Metric space:

let X be a non-empty set. A map
$d: X \times X \longrightarrow R_+ = [0, \infty)$ s.t.
(i) $d(x, y) = 0$ iff $x = y$; $x, y \in X$
(ii) $d(x, y) = d(y, x)$ (Symmetric)
(iii) $d(x, z) \leq d(x, y) + d(y, z)$
(triangle inequality)

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is called a metric on X, and the pair (X, d) is called metric space

Ex. If $X = \mathbb{R}^n$, then for $x, y \in \mathbb{R}^n$, $1 \leq p < \infty$,
(i) $d_p(x, y) = \left(\sum_{1}^{n} |x_i - y_i|^p \right)^{1/p}$. $x = (x_1, \ldots x_n)$ etc
is a metric on $\mathbb{R}^n$ (we proof it later)
(ii) $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ is a metric on $\mathbb{R}_n$. (It follows easily)

Ex. Let (X, d) be a metric space. Show that $d'(x, y) = \min\{1, d(x, y)\}$ defines a metric.

Ex. If $X = C[0,1]$, the space of continuous functions on $[0,1]$. Then for $f, g \in X$,

$$d_\infty(f,g) = \sup_{0 \le t \le 1} |f(t) - g(t)|$$ (48)

defines a metric on $\mathbb{R}$.

Hint: f is cont on $[0,1]$. So f is bounded. and $|f(t)-h(t)| \le |f(t)-g(t)| + |g(t)-h(t)|$)

Ex. For $f, g \in C[0,1]$, define

$$\rho(f,g) = \int_0^1 \min\{|f(t)-g(t)|, 1\}\, dt.$$

Then ρ is a metric on $C[0,1]$.

Ex. If $X \neq \emptyset$, then for $x, y \in X$,

$$d_0(x,y) = \begin{cases} 1 & x \neq y, \\ 0 & x = y; \end{cases}$$

define a metric on X, and is called discrete metric. Thus, every non-empty set has a metric.

Note that for $d(x,z) \le d(x,y) + d(y,z)$ to hold we need to verify ~~five~~ three cases

(i) $x = y$ and $y \neq z$;

(ii) $x \neq y$, $y = z$;
(iii) all of x, y, z are distinct.

Question: If (X, d) is a metric space & $f : [0,\infty) \to [0,\infty)$ is a map, does it imply that $f \circ d$ is a metric on X?

Ex. Let $f(t) = \frac{t}{1+t}$; then $f'(t) = \left(1 - \frac{1}{1+t}\right)'$

ie $f'(t) = \frac{1}{(1+t)^2} > 0$, $\forall t \in [0,\infty)$.

Hence f is strictly ~~inct.~~ increasing

and $f''(t) = -\frac{2}{(1+t)^3} < 0$, hence concave.

Also, $f(t) = 0$ iff $t = 0$.

Note that
$$\frac{t+s}{1+t+s} \leq \frac{t}{1+t} + \frac{s}{1+s}$$

Let $s = d(x,y)$, $t = d(y,z)$, $r = d(x,z)$.

Then $r \leq s+t$,
$$f(r) \leq f(s+t) \leq f(s) + f(t)$$

$$\Rightarrow f\circ d(x,z) \leq f\circ d(x,y) + f\circ d(y,z).$$

Thus, $f \circ d$ is a metric on X.

This result is true for a large class of ~~convex fun~~ concave function.

Ex. Let $f : [0,\infty) \to [0,\infty)$ be concave and $f(0) \geq 0$. Then

$$f(x+y) \leq f(x) + f(y) \quad (\text{Sub-additive}).$$

Hence,

$$\frac{y}{x+y} f(0) + \frac{x}{x+y} f(x+y) \leq f\left(\frac{y}{x+y}\cdot 0 + \frac{x}{x+y}(x+y)\right).$$

$$(\because f \text{ is concave})$$

$$\Rightarrow \frac{x}{x+y} f(x+y) \leq f(x)$$

Replacing $x \to y$, we get $\frac{y}{x+y} f(x+y) \leq f(y)$

$$\Rightarrow f(x+y) \leq f(x) + f(y).$$

Result: Let (X,d) be a metric space and $f : [0,\infty) \to [0,\infty)$ be a monotone increasing function with $f(t) = 0$ iff $t = 0$. If f is concave, then $f \circ d$ is a metric on X.

(Hint: Conclude from the example and the previous result)

Ex. Let H^∞ (Hilbert cube) be the space of seq^{ns} $x = (x_n) = (x_1, x_2, \dots x_n, \dots)$ s.t. $|x_n| \le 1$. Then 51

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$$

defines a metric on H^∞.

(i) $d(x,y) \le \sum \frac{2}{2^n} < \infty$

(ii) $|x_n - z_n| \le |x_n - y_n| + |y_n - z_n|$

$$\Rightarrow \sum_{n=1}^{K} \frac{|x_n - z_n|}{2} \le \sum_{n=1}^{K} \frac{|x_n - y_n|}{2^n} + \sum_{n=1}^{K} \frac{|y_n - z_n|}{2^n}.$$

$$\le d(x,y) + d(y,z) < \infty$$

Since LHS is an ↑ seq^n which is bound above, it follows that

$$\lim_{K \to \infty} \sum_{n=1}^{K} \frac{|x_n - z_n|}{2^n} \le d(x,y) + d(y,z)$$

ie $d(x,z) \le d(x,y) + d(y,z)$.

Ex. Show that $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ defines a metric on $(0,\infty)$.

(Hint: $\left|\frac{1}{x} - \frac{1}{z}\right| = \left|\frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z}\right| \le \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right|$)

Def^n: $B_r(x) = \{y \in X : d(y,x) < r\}$ is called an open ball in the metric space (X,d).

$B_r[x] = \{y \in X : d(y,x) \leq r\}$ (52)

is called closed ball in (X,d).

Def^n: A set O in metric space (X,d) is called open if for each $x \in O$, $\exists\ r > 0$ s.t. $B_r(x) \subseteq O$.

Let J be the collection of all open set in X w.r.t metric d. Then

(i) $\phi, X \in J$ (why?)

(ii) $\bigcup\limits_{i \in I} O_i \in J$, for $O_i \in J$, and for any index set I.

(iii) $\bigcap\limits_{i=1}^{n} O_i \in J$, for $O_i \in J$.

(Proof: Follows from def^n of open set)

Def^n: A function $f:(X,d) \to \mathbb{R}$ is said to be continuous at $x \in X$, if $\forall \epsilon > 0$, $\exists$ $\delta > 0$ s.t.

$$d(x,y) < \delta \Rightarrow |f(x)-f(y)| < \epsilon \text{ — } (*)$$

If it happens for each $x \in X$, we say that f is continuous on X.

From $(*)$, it follows that

$$y \in B_\delta(x) \Rightarrow f(y) \in (f(x)-\epsilon, f(y)+\epsilon)$$

$$\text{i.e.} \quad B_\delta(x) \subseteq f^{-1}\{(f(x)-\epsilon, f(x)+\epsilon)\}.$$

Since $x \in$ RHS, it follows that RHS is open around x.

<u>Result</u>: A function $f:(X,d) \to \mathbb{R}$ is continuous iff $f^{-1}(O) \in \mathcal{J}$, for each open set O in $\mathbb{R}$.

Pf: Suppose f is continuous. Let $O \subset \mathbb{R}$ be open. Claim $f^{-1}(O)$ is open in X. Let $x \in f^{-1}(O)$. Then $f(x) \subset O$. Hence,

~~for $\epsilon>0$,~~ $\exists$ some $\epsilon_0 > 0$ s.t.

$$(f(x)-\epsilon_0,\ f(x)+\epsilon_0) \subseteq O.$$

Given f is cont at x. For $\epsilon_0>0$, $\exists\, \delta>0$ s.t. $B_\delta(x) \subseteq f^{-1}(f(x)-\epsilon_0,\ f(x)+\epsilon_0)$

$$\subseteq f^{-1}(O).$$

$\Rightarrow f^{-1}(O)$ is open in X.

Conversely, let $f^{-1}(O) \in \mathcal{J}$ for each open set O in $\mathbb{R}$. For $\epsilon>0$, it follows that

$$x \in f^{-1}(f(x)-\epsilon,\ f(x)+\epsilon) \in \mathcal{J}.$$

Since $f^{-1}(f(x)-\epsilon, f(x)+\epsilon)$ is open in X, it follows that $\exists\, \delta>0$ s.t.

$$y \in B_\delta(x) \subset f^{-1}(f(x)-\epsilon,\ f(x)+\epsilon)$$

i.e. $d(x,y)<\delta \Rightarrow f(y) \in f(B_\delta(x)) \subset (f(x)-\epsilon, f(x)+\epsilon)$.

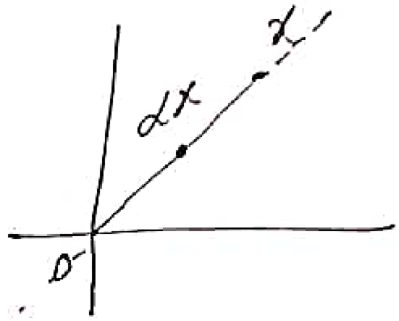
$$\Rightarrow |f(x)-f(y)| < \epsilon.$$

For a metric space (X,d), we call $(X,\mathcal{J})$ the topology of X generated by d.

Normed linear space:

Normed linear space is eventually mixing of linear structure of a space with its some of topological structure.

Let $(X,+,\cdot)$ be a linear space over the field $F(=\mathbb{R}$ or $\mathbb{C})$. Let $(X,\mathcal{J})$ be the topological structure given by some metric d on X. Now, the question is: how to mix linear structure with top. structure?



Note that a linear space is mainly concerned about two maps:

(i) $(x,y)\mapsto x+y$ $(X\times X\to X)$

(ii) $(\alpha,x)\mapsto \alpha x$ $(F\times X\to X)$.

Therefore, a linear space X can be thought of made by these two maps.

And topology is all about continuity of maps. Thus, we can think of continuity of "+" & "." on $X \times X$ and $F \times X$ respectively in their respective product topology $\mathcal{J} \times \mathcal{J}$ and $\mathcal{U} \times \mathcal{J}$. ($\because \mathcal{U}$ is usual top. on $\mathbb{R}$ & $\mathbb{C}$) (56)

A linear space with such property is called topological vector (linear) space.

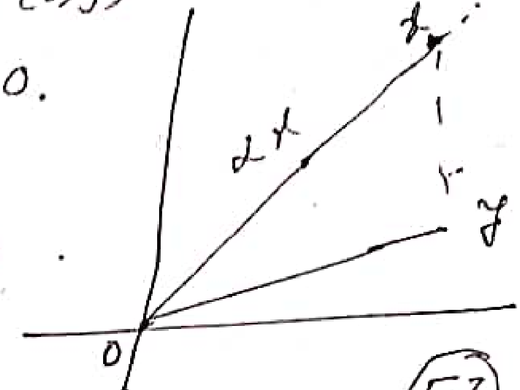
Note that an open set in $\mathcal{J} \times \mathcal{J}$ is union of sets of the form $O_1 \times O_2$, where $O_1, O_2 \in \mathcal{J}$. And open set in $\mathcal{U} \times \mathcal{J}$ is of union of the sets $O_1 \times O_2$ with $O_1 \in \mathcal{U}$, and $O_2 \in \mathcal{J}$.

Now, because of linearity and homogeneity of the space X, we can opt a sense of distance that should satisfies the following set of rules.

(i) $dist(0, \alpha x) = |\alpha|\, dist(0, x)$

(ii) $\text{dist}(x,y) \leq \text{dist}(0,x) + d(0,y)$

(iii) When $\alpha = 0$, $\text{dist}(0,0) = 0$.



Let $P := \text{dist} : X \to [0,\infty)$ be defined by

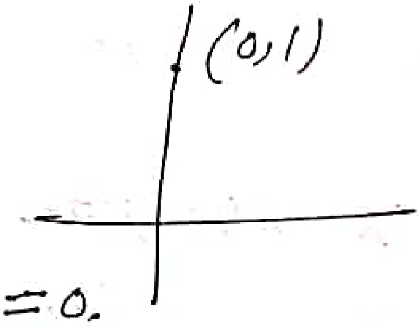
$$p(x) = \text{dist}(0,x).$$

Then (i) $P(x) = 0$ for $x = 0$.

(ii) $P(\alpha x) = |\alpha| P(x)$ (absolute homogeneity)

(iii) $P(x+y) \leq P(x) + P(y)$ (triangle inequality)

Here, P is known as semi-norm, because it is little away from the natural sense of usual distance. For example,



$P : \mathbb{R}^2 \to [0,\infty)$ by

$P(x_1, x_2) = |x_1|$. Then P is a semi-norm and $P(0,1) = 0$.

That is, points on the y-axis is at 0 distance from origin. This does not looks convincing as long as natural distance (or usual distance) is concerned.

Let $\|\cdot\| : X \longrightarrow [0,\infty)$ be a map (58)
s.t.
(i) $\|x\| \geq 0$ for each $x \in X$, and
$\|x\| = 0$ iff $x = 0$
(ii) $\|\alpha x\| = |\alpha| \|x\|$ for each $(\alpha, x) \in F \times X$
(absolute homogeneity)
(iii) $\|x+y\| \leq \|x\| + \|y\|$, for each $x, y \in X$.
(triangle inequality)

The map $\|\cdot\|$ is called a norm on X.
Note that $\|\cdot\|$ induces a metric on X by $d(x,y) = \|x-y\|$, that produces a top. on X. For $r>0$, $x \in X$, open ball
$$B_r(x) = \{y \in X : \|x-y\| < r\}.$$
Hence, open sets can be defined accordingly.

Note that every metric on a linear space needs not produce a norm.

For example, discrete metric on any linear space is not normable, because it fails to satisfy the absolute homogeneity.

For $x, y \in X$, define

$$d_0(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

If we write $\|x\| = d(0,x)$, then for $\alpha \in F$, $\|\alpha x\| \neq |\alpha|\,\|x\|$ $(x \neq 0)$ unless $\alpha = 1$.

However, if d is a metric on a linear space X s.t. $d(x,y) = d(x-y, 0)$ and $d(\alpha x, \alpha y) = |\alpha| d(x,y)$, then $d(x,0) = \|x\|$ defines a norm on X.

(i) $\|x\| = 0$ iff $d(x,0) = 0$ iff $x = 0$.

(ii) $\|\alpha x\| = d(\alpha x, 0) = |\alpha| d(x,0) = |\alpha|\,\|x\|$.

(iii) $\|x+y\| = d(x+y, 0) = d(x, -y)$
$\leq d(x,0) + d(-y,0)$
$= \|x\| + \|y\|$.

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be convex if $f(t_1 x_1 + \cdots + t_n x_n)$
$\leq t_1 f(x_1) + \cdots + f(x_n)$,
when $0 \leq t_i \leq 1$, & $x_i \in \mathbb{R}^n$?

Ex. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying $f(\alpha x) = \alpha f(x)$, $\forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$. (60)
prove that
(i) $f(x+y) \leq f(x) + f(y)$
(ii) $f(0) = 0$
(iii) $f(-x) \geq -f(x)$
(iv) $f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$.

Further, what requires to make f a norm on $\mathbb{R}^n$?

<u>Convergence of seq^n in metric space.</u>

A seq^n (x_n) in a metric space (X, d) is said be converging to $x \in X$ if $\forall \epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t
$n \geq n_0 \Rightarrow d(x_n, x) < \epsilon$.

~~Ex. X = R, & d.E~~
Ex. Let $X = (0, \infty)$ and $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$.

Then $x_n = n$ does not converge to pt of X.

However, this seq^n is not so bad as $x_n = n \to 0$, which is not in X. (61)

Such sequences can be classified as Cauchy sequences.

Def^{n}: A seq^n x_n in (X, d) is said to be a Cauchy seq^n (C.C. in ~~fort~~ short) if $\forall \epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

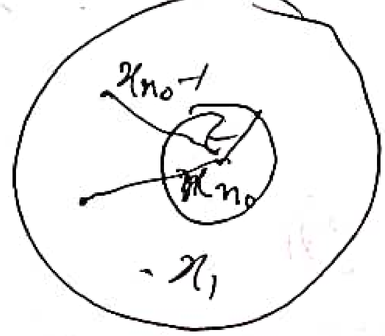
$$\forall\, m, n \geq n_0 \Rightarrow d(x_n, x_m) < \epsilon.$$

Ex. Show that every Cauchy seq^n in a metric space is bounded.

(Hint: A set $A \subset X$ is said to be bounded if $A \subseteq B_r(x)$ for some fixed x & $r > 0$)

$$x_n \in B_\epsilon(x_{n_0}) \quad \text{for } n \geq n_0.$$

Let $r = \max\{\epsilon, d(x_{n_0}, x_i);\ i = 1, 2, \ldots n_0 - 1\}$.



Then $x_n \in B_r(x_{n_0})$, $\forall\, n \geq 1$.

We need certain inequalities to deal with sequence spaces.

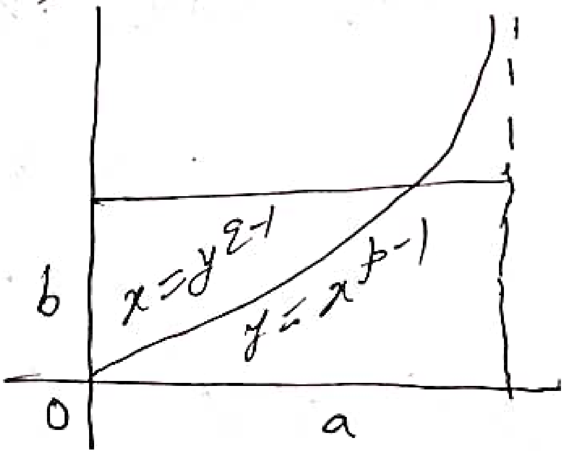
<u>Young's inequality:</u>

Let $1<p<\infty$ and $a,b>0$. Then for $\frac{1}{p}+\frac{1}{q}=1$, $ab \le \frac{a^p}{p}+\frac{b^q}{q}$ —— (*)

proof: let $y=x^{p-1}$, then $x=y^{q-1}$
($\because p-1=\frac{1}{q-1}$ by $\frac{1}{p}+\frac{1}{q}=1$).

Now, from figure, it is clear that

$$ab \le \int_0^a x^{p-1}\,dx+\int_0^b y^{q-1}\,dy$$
$$= \frac{a^p}{p}+\frac{b^q}{q}.$$



Note that equality in (*) holds iff $a^p=b^q$ (or $a=b^{q-1}$).

For this, consider
$$ab = \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p}+\frac{1}{q}=1.$$
Replace $a \to a^{1/p}$, $b \to b^{1/q}$ & $\frac{1}{p} = \alpha$.

(63)

Then, we get
$$a^{\alpha} b^{1-\alpha} = \alpha a + (1-\alpha) b$$
or $t^{\alpha} - \alpha t - (1-\alpha) = 0$ if $t = a/b$.

Let $f(t) = t^{\alpha} - \alpha t - (1-\alpha)$, $t \in [0, \infty)$.

Then $f(1) = 0$, $f'(t) = \alpha(t^{\alpha-1} - 1) = 0$ iff $t=1$.

Since $f'(t) < 0$ if $t > 1$ and $f'(t) > 0$ for $0 < t < 1$.

Hence, f is strictly increasing in $(0,1)$ and strictly decreasing in $(1, \infty)$. Thus, $t=1$ is the ~~ab~~ point of absolute maxi. of f. Therefore $f(t) \leq f(1) = 0$, which another proof of inequality. On the other hand $f(t) = 0$ iff $t = 1$. This completes the proof.

Ex. Let $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$. Write $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. Then $(\mathbb{R}^n, \|\cdot\|_1)$ is a normed linear space (n.l.s). If $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$, then by Cauchy-Schwarz inequality $(\mathbb{R}^n, \|\cdot\|_2)$ is a n.l.s.

For $\|x\|_\infty = \sup_i |x_i|$, $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a normed linear space.

For $1 \le p < \infty$, write $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$. Then $\ell_n^p := (\mathbb{R}^n, \|\cdot\|_p)$ will be a normed linear space.

<u>Space of Sequences:</u>

Let $1 \le p < \infty$, and let ℓ^p denote the space of all sequences that satisfies
$$\sum_{i=1}^{\infty} |x_i|^p < \infty \; ; \; x = (x_1, x_2, \dots x_n, \dots).$$
Then $(\ell^p, \|\cdot\|_p)$ or simply ℓ^p is will be a normed linear space

If $p=\infty$, $\|x\|_\infty = \sup_{1\le i<\infty} |x_i| < \infty$, then $(l^\infty, \|\cdot\|_\infty)$ is a normed linear space. (follows from defn of sup).

For $1\le p<\infty$, showing l^p is a n.l.s. required the following inequalities.

Holder's Inequality:

Let $1\le p\le\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then for $x\in l^p$, and $y\in l^q$, it follows that $x\cdot y (= x_1y_1+\cdots+x_ny_n+\cdots)\in l^1$, and

$$\|x\cdot y\|_1 \le \|x\|_p \|y\|_q \quad \text{---} (*)$$

(where $\frac{1}{\infty}=0$ (adopted)).

When $p=1$, $q=\infty$. In this case $(*)$,

$$\|x\cdot y\|_1 = \sum_{i=1}^{\infty} |x_i y_i| \le \sum |x_i| \|y\|_\infty = \|x\|_1 \|y\|_\infty.$$

Now, let $1<p<\infty$. Then $1<q<\infty$.

Substitute $a=a_j = \frac{|x_j|}{\|x\|_p}$ & $b=b_j=\frac{|y_j|}{\|y\|_q}$ in the Young's Inequality. Then

$$\sum_{j=1}^{n} \frac{|x_j y_j|}{\|x\|_p \|y\|_q} \leq \sum \left(\frac{|x_j|^p}{p\,\|x\|_p^p} + \frac{|y_j|^q}{q\,\|y\|_q^q} \right)$$
$$\leq \left(\frac{\|x\|_p^p}{p\,\|x\|_p^p} + \frac{\|y\|_q^q}{q\,\|y\|_q^q} \right)$$
$$= \frac{1}{p} + \frac{1}{q} = 1.$$

That is, $\sum_{j=1}^{n} |x_j y_j| \leq \|x\|_p \|y\|_q$, $\forall n \geq 1$.

Since LHS is an $\uparrow$ seq^n which is bounded above, hence

$$\|xy\|_1 \leq \|x\|_p \|y\|_q.$$

Notice that if $\|x\|_p = 1 = \|y\|_q$. Then

$$\|xy\|_1 \leq 1,$$

and equality holds iff $|x_j|^p = |y_j|^q$ $\forall j$.

This follows from Young equality. For $ab = \frac{a^p}{p} + \frac{b^q}{q}$, we must have $a^p = b^q$.

<u>Minkowski Inequality</u>:

Let $1 \leq p \leq \infty$. Then for $x, y \in \ell^p$, $x+y \in \ell^p$, and $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

Proof: For $p=1$ or ∞, the proof is trivial.

Let $1<p<\infty$. Then

$$\|x+y\|_p = \left(\sum_{i=1}^{\infty} |x_i+y_i|^p\right)^{1/p}$$

$$\le \left(\sum_{i=1}^{\infty} (|x_i|+|y_i|)^p\right)^{1/p} \quad \text{--- (1)}$$

Since

$$(|x_i|+|y_i|)^p = (|x_i|+|y_i|)(|x_i|+|y_i|)^{p-1},$$

by Hölder's inequality,

$$\sum (|x_i|+|y_i|)^{p-1}|x_i| \le \left(\sum (|x_i|+|y_i|)^{(p-1)q}\right)^{1/q}\left(\sum |x_i|^p\right)^{1/p}.$$

Thus,

$$\sum (|x_i|+|y_i|)^p \le \left(\sum (|x_i|+|y_i|)^p\right)^{1/q}(\|x\|_p+\|y\|_p).$$

That is,

$$\left(\sum (|x_i|+|y_i|)^p\right)^{1-\frac{1}{q}} \le \|x\|_p+\|y\|_p.$$

From (1), we get

$$\|x+y\|_p \le \left(\sum (|x_i|+|y_i|)^p\right)^{1/p} \le \|x\|_p+\|y\|_p$$

<u>Remark:</u> Equality in $\|x+y\|_p \le \|x\|_p+\|y\|_p$ holds iff $x = \frac{\|x\|_p}{\|y\|_p}\, y$.

(Hint: Consider $\|x\|_p = 1 = \|y\|_p$ etc).

Ex. Since we know that any conv sequence is bounded, it follows that the space c of all conv. sequences is a n.l.s under the norm (6.8)

$$\|x\| = \sup |x_i| < \infty ;$$

where $x = (x_1, x_2, \cdots x_n, \cdots)$.

Further the space ~~of~~ c_0 of all seq^ny converging to "zero" is also a n.l.s.

That is, $x = (x_1, x_2, \cdots x_n, \cdots)$,

$$\lim_{n \to \infty} |x_n| = 0.$$

Thus, $(c_0, \|\cdot\|_\infty)$ is a linear subspace of $(c, \|\cdot\|_\infty)$.

Ex. Show that the followings strict inclusions hold:

$$l^1 \subsetneq l^2 \subsetneq c_0 \subsetneq c \subsetneq l^\infty$$

(Hint: $x = (x_n) \in l^1$ then $\lim x_n = 0 \Rightarrow x \in l^\infty$,
$\sum |x_n|^2 \le \sum \|x\|_\infty |x_n| \Rightarrow \|x\|_2^2 \le \|x\|_\infty \|x\|_1$.)

Ex. For $x = (x_1, \cdots x_n) \in \mathbb{R}^n$ (& $\mathbb{C}^n$), show that $\|x\|_\infty \le \|x\|_1 \le \sqrt{n}\,\|x\|_2 \le n\,\|x\|_\infty$. (69)

~~Ctrant~~:

<u>Geometry of spheres in $(\mathbb{R}^n, \|\cdot\|_p)$:</u>

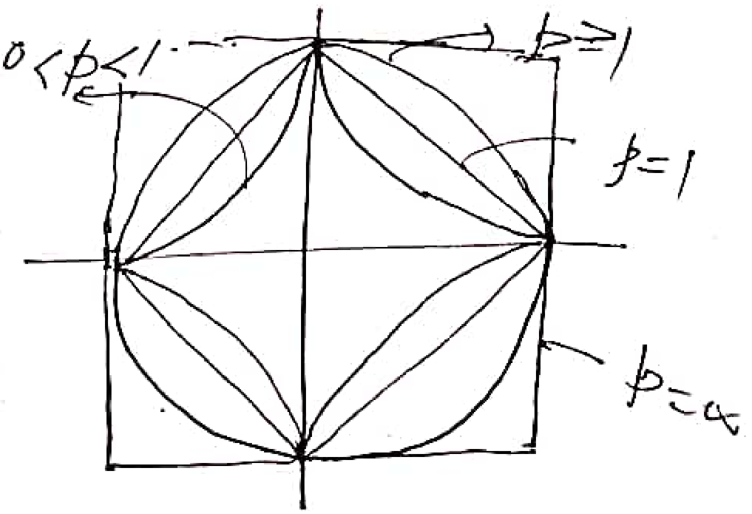
For $0 \le p \le \infty$, and $x \in \mathbb{R}^n$, write

$$\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}.$$

Then $\|\cdot\|_p$ is a norm for $1 \le p < \infty$, and for $0 < p < 1$, $\|x\|_p^p = d_p(0, x)$ is $d_p(x, y) = \|x - y\|_p^p$ is a metric. (we see later)

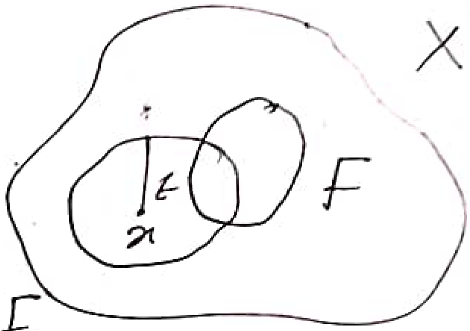
let $S_1^p(0) = \{x : d_p(0, x) = 1\}$.

Then the following figer can be traced for different values of p; $0 < p < \infty$; $p = \infty$.



Closed sets in (X, d):

Def^n: A set $F \subset (X,d)$ is said to be closed if F^c is open:
i.e. $\forall\, x \in F^c = X \setminus F$, $\exists\ \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq F^c$.



On the other hand, if for each $\epsilon > 0$,
$B_\epsilon(x) \cap F \neq \emptyset \Rightarrow x \in F$.

Theorem: Let (X,d) be a metric space and $F \subset X$. Then TFAE:

(i) F is closed set (F^c - open)

(ii) $\forall\, \epsilon > 0$, $B_\epsilon(x) \cap F \neq \emptyset \Rightarrow x \in F$.

(iii) $\forall$ seq^n $(x_n) \in F$ s.t. $x_n \to x \Rightarrow x \in F$.

Proof: (i) $\Rightarrow$ (ii). Suppose F is closed.

Claim $B_\epsilon(x) \cap F \neq \emptyset$ $\forall\, \epsilon > 0 \Rightarrow x \in F$.

Notice that if $x \notin F \Rightarrow x \in F^c$, and

F^c is open $\Rightarrow \exists\ \epsilon_0 > 0$ st

$$B_{\epsilon_0}(x) \subset F^c \Rightarrow B_{\epsilon_0}(x) \cap F = \Phi,$$

which is a contradiction. 71

(ii) $\Rightarrow$ (iii): let $(x_n) \subset F$ & $x_n \to x$.

Then for ~~each~~ $\epsilon > 0$, $x_n \in B_\epsilon(x)\ \forall n \geq n_0$.

$$\Rightarrow x_n \in B_\epsilon(x) \cap F \neq \Phi,\ \forall \epsilon > 0$$

$$\Rightarrow x \in F.$$

(iii) $\Rightarrow$ (i):

Claim F^c is open. let $x \in F^c$.

Then $x \notin F$. By (iii), $\exists\ \epsilon_0 > 0$ st

$$B_{\epsilon_0}(x) \cap F = \Phi \Rightarrow B_{\epsilon_0}(x) \subset F^c.$$

Ex. let $f : (X,d) \to \mathbb{R}$ be function. Then f is continuous at $x \in X$ iff every seq^n $x_n \in X$ with $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Proof: Suppose f is continuous at x ((ϵ,δ)-def^n).

let $x \in X$ & $x_n \in X$ st $x_n \to x$.

Since f is cont at x, for each $\epsilon > 0$, $\exists\ \delta > 0$ s.t. (72)

$$d(y, x) < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Given $x_n \to x$. For $\delta > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t.

$n \geq n_0 \Rightarrow d(x_n, x) < \delta \Rightarrow |f(x_n) - f(x)| < \epsilon$.

[That is, for each $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t.
$n \geq n_0 \Rightarrow |f(x_n) - f(x)| < \epsilon$.]
$\Rightarrow f(x_n) \to f(x)$.]

Conversely, suppose for each seq^n $x_n \in X$ with $x_n \to x \Rightarrow f(x_n) \to f(x)$.

$$d(x_n, x) \to 0 \Rightarrow |f(x_n) - f(x)| \to 0.$$

[That is, for each $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ & $\delta > 0$ s.t.
$n \geq n_0 \Rightarrow d(x_n, x) < \delta \Rightarrow |f(x_n) - f(x)| < \epsilon$]

If f is not cont at x, then $\exists\ \epsilon_0 > 0$ s.t. for each $\delta > 0$, $\exists\ y$ s.t.

$$d(x, y) < \delta \text{ but } |f(x) - f(y)| \geq \epsilon_0.$$

Let $\delta = \frac{1}{n}$, then $\exists\ y_n \in X$ s.t.

$$d(x, y_n) < \frac{1}{n} \text{ but } |f(x) - f(y_n)| \geq \epsilon_0.$$

i.e $y_n \to x$ but $f(y_n) \not\to f(x)$.
is a contradiction

Ex. If $f:(X,d) \to \mathbb{R}$ is continuous, and $f(x_0) \neq 0$ for some $x_0 \in X$, then $\exists\, \delta > 0$ s.t. $f(x) \neq 0 \;\forall\, x \in B_\delta(x)$.
(Hint: take $\epsilon_0 = \frac{1}{2}|f(x_0)| > 0$, $\exists\, \delta > 0$ etc)

Ex. Show that if $f:(X,d) \to \mathbb{R}$ is continuous, then $A = \{x : f(x) > 0\}$ is open (without using the complement should be closed).
(Hint: let $x \in A$, then for $\epsilon = \frac{1}{2}f(x) > 0$, $\exists\, \delta > 0$ st $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$)

Interior in (X,d):

Let $A \subset X$. ~~be a~~ Then interior(A) or Int(A) or A° is the largest open set contained in A:

i.e $A^{\circ} = \bigcup \{ O \overset{\text{open}}{\subset} X : O \subseteq A \}$

$= \bigcup \{ B_{\epsilon}(x) \subset A : \text{for } x \in A,$ and some $\epsilon > 0 \}$.

$=$ union of all open balls ~~in~~ contained in A.

Closure in (X, d):

The closure of set $A \subset X$ is the smallest closed set containing A.

i.e $\bar{A} = \bigcap \{ F \subseteq X : F \text{ is closed } \& \ A \subset F \}$

$= \{ x \in X : \exists x_n \in A \text{ with } x_n \to x \}$

$=$ Collection of limit of all conv. seq^{ns} in A.

(limit need not be in the set A)

Ex. $A = \{ (n, \frac{1}{n}) : n \in \mathbb{N} \}$. Then closure of A in $(\mathbb{R}^2, \mathcal{U})$ is $\bar{A} = A$, and $A^{\circ} = \Phi$. (why?)

Result: Let $A \subset (X, d)$. Then $x \in \bar{A}$ iff $B_\epsilon(x) \cap A \neq \Phi$, $\forall \epsilon > 0$. (75)

Pf: Let $x \in \bar{A}$. If $\exists\, \epsilon_0 > 0$ s.t. $B_{\epsilon_0}(x) \cap A = \Phi$,

then $A \subset (B_{\epsilon_0}(x))^c$ = closed set.

By defn of $\bar{A}$, $\bar{A}$ is the smallest closed set containing A. Hence

$$\bar{A} \subset (B_{\epsilon_0}(x))^c.$$

Since $x \in \bar{A}$, but $x \notin (B_{\epsilon_0}(x))^c$, a contradiction.

Conversely, Suppose $B_\epsilon(x) \cap A \neq \Phi$, $\forall \epsilon > 0$.

Then $B_\epsilon(x) \cap \bar{A} \neq \Phi$, $\forall \epsilon > 0$.

By previous result, $x \in \bar{A}$ ($\because \bar{A}$ is closed).

Result: $x \in \bar{A}$ iff $\exists$ a seqn $x_n \in A$ s.t. $x_n \to x$.

Pf: If $x \in \bar{A}$, then $B_{1/n}(x) \cap A \neq \Phi$, $\forall n \geq 1$

$\Rightarrow \exists\, x_n \in B_{1/n}(x)$ & $x_n \in A$.

$\Rightarrow d(x_n, x) < \frac{1}{n}, \forall n \geq 1$

$\Rightarrow x_n \to x.$

Conversely, if $\exists$ $x_n \in A$ s.t. $x_n \to x$.

Then for $\epsilon > 0$, $d(x_n, x) < \epsilon, \forall n \geq n_0$.

$\Rightarrow x_n \in B_\epsilon(x) \cap A \neq \Phi, \forall \epsilon > 0$

$\Rightarrow x \in \bar{A}.$

(by previous result)

Def: A set $A \subset (X, d)$ is said to be dense if $\bar{A} = X$.

Space of finite Seq^ns:

~~The~~ Space of finite Seq^ns play a vital role as similar to the space of all polynomials:

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$\Rightarrow (a_0, a_1, \cdots a_n) \simeq (a_0, a_1, \cdots a_n, 0, 0, 0)$

Let $c_{00} = \{x = (x_1, \cdots x_n, 0, 0, \cdots): x_i \in F\}$. 77

Then obviously x is a bounded seqn, and

$\|x\|_\infty = \max_{1 \le i \le n} |x_i| < \infty$ defines a norm

on c_{00}.

Notice that the space of all finite seqns c_{00} is dense for all ℓ^p; $1 \le p < \infty$, which we see later. However, the closure of c_{00} is c_0, which is a closed proper subspace of ℓ^∞.

For $x_n = (1, \frac{1}{2}, \cdots \frac{1}{n}, 0, 0, \cdots) \in c_{00}$

& $x = (1, \frac{1}{2}, \frac{1}{n}, \frac{1}{n+1}, \cdots)$,

$\|x - x_n\|_\infty = \sup_{k > n} \frac{1}{k+1} = \frac{1}{n+1} \to 0$,

But $x \notin c_{00}$, hence c_{00} is not a closed subspace of ℓ^∞. ~~However~~ In addition c_{00} is not open in ℓ^∞.

For this, let $\epsilon > 0$ be arbitrarily small. Then for $B_\epsilon(0) \in \ell^\infty$, $(\epsilon/2, \epsilon/2, \cdots) \in B_\epsilon(0)$ but $(\epsilon/2, \epsilon/2, \cdots) \notin c_{00}$. Hence,

$B_\epsilon(0) \not\subseteq C_{00}$, for any $\epsilon > 0$.

For $1 \le p < \infty$, $C_{00} \subsetneq \ell^p$ and C_{00} is neither closed nor open in ℓ^p. For this, let $x_n = \left(\frac{\epsilon^p}{2^{n+1}}\right)^{1/p}$, $1 \le p < \infty$.
and write $x = (x_1, x_2, \cdots)$. Then $x \in B_\epsilon(0) \subset \ell^p$, but $x \notin C_{00}$.

Now, write $X_n = (x_1, \cdots, x_n, 0, \cdots) \in C_{00}$.
Then $\|X - X_n\|_p^p = \sum_{k=n+1} \frac{\epsilon^p}{2^{k+1}} \to 0$, but $X \notin C_{00}$.

Ex. Let M be an open subspace of a n.l.s X. Show that $M = X$.
(Hint: $0 \in M \Rightarrow B_\epsilon(0) \subset M \subset X$, since M is linear $\alpha B_\epsilon(0) \subset M \subset X \Rightarrow B_{\epsilon'}(0) \subset M, \forall \epsilon > 0$.
If $y \in X$, then $y \in B_{\epsilon'}(0) \subset M \subset X$ for some $\epsilon' > 0$).

Notice that for $x = (x_1, x_2, \ldots x_n, \cdots) \in \ell^p$,
$1 \le p < \infty$, $X_n = (x_1, \cdots x_n, 0, \cdots) \in C_{00}$.

And
$$\|X - X_n\|_p^p = \sum_{k=n+1}^{\infty} |x_k|^p \to 0 \quad (\because x \in \ell^p).$$ (79)

Hence $X_n \to X$ in ℓ^p. Thus $\overline{c_{00}} = \ell^p$.

However, c_{00} is not dense in ℓ^∞. but $\overline{c_{00}} = c_0$. For this, let

$x = (x_1, \dots x_n, \dots) \in c_0$. Then

$\lim_{n\to\infty} x_n = 0$. For $\epsilon > 0$, $\exists$ $n_0 \in N$ s.t

$n \geq n_0 \Rightarrow |x_n| < \epsilon/2$. Now, write

$X_n = (x_1, \dots x_n, 0, 0, \dots)$. Then

$X_n \in c_{00}$ & for $n \geq n_0$,

$\|X - X_n\|_\infty = \sup_{i \geq n+1} |x_i| \leq \epsilon/2$.

$\Rightarrow X_n \to X$.

Remark: $\overline{c_{00}} = c_0 \subsetneq \ell^\infty$. That is, c_{00} is not dense in ℓ^∞.

Ex. Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{C}$) be a continuous function. Suppose $\lim_{|x|\to\infty} f(x) = 0$. Then for $\epsilon > 0$, $\exists$ $\delta > 0$ s.t $|f(x)| < \epsilon$ for $|x| > \frac{1}{\delta}$.

Since f is continuous, it follows that f is bounded. Let $\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)| < \infty$.

Then $C_0 = \{ f : \mathbb{R} \xrightarrow{\text{Cont}} \mathbb{R},\ \lim_{|x| \to \infty} |f(x)| = 0 \}$ (80)

is a normed linear space.

Now, for function $f : \mathbb{R} \to \mathbb{R}$, write

$\mathrm{Supp}(f) = \overline{\{ x \in \mathbb{R} : f(x) \neq 0 \}}$, called support of f.

Let $C_c = \{ f : \mathbb{R} \xrightarrow{\text{Cont}} \mathbb{R}$ & $\mathrm{Supp}(f)$ is compact$\}$.

Then $f \in C_c$ is a bounded function, and $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)| = \sup_{x \in \mathrm{Supp}(f)} |f(x)| < \infty$.

Let $K = \mathrm{Supp}(f)$, compact.

Then $(C_c, \|\cdot\|_\infty)$ is a dense subspace of $(C_0, \|\cdot\|_\infty)$.

For this, let $f \in C_0$, then for $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x)| < \epsilon$ for $|x| > \frac{1}{\delta}$.

write $K = \{ x : |x| \leq \frac{1}{\delta} \}$.

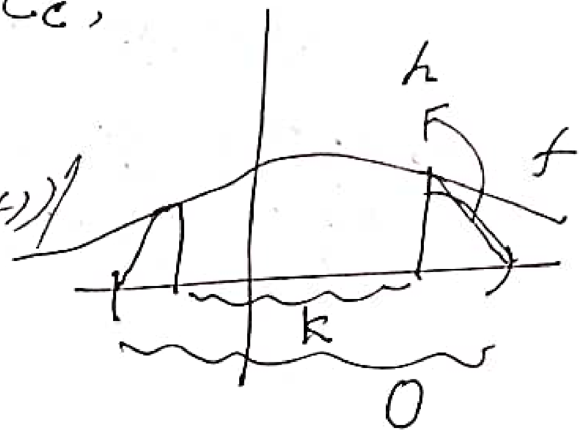
Let O be a bounded open set with $K \subset O$.

Define $g(x) = \dfrac{d(x, O^c)}{d(x, O^c) + d(x, K)}$.

Then g is continuous on $\mathbb{R}$, $0 \le g(x) \le 1$, and $g(x) = 1$ for $x \in K$ & $g(O^c) = \{0\}$.

Let $h = fg$. Then $h \in C_c$, and

$$\|f - h\|_\infty = \|f(1-g)\|_\infty = \sup_{x \in \mathbb{R}} |f(t)(1-g(t))| \le \epsilon.$$



Hence, C_c is dense in C_0.

Note that $d(x, A) = \inf_{y \in A} |x - y|$.

Complete metric spaces:

It has been often seen that there are Cauchy seq^{n}s whose limit need not fall into the space. E.g.: $\frac{1}{n} \in ((0,1), \mathcal{U})$ is C.C. but $\frac{1}{n} \to 0 \notin (0,1)$.

It is always possible to ~~not~~ enlarge the space so that limit of all C.C. can be accommodated. We shall see this later. Known as completion of metric space. However there are many spaces which do accommodate limits of ~~#~~ their ~~C.C.~~ Cauchy seq^ns. (82)

Def^n: A metric space (X, d) is called complete if every C.C. in X has limit in X.

Ex. $(\mathbb{R}, u)$ is complete.

Let $(x_n) \subset \mathbb{R}$ be a C.C. Then it is bounded. And by B-W theorem, $\exists$ subseq^n $x_{n_k} \to x \in \mathbb{R}$. For $\epsilon > 0$, $\exists\, k_0 \in \mathbb{N}$ s.t. $|x_{n_k} - x| < \epsilon$ for all $k \geq k_0$. (1)

But (x_n) is C.C., for $\epsilon > 0$, $\exists\, n_0$

s.t. $|x_n - x_m| < \epsilon$ for all $m, n \geq n_0$. (83)

Let $m \geq n_0$ & $m \geq n_{k_0}$. Then

(2) — $|x_n - x_{n_k}| < \epsilon$ for $n \geq n_0$ & $k \geq k_0$.

From (1) & (2), it follows that

$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < 2\epsilon$

$\therefore$ for $n \geq n_0$ & $n_k \geq n_{k_0}$.

Thus, for $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t.

$n \geq n_0 \Rightarrow |x_n - x| < \epsilon$.

Notice that the above discussion can be used to prove the following result.

<u>Result</u>: Let (x_n) be a C.C in a metric space (X, d). If (x_n) has a conv. subsequence $x_{n_k} \to x$, then $x_n \to x$.

(Proof is similar to above).

Ex. $(\mathbb{R}^n, \|\cdot\|_p)$ is complete for $1 \le p \le \infty$.

Let $1 \le p < \infty$, and $x^k = (x_1^k, \dots x_n^k)$ (84)
be a C.S. in $(\mathbb{R}^n, \|\cdot\|_p)$. Then for
$\epsilon > 0$, $\exists$ $k_0 \in \mathbb{N}$ s.t

$$l, k \ge k_0 \Rightarrow \|x^k - x^l\|_p = \left(\sum_{j=1}^{n} |x_j^l - x_j^k|^p \right)^{1/p} < \epsilon$$

$$\Rightarrow |x_j^l - x_j^k| < \epsilon \quad \forall \; l, k \ge k_0.$$

$$\Rightarrow (x_j^l) \text{ is a C.S in } (\mathbb{R}, 1),$$

Hence, x_j^l is conv., say to x_j. Then
for $\epsilon > 0$, $\exists$ $m_j \in \mathbb{N}$ s.t

$$l \ge m_j \Rightarrow |x_j^l - x_j| < \epsilon / n^{1/p}.$$

Let $m_0 = \max_j \{m_j\}$. Then for $x = (x_1, \dots x_n)$

$$\|x^l - x\|_p < \epsilon \quad \text{for } l \ge m_0.$$

Notice that $p = \infty$ case is similar, we
skip its proof here.

Ex. Let $1 \le p \le \infty$. Then $(l^p, \|\cdot\|_p)$ is complete. (95)

Let $1 \le p < \infty$, and let $x^k = (x_1^k, \ldots x_n^k, \ldots)$ be a C-S in $(l^p, \|\cdot\|_p)$. Then for $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t

$$\forall\, k, l \geq n_0 \Rightarrow \|x^k - x^l\|_p < \epsilon$$

$$\Rightarrow \sum_{j=1}^{n} |x_j^k - x_j^l|^p < \epsilon^p \quad \text{---}(1)$$

Then for each fixed ~~on~~ n, it reduces to $(\mathbb{R}^n, \|\cdot\|_p)$, which we know is complete.

Hence $x_j^k \to x_j$; $j = 1, 2, \ldots n$. Thus, letting $k \to \infty$ in (1), it follows that

$$(2) - \sum_{j=1}^{n} |x_j^l - x_j|^p \le \epsilon^p, \ \forall\ l \geq n_0.$$

But LHS of (2) is ~~decreasing~~ increasing seq^n and bounded above, hence, letting $n \to \infty$, we get

$$\sum_{j=1}^{\infty} |x_j^l - x_j|^p \le \epsilon^p$$

i.e. $\|x^l - x\|_p \le \epsilon$ for $l \geq n_0$, where $x = (x_1, \ldots x_n, \ldots)$.

Notice that
$$\|x\|_p \le \|x - x^{n_0}\|_p + \|x^{n_0}\|_p < \epsilon + \|x^{n_0}\|_p < \infty.$$
$\Rightarrow x \in \ell^p$. (86)

Result: Every closed subset of a complete metric space is complete.

Pf: Let F be a closed subset of a complete metric space (X,d). Then $(x_n) \in F$ to be C.S., it follows that (x_n) is a C.S. in X. Hence $x_n \to x \in X$. But F is closed, it implies that $x \in F$.

In fact, if (X,d) is complete, then F is closed iff F is complete. (Hint: it follows easily)

Ex. Show that c_0 is a proper closed subspace of $(\ell^\infty, \|\cdot\|_\infty)$.

We know that $c_0 \subsetneq \ell^\infty$.

Now, let $x^k = (x_1^k, \ldots x_j^k, \ldots)$ be a seqn in c_0 s.t. $x^k \to x = (x_1, \ldots x_j, \ldots)$.

That is, for $\epsilon > 0$, $\exists$ $K_0 \in \mathbb{N}$ s.t (87)

$\forall$ $k \geq K_0 \Rightarrow \|x^k - x\|_\infty < \epsilon$

(1) $\Rightarrow |x_j^k - x_j| < \epsilon$. for each $j \geq 1$.
& $\forall$ $k \geq K_0$.

Since $x_j^k \in c_0 \Rightarrow \lim_{j\to\infty} x_j^k = 0$, for each k.

for $\epsilon > 0$, $\exists$ $j_0 \in \mathbb{N}$ s.t

(2) $|x_j^k| < \epsilon$ $\forall$ $j \geq j_0$ & $k \geq K_0$.

It follows from (1) & (2) that

$|x_j| < |x_j^{K_0} - x_j| + |x_j^{K_0}| < 2\epsilon$ $\forall$ $j \geq j_0$.

i.e $|x_j| < 2\epsilon$, ~~for $\epsilon > 0$~~ $\Rightarrow \lim_{j\to\infty} |x_j| = 0$
$\forall$ $j \geq j_0$.

Thus, $x_j \to 0$ as $j \to \infty$. Hence, c_0 is a closed subspace of l^∞. Thus, c_0 is complete in its own right.

Ex- The space $(C[a,b], \|\cdot\|_\infty)$ is a Complete n.l.s.

Let (f_n) be a C.S. in $(C[a,b], \|\cdot\|_\infty)$.

Then for $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ st

$\forall m, n \geq n_0 \Rightarrow \|f_n - f_m\|_\infty < \epsilon$

(1) $\Rightarrow |f_n(t) - f_{n_0}(t)| < \epsilon$, $\forall\ n \geq n_0$, $\forall\ t \in [a,b]$.

$\Rightarrow f_n(t)$ is a C.S. in $(\mathbb{R}, d)$

for each fixed $t \in [a,b]$. Hence,

$f_n(t) \to f(t)$.

letting $n \to \infty$ in (1), we get

$|f(t) - f_{n_0}(t)| \leq \epsilon\ \forall\ t \in [a,b]$.

(notice that n_0 is free of choice of t.

Since f_{n_0} is cont. for each fixed t & $\epsilon > 0$, $\exists\ \delta > 0$.

st $|s - t| < \delta \Rightarrow |f_{n_0}(t) - f_{n_0}(s)| < \epsilon$

thus, $|f(s) - f(t)| < |f(s) - f_{n_0}(s)| + |f_{n_0}(s) - f_{n_0}(t)|$
$+ |f_{n_0}(t) - f(t)| < 3\epsilon$

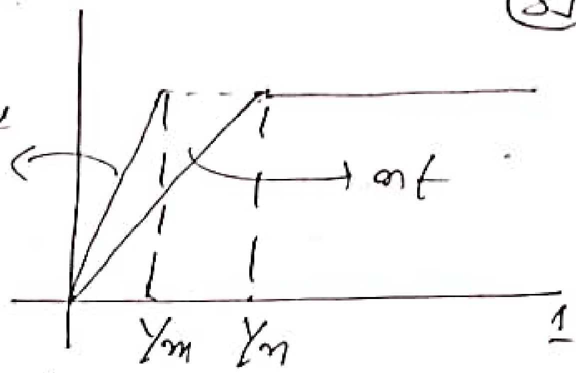
$\Rightarrow f$ is cont on $[a,b]$.

However, $(C[0,1], \|\cdot\|_2)$ is not complete.

for this, we consider the following,

Consider

$$f_n(t) = \begin{cases} nt & 0 \le t < \frac{1}{n}; \\ 1 & \frac{1}{n} \le t \le 1 \end{cases}$$



Then it is easy to see that for $\frac{1}{m} < \frac{1}{n}$,

$$\|f_n - f_m\|_1 = \left(\int_0^{1/m} + \int_{1/m}^{1/n} + \int_{1/n}^{1} \right) |f_n(t) - f_m(t)|\, dt$$

$$= \int_0^{1/m} (mt - nt) + \int_{1/m}^{1/n} (1 - nt) + \int_{1/n}^{1} (1 - 1)$$

$$= \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right) \longrightarrow 0 \text{ as } n < m \to \infty.$$

Thus, (f_n) is a C-C. in $(C[0,1], \|\cdot\|_1)$.

But $f(t) = \lim f_n(t) = \begin{cases} 1 & 0 < t \le 1 \\ 0 & t = 0 \end{cases}$.

Hint: $f_n(0) = 0$ & $f_n(1) = 1 \Rightarrow f(0) = 0$ & $f(1) = 1$.

For $0 < t_0 < 1$, we can find large n s.t. $0 < \frac{1}{n} < t_0 < 1$. Hence $f_n(t_0) = 1$ for large n.

Thus $f(t_0) = 1$).

However, f is not cont, hence $(C[0,1], \|\cdot\|_1)$ is not complete.