformly cont. on A iff $\forall$ pair of seq^{ns} $x_n, y_n \in A$ with $d(x_n,y_n) \to 0$, implies $|f(x_n)-f(y_n)| \to 0$.

Pf: Suppose f is unif. cont on A. Then for $\epsilon>0$, $\exists$ $\delta>0$ s.t.

$$d(x,y)<\delta \Rightarrow |f(x)-f(y)|<\epsilon \quad \text{—(1)}$$

Let $x_n, y_n \in A$ s.t $d(x_n,y_n) \to 0$. Then for $\delta>0$, $\exists$ $n_0 \in \mathbb{N}$ s.t for $n \geq n_0$,

$$d(x_n,y_n)<\delta \Rightarrow |f(x_n)-f(y_n)|<\epsilon \quad (\text{from (1)}).$$

That is, if $d(x_n,y_n) \to 0$, then $|f(x_n)-f(y_n)| \to 0$.

Conversely, suppose that f is not unif. cont.

Then $\exists\ \epsilon_0 > 0$ s.t. $\forall\ \delta > 0$, $\exists\ x, y \in A$ with $d(x,y) < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$.

Now, let $\delta = \frac{1}{n}$, $n \in \mathbb{N}$. Then $\exists\ x_n, y_n \in A$ s.t. $d(x_n, y_n) < \frac{1}{n}$, $\forall\ n \geq 1$, but

$$|f(x_n) - f(y_n)| \geq \epsilon_0$$

That is, $d(x_n, y_n) \to 0$, but $\lim |f(x_n) - f(y_n)| \geq \epsilon_0$, is a contradiction. Hence, f is unif. cont.

Ex. Show that uniform cont. function on a metric space (X, d) send C.C to C.C.
(Hint: $f : (X, d) \longrightarrow \mathbb{R}$ uniformly cont. so for $d(x_n, y_n) \to 0 \Rightarrow |f(x_n) - f(y_n)| \to 0$)

Result: Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then f is uniformly continuous.

Pf: On contrary, suppose f is not unif. cont on $[a,b]$. Then $\exists\ \epsilon_0 > 0$ s.t. $\forall\ \delta > 0$, $\exists\ x, y \in [a,b]$ with $|x - y| < \delta$, but

$$|f(x) - f(y)| \geq \epsilon_0.$$

For $\delta = 1/n$, $\exists\ x_n, y_n \in [a,b]$ s.t.

$|x_n - y_n| < 1/n$, but $|f(x_n) - f(y_n)| \geq \epsilon_0$.

By B-W theorem, x_n & y_n have conv. subseq's, say $x_{n_k} \to x$ & $y_{n_k} \to y$.

Then $|x-y| = \lim |x_{n_k} - y_{n_k}| \leq \lim \frac{1}{n_k} = 0$.

Since f is cont, and $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0$

$\Rightarrow |f(x) - f(y)| \geq \epsilon_0$

i.e. $0 \geq \epsilon_0$. $\times$

Ex. Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. s.t. $\lim_{|x|\to\infty} f(x) = 0$. Then f is uniformly cont.

For $\epsilon > 0$, $\exists\ [-a, a]$ s.t.

$|f(x)| < \epsilon/2$ if $x \in [-a,a]^c$.

Hence, if $x, y \in [-a,a]^c$, then

$|f(x) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$ ——— (1)

Since f is uniformly cont on $[-a,a]$.

For $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon$ —— (2)

Since (1) holds true for x, y with $|x-y|<\delta$.
It follows that for $\epsilon>0$, we get $\delta>0$
st $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon$ (105)
(for $x, y \in \mathbb{R}$).
Hence f is uniformly Cont on $\mathbb{R}$.

Notice that $f \in C_0(\mathbb{R}) \Rightarrow f$ is Cont &
$\lim_{|x|\to\infty} f(x)=0$, and hence f is uniformly Cont.
But if f is Cont & bdd, then f need not be uniformly cont on $\mathbb{R}$.

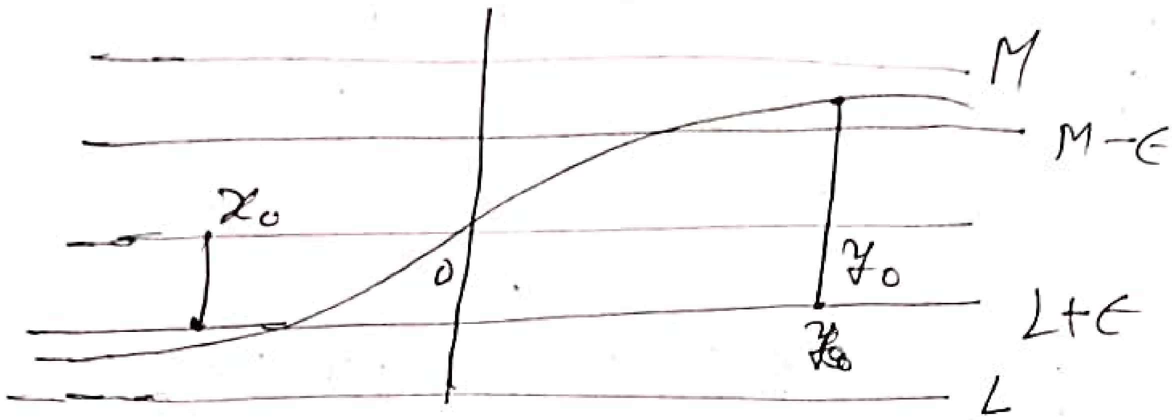
Ex. $f(x)=\sin x^2$, is Cont & bdd but not unif. Cont. on $\mathbb{R}$.

Ex. Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded Cont. function. If f is monotone, then f is Uniformly Continuous on $\mathbb{R}$.

Since f is bounded, let

$$\inf f(x)=L \quad \& \quad \sup f(x)=M.$$

For $\epsilon>0$, $\exists\, x_0, y_0 \in \mathbb{R}$ s.t $f(x_0)<L+\epsilon$ & $f(y_0)>M-\epsilon$



If f is monotone increasing, then

for $x, y \in [x_0, y_0]^c$ & $x, y \geq y_0$

$f(y) - f(x) \leq M - f(y_0) < M - (M-\epsilon) = \epsilon.$

Similarly, if $x, y \leq x_0$, then

$f(y) - f(x) < L+\epsilon - f(x_0) < L+\epsilon - L = \epsilon.$

Thus, for $x, y \in [x_0, y_0]^c$ we get

$|f(x) - f(y)| < \epsilon$ ——— (1)

Since f is cont on $[x_0, y_0]$, f is unif. cont on $[x_0, y_0]$. For $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

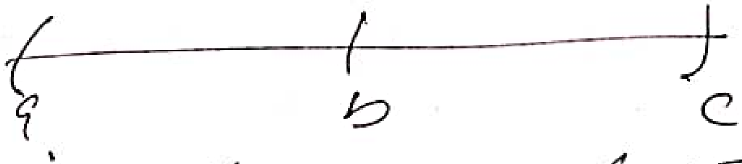
$x, y \in [x_0, y_0]$, & $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ ——— (2)

Notice that (1) also holds for $x, y \in [-x_0, x_0]^c$ with $|x-y|<\delta$. Thus, we set single $\delta>0$ s.t. $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon$ 107

Ex. If $f:\mathbb{R}\to\mathbb{R}$ is a bounded cont. f^n, then for f monotone, it follows that
$$\lim_{x\to\infty} f(x) = \text{finite} \quad \& \quad \lim_{x\to-\infty} f(x) = \text{finite}.$$

(Hint: For any seq^n $x_n\to\infty$, $f(x_n)$ is bounded & $\lim_{n\to\infty} f(x_n)=\sup_n f(x_n)$, for f is increasing.)

Ex. Let $f:(a,b]\to\mathbb{R}$ & $f:(b,c)\to\mathbb{R}$ be ~~continuous~~ uniformly cont. Then $f:(a,c)\to\mathbb{R}$ is uniformly continuous.



Since f is unif cont on $(a,b]$ & (b,c), for $\epsilon>0$, $\exists\,\delta>0$ s.t. if $x,y\in(a,b]$ or $x,y\in(b,c)$, with $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon$.

Now, let $x, y \in (a, c)$, with $|x-y|<\delta$.
Then $|x-b|<\delta$ & $|y-b|<\delta$. Hence,

$$|f(x)-f(y)| < |f(x)-f(b)| + |f(b)-f(y)|$$
$$< 2\epsilon.$$

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Thus, f is unif. cont on (a, c).

We see that a uniformly continuous function can be extended uniformly to ~~the lens~~ the closure of the set

<u>Theorem</u>: Let $f: A \subset \mathbb{R} \to \mathbb{R}$ be uniformly cont. on A. Then f can be extended uniformly to $\bar{A}$, & this extension is unique.

Proof: Let $x \in \bar{A}$. Then $\exists$ $x_n \in A$ s.t. $x_n \to x$. Now, $f(x_n)$ is a bounded sequence in $\mathbb{R}$. Hence by ~~B~~-W theorem $f(x_n)$ has a conv. subsequence. W.l.o.g. we can assume that $f(x_n)$ is conv.

Let $\bar{f}(x) = \lim f(x_n)$. ($\because \lim f(x_n)$ exists)

Notice, that $\bar{f}$ is well defined, because

f is uniformly continuous on A. If $x_n, y_n \to x$ then $x_n - y_n \to 0$ & $\Rightarrow f(x_n) - f(y_n) \to 0$ (109)

i.e. $\lim f(x_n) = \lim f(y_n)$

($\because$ $\lim f(x_n)$ & $\lim f(y_n)$ both exist).

Hence $\tilde{f} : \bar{A} \to \mathbb{R}$ is well defined.

Suppose $x, y \in \bar{A}$ and they are closed enough to each other. Then $\exists$ $x_n, y_n \in A$ s.t. $x_n \to x$ & $y_n \to y$.

Hence,

$$\tilde{f}(x) - \tilde{f}(y) = \tilde{f}(x) - f(x_n) + f(x_n) - f(y_n) + f(y_n) - \tilde{f}(y)$$

$$\Rightarrow |\tilde{f}(x) - \tilde{f}(y)| \le |\tilde{f}(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - \tilde{f}(y)|.$$

Notice that $|\tilde{f}(x) - f(x_n)| < \epsilon$ & $|\tilde{f}(y) - f(y_n)| < \epsilon$ for $n \ge n_0$ (say).

Let $|x-y| < \delta$ (small enough). Then $\exists$ $n' \in \mathbb{N}$ s.t. $|x_n - y_n| < \delta$ for $n > n'$!

Since f is ~~co~~ unif cont. on A, it follows that $|f(x_n) - f(y_n)| < \epsilon$ for $n \geq n'$.

thus for large $n \geq \max(n_0, n')$. (110)

$|\tilde{f}(x) - f(y)| \leq 3\epsilon$, when $|x-y| < \delta$.

Hence $\tilde{f}$ is uniformly cont. on $\bar{A}$.

This extension of f is unique.

If $\tilde{g} : \bar{A} \xrightarrow[\text{cont}]{\text{unif}} \mathbb{R}$ & $\tilde{g} = f$ on A,

then for $x \in \bar{A}$, $\exists$ $x_n \in A$ s.t. $x_n \to x$.

Hence, $\tilde{f}(x) = \lim f(x_n) = \lim g(x_n) = \tilde{g}(x)$

($\because$ g is uniform cont. extension).

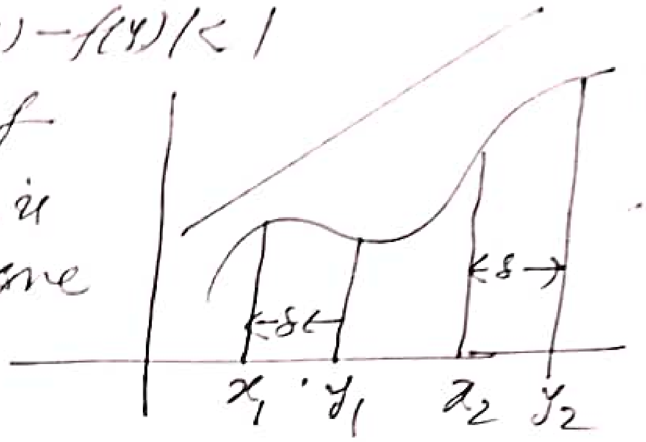
Next, we shall see that uniform cont. function grows slower than a straight line.

__Theorem:__ Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly cont., then $\exists$ $A, B \geq 0$ s.t. $|f(x)| \leq A|x| + B$ for all $x \in \mathbb{R}$.

Proof: For $\epsilon = 1$, $\exists$ $\delta > 0$ s.t.

$|x-y|<\delta \Rightarrow |f(x)-f(y)|<1$

We divide the proof into two parts: one is near "0" & other one away from "0".

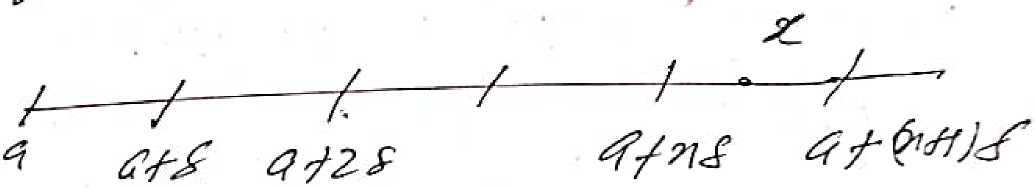


Let $a>0$, then

$$|f(x)|\le A<\infty \quad \forall x\in[-a,a].$$

Now, consider $f:[a,\infty)\longrightarrow\mathbb{R}$. Then for $x\in[a,\infty)$, we can find $n\in\mathbb{N}$ s.t.

$$x\in[a+n\delta,\ a+(n+1)\delta].$$



Then

$$f(x)-f(a)=f(x)-f(a+n\delta)+f(a+n\delta)-f(a)$$

$$=f(x)-f(a+n\delta)+\sum_{j=1}^{n}\{f(a+j\delta)-f(a+(j-1)\delta)\}$$

$$\Rightarrow |f(x)|<1+n+|f(a)|$$

$$\Rightarrow \left|\frac{f(x)}{x}\right|<\frac{(n+1)+|f(a)|}{a+n\delta}<\frac{(n+1)+|f(a)|}{n\delta}$$

$$<\left(1+\frac{1}{n}\right)\frac{1}{\delta}+\frac{|f(a)|}{n\delta}\le B<\infty$$

Notice that B is independent of n, hence B is independent of x.

that is, $|f(x)| \leq B|x|$ if $x > a$. (112)

Hence, we can summerize that

$$|f(x)| \leq B|x| + A \text{ for } x \in \mathbb{R}.$$

Ex. Notice that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, as it cannot satisfy the conclusion of the above theorem.

Ex. Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable and its derivative is bounded. Then f is uniformly cont. on $\mathbb{R}$.

For $x, y \in \mathbb{R}$, by MVT,

$$|f(x) - f(y)| = |f'(t)(x-y)| \leq M|x-y| \;)$$

However, $f(x) = \sqrt{x}$, $x \in (0, \infty)$ is unif. cont, but its derivative $f'(x) = \frac{1}{2\sqrt{x}}$ is not bounded.

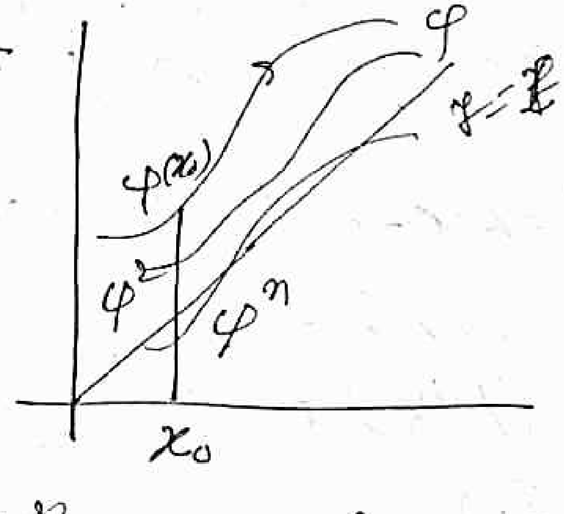
Fixed points:

Fixed point searching is an idea to solve equation of type $\varphi(x)=x$. This helps solving a range of problems, including approximation theory, diff. equations etc. Fixed points can be obtained via iterations, i.e. if the function "shrinks nicely", then we get fixed points via iterations.

That is, if x_0 is a point in the space X, then

$$x_0 \to \varphi^1(x_0) \to \varphi^2(x_0) \cdots$$

where $\varphi^n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n}$



If $\varphi^n(x_0)$ is a conv. seq^n & φ is continuous then $\varphi^n(x_0) \to x \Rightarrow \varphi(x) = \varphi(\lim \varphi^n(x_0)) = x$.

However, if the space is complete, we

only need to verify ~~that~~ $\varphi^n(x_0)$ to be a C.S.

Intially Shrinking function, we mean here with contraction mapping. (114)

Def^n: A function $\varphi:(X,d)\to(X,d)$ is called contraction if $\exists\ 0<\alpha<1$ s.t. $d(\varphi(x),\varphi(y))\le\alpha\, d(x,y)$, $\forall x,y\in X$.

<u>Theorem</u>: Let (X,d) be a complete metric space. If $\varphi:(X,d)\to(X,d)$ is a contraction, then φ has a unique fixed point.

Proof: Let $0<\alpha<1$ be such that

$$d(\varphi(x),\varphi(y))\le\alpha\, d(x,y),\ \forall\, x,y\in X.$$

For a ~~fixed~~ point $x_0\in X$, let

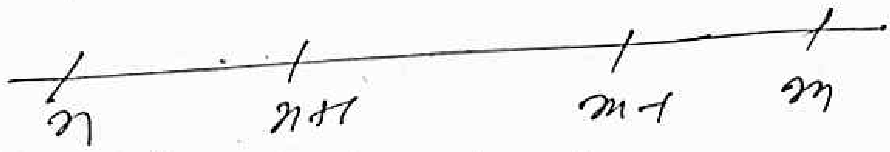
$$\varphi^0(x_0)=x_0,\quad \varphi^1(x_0)=\varphi(x_0)\ldots \text{etc}$$

Then $d(\varphi^{n+1}(x_0),\varphi^n(x_0))\le\alpha\, d(\varphi^n(x_0),\varphi^{n-1}(x_0))$

$$\le\alpha^n d(\varphi(x_0),x_0).$$

We show that $\varphi^n(x_0)$ is a C.S.

Let $m > n$



Then
$$d(\varphi^n(x_0), \varphi^m(x_0)) \le (\alpha^n + \cdots + \alpha^{m-1})\, d(\varphi(x_0), x_0).$$
$$\le \frac{\alpha^n}{1-\alpha}\, d(\varphi(x_0), x_0) \quad (\because 0 < \alpha < 1).$$
$$\to 0 \text{ as } m \to \infty.$$

Since (X, d) is complete,
$$\varphi^n(x_0) \to x \in X \text{ (say)}$$
$$\Rightarrow \varphi(x) = \varphi(\lim \varphi^n(x_0)) = \lim \varphi^{n+1}(x_0)$$
$$\Rightarrow \varphi(x) = x.$$

If $\exists\, y \in X$ st $\varphi(y) = y$, then
$$d(x, y) = d(\varphi(x), \varphi(y)) \le \alpha\, d(x, y)$$
$$\Leftrightarrow x = y \quad (\because 0 < \alpha < 1).$$

This establishes that φ has unique fixed point.

Notice that completeness property of the space is a sufficient condition for existence of fixed point. For example,

$$\varphi : (0,\infty) \to (0,\infty)$$

$$\varphi(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \quad a > 0.$$

satisfies $\varphi(\sqrt{a}) = \sqrt{a}$.

Also, contraction is a sufficient condition for existence of fixed pt.

Notice that φ above is not a contraction map, since

$$|\varphi(x) - \varphi(y)| = \frac{1}{2}\left|1 - \frac{a}{xy}\right||x-y|$$

because the function $\left|1 - \frac{a}{xy}\right|$ is not bounded near "0".

Ex. If (X,d) is a complete metric space & $f : X \to X$ is such that f^K is a contraction, then show that f has a unique fixed point.

(Hint: do for $K=2$, use the fact that f^2 cannot have two fixed points;

ie $f^2(x_0) = x_0$ ~~&~~ $y_0 = f(x_0)$ (say).
implies that $f^2(y_0) = y_0 \Rightarrow y_0 = x_0$ etc.)

Ex. Let $T: C[0,1] \longrightarrow C[0,1]$ be (117)
defined by $T(f)(x) = \int_0^x f(t)\,dt$.
Show that T^2 is a contraction but T is not a contraction.

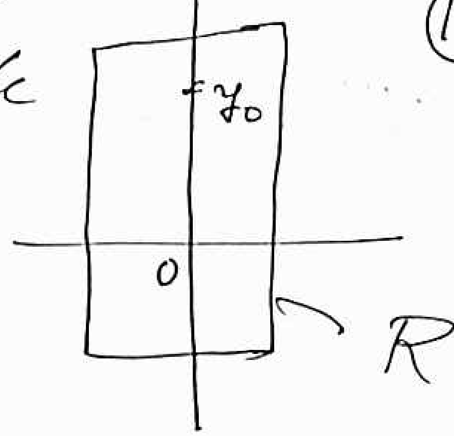
Notice that the above fact in these example is also clear from the fact that in the convergence of $\varphi^n(x_0)$, we can ignore finitely many steps.

Now, we shall try to understand the existence & uniqueness of the initial value problem:

$$(*) \quad \left.\begin{array}{l} y' = f(x, y) \\ y(0) = y_0 \end{array}\right\}$$

with the help of fixed point theorem.

Suppose f is a continuous function in some rectangle containing the interval $(0, y_0)$ ~~in~~ in its interior, & f is Lipschitz in the 2nd variable. i.e.



$$|f(x,y_1) - f(x,y_2)| \le K|y_1 - y_2|,$$ ~~for~~ where K is a fixed constant

Then (*) has a unique solution in some nbd of $x=0$.

Notice that solving (*) is equivalent to solve

$$\int_0^x y'(t)\,\cancel{dt = y(0) +} \int_0^x f(t, y(t))\,dt$$

i.e. $$y(t) = y_0 + \int_0^x f(t, y(t))\,dt \qquad \text{—— } (**)$$

That is, we want $y(t)$ s.t. $(**)$ holds.

In other words, we want to get fixed pt for the map $\varphi \mapsto F(\varphi)$,

where

$$F(\varphi)(t) = y_0 + \int_0^x f(t, \varphi(t))\,dt,$$

with $\varphi \in C[-\delta, \delta]$, for some $\delta > 0$, which we get very soon. (119)

Now,

$$|F(\varphi)(x) - F(\psi)(x)| \leq \int_0^x |f(t, \varphi(t)) - f(t, \psi(t))|\,dt$$

$$\leq K \int_0^x |\varphi(t) - \psi(t)|\,dt$$

$$\leq K \cdot 2\delta \cdot \|\varphi - \psi\|_\infty.$$

Thus, $F : C[-\delta, \delta] \to C[-\delta, \delta]$ is a contraction as long as $2K\delta < 1$, i.e if $\delta < 1/2K$.

Hence F has a unique fixed point in $C[-1/2K, 1/2K]$.

That is, (*) has unique solution in $|x| < \frac{1}{2K}$.

Ex. Consider

$$y' = 2x(1+y), \quad y(0) = 0.$$

Then $\varphi(x) = \int_0^x 2t(1 + \varphi(t))\,dt.$

with an initial guess ~~$\varphi^0(x) = 0 \ \forall$~~ $\varphi^0 \equiv 0$, we get

$$\varphi^1(x) = \int_0^x 2t(1+0)\,dt = x^2.$$

$$\varphi^2(x) = \int_0^x 2t(1+t^2)\,dt = x^2 + \frac{x^4}{2}$$

Also $\varphi^3(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{6}$.

Thus, by induction,

$$(*) \qquad \varphi^n(x) = \sum_{k=1}^{n} \frac{x^{2k}}{k!} \to e^{x^2} - 1,$$

and $\varphi(x) = e^{x^2} - 1$ is a solution, which is same as method of separation of variable etc.

Notice that the series $(*)$ converges ~~for each~~ uniformly on every interval $[-a, a]$ or on any interval $[a, b]$.

On the other hand, $\varphi'(x) = 2x(1 + \varphi(x))$ has unique solution in nbd of any point x_0, i.e., $[x_0 - \delta, x_0 + \delta]$ with $\delta < \frac{1}{4}$.

(Hint: Lipschitz const $= 2$.)