Preliminary:

$\mathbb{Q}$ = Set of rationals:
$= \{\frac{p}{q} : p, q \in \mathbb{Z}, (p,q)=1\}$.
$\mathbb{Z}$ - the set of integers.

There are numbers other than rationals.
Consider $(p/q)^2 = 2$, $(p,q)=1$.
$p^2 = 2q^2 \Rightarrow p = 2m$, for some $m \in \mathbb{Z}$. Then $2m^2 = q^2 \Rightarrow q = 2n$
$\Rightarrow (p,q) \geq 2$, which is a contradiction. Thus, $\sqrt{2}$ is not a rational number.
Such nos we say, Irrational & we denote $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ as the set of Irrationals.

* The set of rationals is not complete in the following sense.

Defⁿ: Let $A \subseteq \mathbb{R}$. A number $x_0 \in \mathbb{R}$ is called an upper bound for A if $a \leq x_0$, $\forall\, a \in A$. Similarly, y_0 is called a lower bound for A if $a \geq y_0$, $\forall\, a \in A$.

Defⁿ: An upper bound x_0 of A is called least upper bound (l.u.b.) or supremum for A if z is any upper bound for A, implies $x_0 \leq z$. Similarly, greatest lower bound (g.l.b.) (or infimum) is defined.

Ex: $A = \{1 - \frac{1}{n} : n \in \mathbb{Z}\}$. Show that $\inf A = 0$ & $\sup A = 1$.

* Every non-empty subset of $\mathbb{R}$ having an upper bound has l.u.b (sup), and every non-empty subset of $\mathbb{R}$ having a lower bound has g.l.b (~~sup~~ inf).

This is known as Completeness property of real line $\mathbb{R}$. (For a proof, see Chap I. Rudin PMA)

Ex. If $A (\neq \Phi) \subseteq \mathbb{R}$ is not bounded above, we write $\operatorname{Sup} A = \infty$. Similarly, if $B (\neq \Phi) \subseteq \mathbb{R}$ is not bounded below, we write $\operatorname{Inf} B = -\infty$.

If $A = \Phi$ (empty), then we write $\operatorname{Inf} A = \infty$, and $\operatorname{Sup} A = -\infty$.

(Hint: $\{a\} \subset \{a, b\} \Rightarrow \operatorname{Inf}\{a\} = a \geq \operatorname{Inf}\{a, b\}$.
$\therefore \phi \subset \{a\} \Rightarrow \operatorname{Inf} \Phi \geq a, \ \forall a \in \mathbb{R}$)

Ex. $A \subseteq B \subseteq \mathbb{R} \Rightarrow \operatorname{Inf} A \geq \operatorname{Inf} B$ & $\operatorname{Sup} A \leq \operatorname{Sup} B$.

<u>Archimedean property</u>:

Let $x > 0$ & y be any real no. Then $\exists$ a positive integer n s.t. $nx > y$.

(implies any two real nos can be compared).

Proof: If $\nexists$ any $n \in \mathbb{N}$ s.t. $nx > y$, then $nx \leq y$, $\forall n \in \mathbb{N}$. Thus, y is an upper bound of the set

$$\{nx : n \in \mathbb{N}\}.$$

By completeness property of $\mathbb{R}$,
$\exists \alpha \in \mathbb{R}$ s.t. $\alpha = \sup\{nx : n \in \mathbb{N}\}$.

Note that $\alpha \leq y$.

Since α is the least upper bound,

$\exists n \in \mathbb{N}$ s.t. $\alpha - x < nx < \alpha$
$\Rightarrow \alpha < (n+1)x$, which contradicts the fact that α is a supremum.

Ex. Let $A = \{r \in \mathbb{Q} : r > 0, r \in \mathbb{Q}\}$. Show that $\sup A = \sqrt{2} \, (\notin \mathbb{Q})$.

* If $x, y \in \mathbb{R}$, then $x<y$ or $x>y$.
If $y-x>0$, by comparing $y-x$ with 1 (using Archimedean property or AP), we get $n(y-x)>1$.

$\Rightarrow \exists$ integer m s.t $ny>m>nx$

$\Rightarrow x<\frac{m}{n}<y$.

That is, between any two real, there is a rational. Similarly, $\frac{x}{\sqrt{2}}<\frac{m}{n}<\frac{y}{\sqrt{2}}$.

$\Rightarrow x<\frac{m}{n}\sqrt{2}<y$.

i.e between any two real, there is an irrational.

Ex. Find Inf & Sup of $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$.

Let $A=\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$. Clearly,

$\left\{\frac{1}{1+n} : n \in \mathbb{N}\right\} \subset A$ & $\frac{1}{1+n}$ approaches to 0 for large n. If $\alpha = \inf A > 0$, then by AP, $\exists\, n \in \mathbb{N}$ st. $(n+1)\alpha > 1$.

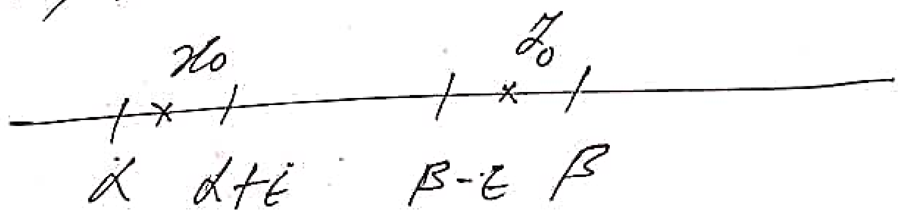
$\Rightarrow \alpha > \frac{1}{n+1}$,

which contradicts that α is $\inf A$.

If $\beta = \sup A < 1$. Then $(m+1)(1-\beta) > 1$ (by AP).

$\Rightarrow \beta < \frac{m}{m+1}$ ✗.

Ex. If $\alpha = \inf A$ & $\sup A = \beta$. Then for $\epsilon > 0$, $\exists\ x_0, y_0 \in A$ s.t. $x_0 < \alpha + \epsilon$ and $y_0 > \beta - \epsilon$.



<u>Proof</u>: Suppose for a given $\epsilon > 0$, $\nexists\ a \in A$ s.t. $a < \alpha + \epsilon$. Then $a \geq \alpha + \epsilon$, $\forall\ a \in A$.

$\Rightarrow a \geq \alpha + \epsilon > \alpha \Rightarrow \alpha + \epsilon$ is a lower bound, which contradicts the fact that α is the greatest lower bound.

Similar argument for β works.

Defn: A function $f: \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a sequence, and we write $\{f(1), f(2), \dots, f(n), \dots\}$ or $\{f_n\}$.

Defn: A seqn $(a_n) \subset \mathbb{R}$ is said to be conv. to l if $\forall \epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$n \geq n_0 \Rightarrow |a_n - l| < \epsilon$

or $a_n \in (l-\epsilon, l+\epsilon)$, $\forall n \geq n_0$.

Ex. $a_n = \frac{1}{n} \to 0$. For this, let $\epsilon > 0$,

$\frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon} > [\frac{1}{\epsilon}]$

$\Rightarrow \forall\, n > [\frac{1}{\epsilon}] + 1 = n_0$,

$|a_n - 0| < \epsilon$.

Result: Every conv. seqn is bounded.

Pf. Let $a_n \to a$. Then for $\epsilon = 1 > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t. $|a_n - a| < 1 \Rightarrow a_n \in (a-1, a+1)$, $n \geq n_0$.

Let $m = \inf[(a-1, a+1) \cup \{a_1, \dots, a_{n_0-1}\}]$

& $M = \sup[(a-1, a+1) \cup \{a_1, \dots, a_{n_0-1}\}]$.

Then $m \leq a_n \leq M$, $\forall\, n \geq 1$.

Result: If x_n is $\uparrow$ & bounded above, then x_n is conv & and $\lim x_n = \sup\limits_{n \geq 1} x_n$.

proof: Let $\alpha = \sup x_n$. Then for $\epsilon > 0$, $\exists$ x_{n_0} s.t. $x_{n_0} > \alpha - \epsilon$.

$$\Rightarrow \alpha + \epsilon > x_n \geq x_{n_0} > \alpha - \epsilon, \ \forall n \geq n_0.$$

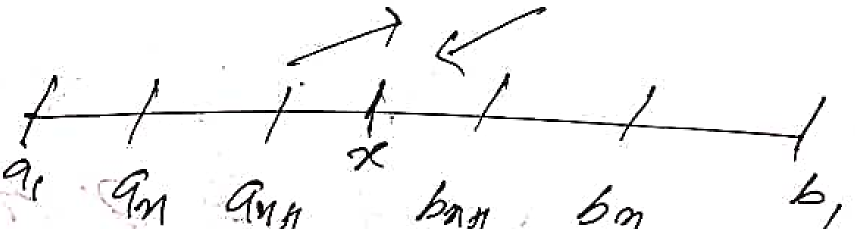
Thus, $x_n \to \alpha = \sup x_n$.

Similarly, if $x_n \downarrow$ & bounded below, then x_n is conv & $\lim x_n = \inf x_n$.

Nested interval theorem:

If $I_n \supset I_{n+1} \supset \cdots$ & $\lim l(I_n) = b_n - a_n = 0$, where $I_n = [a_n, b_n]$. Then $\cap I_n = \{x\}$.

Proof:



It is clear that $a_n \uparrow$ & $< b_1$ and $b_n \downarrow > a_1$. Hence, $\{a_n\}$ & $\{b_n\}$ are convergent. Let $a_n \to a$ & $b_n \to b$.

Then $b-a = \lim (b_n - a_n) = 0 \Rightarrow a = b$.

Notice that $a_n \le a$ & $b_n \geq a$

$\Rightarrow a_n \le a \le b_n \Rightarrow a \in \cap I_n$.

If $x \in \cap I_n$, then $a_n \le x \le b_n \Rightarrow x = a$.

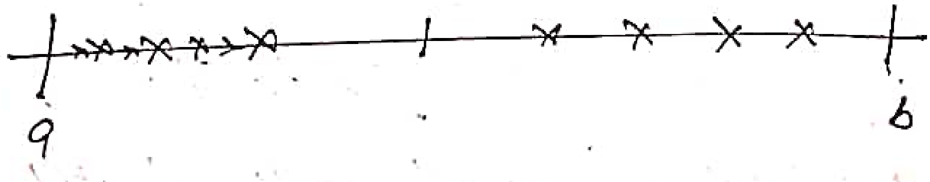
If $\{x_n\}$ is a seq^n & $n_1 < n_2 < \cdots n_k < \cdots$ where $n_k \in N$, then $\{x_{n_k}\}$ is called a subsequence of seq^n $\{x_n\}$.

Ex. $\{\frac{1}{K^2}\}$, $\{\frac{1}{2^K}\}$ are subsequences of $\{\frac{1}{n}\}$ with $n_K = K^2$ & $n_K = 2^K$ respectively.

Bolzano-Weierstrass theorem:

Every bounded sequence in R has a conv. subsequence

proof: Let (x_n) be a bounded seq^n in R. Then $\exists$ $a, b \in R$ s.t. $x_n \in [a, b]$, $\forall n \geq 1$.



Divide $[a,b]$ into two parts, say $[a,b_1]$ & $[b_1,b]$, and write

Suppose $I_1 = [a,b_1]$ contains ∞ly many terms of (x_n). ~~Further~~ Choose $x_{n_1} \in I_1$.

Further, divide $I_1 = I_2 \cup I_2'$ & suppose that I_2 contains ∞ly terms of (x_n).

Choose $x_{n_2} \in I_2$ s.t. $n_1 < n_2$.

Then $x_{n_K} \in I_K$ & $I_K \supset I_{K+1} \supset \cdots$

Then $l(I_K) \to 0$. By NIT,

$\bigcap I_K = \{x\}$.

Thus for each $\epsilon > 0$, $\exists\, K_0 \in \mathbb{N}$ s.t.

$\forall\, K \geq K_0 \Rightarrow I_K \subset (x-\epsilon, x+\epsilon)$ (??)

i.e. $x_{n_K} \in (x-\epsilon, x+\epsilon)$, $\forall\, K \geq K_0$.

$\Rightarrow x_{n_K} \to x$.

Remark:- Suppose $(x_n) \subset [a,b]$,

let $x_{n_K} = \inf_{n \geq K} x_n = \inf\{x_K, x_{K+1}, \ldots\}$

Then $x_{n_k} \uparrow$ & $< b \Rightarrow x_{n_k} \to \sup_{k \geq 1} \left(\inf_{n \geq k} x_n \right)$

i.e. $\lim x_{n_k} = \lim_{k \to \infty} \left(\inf_{n \geq k} x_n \right) = \underline{\lim}\, x_n$ (say).

Similarly, $y_{n_k} = \sup_{n \geq k} x_n = \sup\{x_k, x_{k+1}, \dots\}$.

Then $y_{n_k} \downarrow$ & $> a \Rightarrow y_{n_k} \to \inf_{k \geq 1} \left(\sup_{n \geq k} x_n \right)$.

i.e. $\lim y_{n_k} = \lim_{k \to \infty} \left(\sup_{n \geq k} x_n \right) = \overline{\lim}\, x_n$ (say).

Notice that ~~x_{n_k}~~ subsequences (x_{n_k}) & (y_{n_k}) need <u>not</u> be subsequences of (x_n).

Also, $\inf_{n \geq 1} x_n \leq x_{n_k} \leq y_{n_k} \leq \sup_{n \geq 1} x_n$.

Thus, limit of seq^n (x_{n_k}) can be thought as ~~limit~~ lower limit of (x_n) and similarly limit of (y_{n_k}) can be as upper limit of (x_n).

Since both (x_{n_k}) & (y_{n_k}) are conv,

it follows that $\lim x_{n_k} \leq \lim y_{n_k}$.

That is, $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$.

Example, $x_n = (-1)^n$, then $\underline{\lim}\, x_n = -1 < 1 = \overline{\lim}\, x_n$.

Ex. If $x_n \to x$, then show that $\underline{\lim}\, x_n \geq \overline{\lim}\, x_n$.
Thus, deduce that a bounded seq^n (x_n) is conv. iff $\underline{\lim}\, x_n = \overline{\lim}\, x_n$.

Ex. If $X_n = (x_n, y_n) \in \mathbb{R}^2$ is a bounded seq^n, then $\sqrt{x_n^2 + y_n^2} \leq M$, $\forall\, n \geq 1$.

$\Rightarrow |x_n| \leq M$ & $|y_n| \leq M$, $\forall\, n \geq 1$.

By B-W-T, $\exists$ x_{n_k} s.t. $x_{n_k} \to x \in \mathbb{R}$.
Now, y_{n_k} is also a bounded seq^n, hence by B-W-T, $\exists$ $y_{n_{k_\ell}} \to y$.

Thus, $(x_{n_{k_\ell}}, y_{n_{k_\ell}}) \to (x, y) \in \mathbb{R}^2$.

<u>Remark</u>: Similar argument can be produced for seq^n in $\mathbb{R}^n$.

(13)

Defn: A set $A \subseteq \mathbb{R}$ is said to be open if every pt $x \in A$ encloses an open interval $I_x \subset O$.
i.e. for each $x \in O$, $\exists\ \epsilon > 0$ s.t.
$(x-\epsilon, x+\epsilon) \subset O$.
Thus, a countable union of open intervals is an open set.
On the other hand, any open set in $\mathbb{R}$ can be written as countable union of disjoint open intervals.

<u>Theorem</u>: Let O be an open set in $\mathbb{R}$, then $\exists$ disjoint family of countably many open intervals I_n s.t.

$$O = \bigcup_{n=1}^{\infty} I_n.$$

Proof: Since O is open, for $x \in O$, $\exists$ an open interval ~~x $(a,b) \in O$ s.t~~ (a,b) s.t $x \in (a,b) \subset O$.
Now, we extract the largest open interval containing x and contained in O.

Let $a_x = \inf\{a : (a,x] \subset O\}$, (14)

and $b_x = \sup\{b : [x,b) \subset O\}$.

Then $I_x = (a_x, b_y)$ will be the largest open interval containing x & contained in O.

Note that $I_x = (a_x, b_y) \subset O$. For this, let

$a_x < y < b_y$, then $a_x < y + \epsilon$ for small $\epsilon > 0$.

$\Rightarrow$ $a_x + \epsilon < y$. But by def^n of infimum, $\exists$ $a < a_x + \epsilon$ & $(a, x] \subset O$

$\Rightarrow$ $(a_x + \epsilon, x] \subset O$.

Similarly, $[x, b_x - \epsilon) \subset O$

$\Rightarrow$ $(a_x + \epsilon, b_x - \epsilon) \subset O$, $\forall$ small $\epsilon > 0$.

$\Rightarrow$ $(a_x, b_x) \subset O$.

Now, if $x, y \in O$, & $x \neq y$, then either

$I_x \cap I_y = \Phi$ or $I_x = I_y$.

If $I_x \cap I_y \neq \Phi$, then $I_x \cup I_y$ is an open interval containing x & y.

Therefore, by maximality of I_x for x & I_y for y, it follows that

$$I_x \cup I_y \subseteq I_x \Rightarrow I_y \subseteq I_x$$

Since $y \in I_x \Rightarrow I_y = I_x$ ($\because I_y$ is maximal)

Now, $O = \bigcup_{x \in O} I_x$. Since I_x & I_y are disjoint ($\forall x \neq y$), we can assign distinct rational to each of them. That is, choose $r_x \in I_x$ & $r_y \in I_y$. Then $r_x \neq r_y$.

Thus. $\{I_x : x \in O\} \xrightarrow{1-1} \mathbb{Q}$ - set of rationals

via $I_x \to r_x$.

Hence $O = \bigcup_{i=1}^{\infty} I_{r_i}$ ——— (1)

The rep^n (1) is unique.

Let $O = \bigcup_{n=1}^{\infty} I_n = \bigcup_{m=1}^{\infty} J_m$.

Then $I_n = I_n \cap O = \bigcup_{m=1}^{\infty} (I_n \cap J_m)$.

Since $\{I_n \cap J_m : m \in \mathbb{N}\}$ is a disjoint family & I_n is an open interval, 16

$I_n \subseteq I_n \cap J_{m_0}$ for some m_0.

But then $I_n \subseteq J_{m_0}$ & given I_n is maximal $\Rightarrow I_n = J_{m_0}$.

Thus, the rep^n (1.5) is unique upto change in order of union.

<u>Closed set:</u>

A set $A \subseteq \mathbb{R}$ is said to be closed if for each sequence $x_n \in F$ with $x_n \to x$, implies $x \in F$.

Ep: A set $F \subseteq \mathbb{R}$ is closed iff F^c is open.

Proof: Let F be a closed set. Suppose F^c is not open. Then for some $x \in F^c$, $\nexists\ \epsilon > 0$ s.t $(x-\epsilon, x+\epsilon) \subset F^c$

Take $\epsilon = \frac{1}{n}$, then $x_n \in (x-\frac{1}{n}, x+\frac{1}{n})$ & $x_n \in F$. Thus, $x_n \to x$ & F is closed, implies $x \in F$, which is a contradiction. (17)
Hence F^c is open.

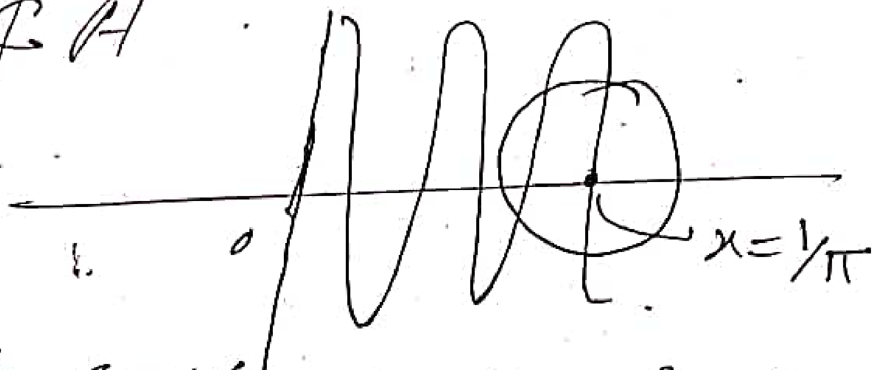
Conversely, suppose F^c is open. Let $x_n \in F$, & $x_n \to x$. Claim $x \in F$.
If $x \notin F$, then $x \in F^c$, which is open.
Then $\exists\ r > 0$ s.t. $(x-r, x+r) \subset F^c$.
Since, $x_n \to x$, $\exists\ n_0 \in \mathbb{N}$ st
$n \geq n_0 \Rightarrow x_n \in (x-r, x+r) \subset F^c$,
which is absurd. Thus $x \in F$.

Notice that we can define open & closed sets in $\mathbb{R}^n$ in a similar way.

Ex. The set $A = \{(x,y) : y = \sin\frac{1}{x},\ x \neq 0\}$ is neither open nor closed in $\mathbb{R}^2$ (in the usual metric).
Let $x_n = \frac{1}{n\pi}$; $n \in \mathbb{N}$, then $(x_n, y_n) = (\frac{1}{n\pi}, 0) \in A$

But $\lim (x_n, y_n) = (0,0) \notin A$.

$B_{1/n}(\frac{1}{\pi}, 0) \not\subset A$



($\because$ A is the graph of the function $f(x) = \sin\frac{1}{x}$, $x \neq 0$)

Interior of a Set:

Let $A \subseteq \mathbb{R}$, then $\exists$ open set $O \subseteq \mathbb{R}$ s.t. $A \subset O = \bigcup_{n=1}^{\infty} I_n$, $I_n = (a_n, b_n)$.

Let us collect all open intervals which are contained in A.

Interior of A (or A^{0}) = union of all open intervals contained in A.

i.e. interior of A is the largest

open set A^0 contained in A.
(That is, O is open & $O \subset A \Rightarrow O \subseteq A^0$.) (19)

Ex. $N^0 = \Phi = Q^0 = (\mathbb{R} \setminus Q)^0$, and
$\{(x, y) : y = \sin\frac{1}{x}, x \neq 0\}^0 = \Phi$.

<u>Closure of a set:</u>

Let $A \subseteq \mathbb{R}$, and $x_n \in A$ s.t. $x_n \to x$.
closure of A (or $\bar{A}$) is the collection of x which is limit of a seq^n in $x_n \in A$.
That is, closure of a set A is the smallest set $\bar{A}$ that contains A. That is, if B is closed & $A \subseteq B \Rightarrow \bar{A} \subseteq B$.

Ex. Show that closure of $A = \{(x, \sin\frac{1}{x}) : x \neq 0\}$ is the set $A \cup (\{0\} \times [-1, 1])$.
Notice that $A \cup (\{0\} \times [-1, 1])$ is a closed set containing A, hence $\bar{A} \subseteq A \cup (\{0\} \times [-1, 1])$.

Here $\left(\frac{1}{n\pi}, 0\right) \to (0,0)$ & $\left(\frac{1}{\pm(2n+1)\pi/2}, \pm 1\right) \to (0, \pm 1)$.

Hence $(0,0), (0, \pm 1) \in \bar{A}$.

Next is to for $y \in (-1,1) \setminus \{0\}$, find seqn $x_n \in \mathbb{R} \setminus \{0\}$ s.t. $\left(x_n, \sin\frac{1}{x_n}\right) \to (0, y)$.

i.e. $x_n \to 0$ & $\sin\frac{1}{x_n} \to y$ etc.

Defn: A closed & bounded subset of $\mathbb{R}^n$ is called compact of $\mathbb{R}^n$. (21)

Ex: The set $\{(x,y): y = \sin\frac{1}{x}, x \neq 0\}$ is closed but not bounded.

Note that if $K \subseteq \mathbb{R}$, $\exists$ open set $O \subset \mathbb{R}$ s.t. $K \subset O = \bigcup_{n=1}^{\infty} I_n$ (open cover)

using B-W thm, it can be deduced that the set $K \subset \mathbb{R}$ is compact iff every open cover of K reduce to finite subcover i.e.

$$K \subset \bigcup_{n=1}^{l} I_n.$$

Similar arguments hold for $K \overset{cpt.}{\subset} \mathbb{R}^n$.

Ex. A subset $F \subseteq \mathbb{R}$ is closed iff $\forall \epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap F \neq \Phi \Rightarrow x \in F$.

Suppose F is closed and $\forall\, \epsilon > 0$ $(x-\epsilon, x+\epsilon) \cap F \neq \Phi$. Then for $\epsilon = \frac{1}{n}$, $\exists\, x_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap F$

$\Rightarrow |x_n - x| < \frac{1}{n}$, $\forall\, n \geq 1$

$\Rightarrow x_n \to x$ & F is closed

$\Rightarrow x \in F$.

Conversely, let $\forall\, \epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap F \neq \Phi$

$\Rightarrow x \in F$.

Claim F is closed. Let $x_n \in F$ & $x_n \to x$. Then for $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$

$n \geq n_0 \Rightarrow x_n \in (x-\epsilon, x+\epsilon) \cap F \neq \Phi$

$\Rightarrow x \in F$.

~~Defn: Let A ⊆~~

Dense Set:

Let $A \subseteq \mathbb{R}$, and $x_n \in A$ s.t. $x_n \to x$.

Then $\bar{A} = \{x \in \mathbb{R} : \exists\, x_n \in A \text{ with } x_n \to x\}$

If $\overline{A} = \mathbb{R}$, then A is called dense in $\mathbb{R}$.

Ex: Let $x \in \mathbb{R}$, then

$$x = x_0 + \frac{x_1}{10} + \cdots + \frac{x_n}{10^n} + \cdots \infty \quad \text{—} (*),$$

where $x_i \in \{0, 1, 2, \ldots, 9\}$

Let $S_n = x_1 + \cdots + \frac{x_n}{10^n} \in \mathbb{Q}$. Then

$S_n \to x$. Thus $\overline{\mathbb{Q}} = \mathbb{R}$.

Also, $x_n = x + \frac{1}{(1+n^3)^{1/3}} \in \mathbb{R} \setminus \mathbb{Q}$ (??)

& $x_n \to x$. Thus $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$.

Note that rep.^{n} $(*)$ is not unique, e.g.

$0.5 = 0.4999\ldots$

<u>Theorem</u>: Let $p \in \mathbb{Z}$, $p \geq 2$ and $0 \leq x \leq 1$.

Then $\exists$ a seq^n of integers (a_n) with $0 \leq a_n \leq p-1$ s.t.

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}.$$

Proof: Choose a_1 be the largest integer s.t. $a_1/p < x$ ($\exists$ Archimedean property) ($\because \frac{1}{p}, x$, AP).

Since $0 < x \leq 1 \Rightarrow a_1 < p$. Given a_1 is an integer, $a_1 \leq p-1$. Also, a_1 is the largest, we must have

$$\frac{a_1}{p} < x \leq \frac{(a_1+1)}{p}.$$

Next, choose a_2 s.t.

$$\frac{a_1}{p} + \frac{a_2}{p^2} < x \quad \left(\because \left(\frac{1}{p}, px - a_1\right)\right)$$

$\Rightarrow 0 \leq a_2 \leq p-1$ and $\left[\frac{a_2}{p} < p - a_1 < 1, \because a_1 \text{ is largest}\right]$

$$\frac{a_1}{p} + \frac{a_2}{p^2} < x \leq \frac{a_1}{p} + \frac{a_2+1}{p^2}$$

By induction, $\frac{a_1}{p} + \cdots + \frac{a_n}{p^n} < x \leq \frac{a_1}{p} + \cdots + \frac{a_n+1}{p^n}$

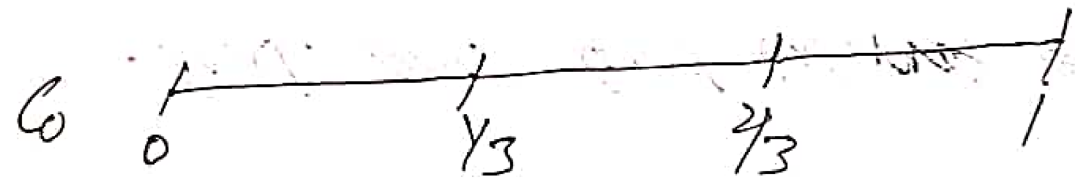
$\Rightarrow x = \sum_{n \geq 1}^{\infty} \frac{a_n}{p^n}$ (p-adic) decimal exp.

Ex. Show that $\{\frac{k}{2^n} : k = 0,1,2,\dots,2^n;\ n = 1,2,\dots\}$ is dense in $[0,1]$.
(Hint: use binary expansion)

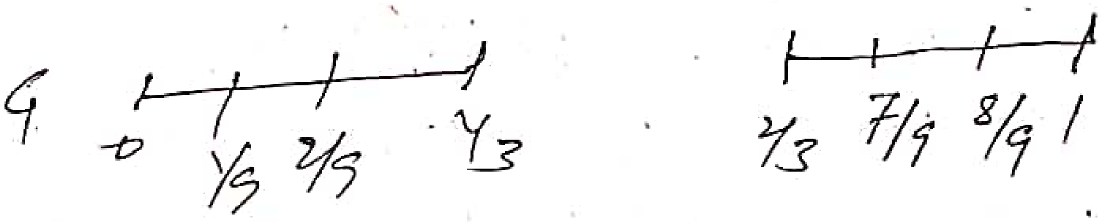
Cantor Set:

Cantor set is an uncountable set in $[0,1]$ having zero length with many peculiar properties answering some of the difficult questions related to topology of real line.

Let $C_0 = [0,1]$.

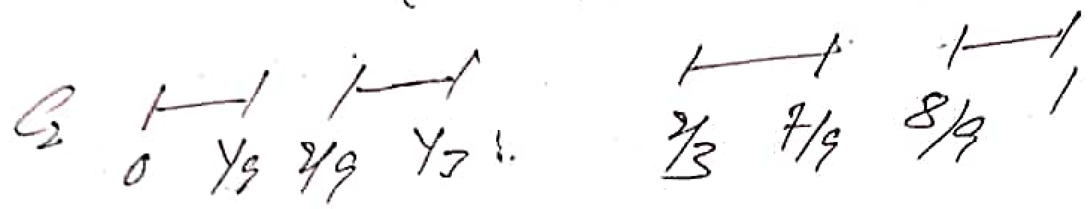


Delete middle one third open interval $J_1 = (\frac{1}{3}, \frac{2}{3})$ from C_0. Then



Delete one-third open interval from each section of C_1, and let

$J_2 = (1/9, 2/9) \cup (7/9, 8/9)$.



Thus, $C_0 = [0,1]$, one closed interval of length $= 1$

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, two ~~clos~~ disjoint closed intervals each of length $1/3$.

$C_2 = [0, \frac{1}{9}] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ has four disjoint closed intervals, each of length $1/9$.

By induction, we can construct C_n, ~~with~~ having 2^n disjoint closed intervals each of length 3^{-n}.

(i) C_n is a sequence of closed & bdd intervals, hence by nested intervals theorem $\bigcap C_n \neq \Phi$
(Hint: use NIT for each chain in the construction of C_n).

(ii) let $C = \bigcap_{n=0}^{\infty} C_n$, then C contains all the end pts of the deleted open intervals.

(iii) $C = [0,1] \setminus J_1 \cup J_2 \cdots J_n \cdots$
$= [0,1] \setminus \bigcup_{n=1}^{\infty} J_n$.

(iv) Since $C \subset C_n$, $\forall\, n \geq 0$,
$l(C) \leq l(C_n) = 2^n \cdot \frac{1}{3^n} \to 0$.
Thus, the total length of $C = 0$.
This shows that the set C is small.
On the other hand, we shall see that C is uncountable.

(v) The Cantor ternary set C (later we just say Cantor set) is nowhere dense.

i.e. $(\overline{C})^{\circ} = C^{\circ} = \Phi$. If not, then (28)

for $x \in C^{\circ} \Rightarrow \exists\ \epsilon > 0$ s.t. $(x-\epsilon, x+\epsilon) \subseteq C^{\circ} \subset C$

$\Rightarrow l(x-\epsilon, x+\epsilon) \leq l(C) = 0$

i.e. $2\epsilon \leq 0$ X.

Hence C is nowhere dense.

(vi) C is totally disconnected (i.e connected sets in C are singletons only)
(we shall prove it later!)

(vii) Every pt of C is a limit pt of C itself (i.e C is a perfect set).

Let $x \in C = \bigcap C_n \Rightarrow x \in C_n, \forall n \geq 0$.

Then x must belong to one of the closed intervals that constitute to C_n.

That is, $x \in [x_n, y_n]$ with $y_n - x_n = \frac{1}{3^n}$

$\Rightarrow |x_n - x| \leq |y_n - x_n| \leq \frac{1}{3^n} \to 0$.

Note that x_n & y_n are end pts of the deleted open intervals J_n's. Hence, $x_n, y_n \in C$. Thus, if E denotes the set of all end pts, then $\bar{E} = C$. Since E is countable (being subset of rationals), C is separable (we define later).

(viii) Representation of Cantor's set:

Consider the end pt $\frac{1}{3} \in C$. We can write $\frac{1}{3} = \frac{0}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots = (0.022\cdots)_3$

Similarly, $\frac{2}{3} = (0.2)_3$. Inductively, it can be shown that any end pt $x \in E$ can be expressed as

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots, \quad a_i \in \{0, 2\}.$$

Since each $x \in [0,1]$ has ~~binary~~ ternary rep^n, consider the set

$$F = \{x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \; a_i \in \{0,1,2\}\} \setminus E.$$

If $x \in F$, then x is not an end pt,
and $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots$, $a_i \in \{0,1,2\}$.

Notice that $a_1 = 1$ iff $x \in (\frac{1}{3}, \frac{2}{3})$ iff $x \notin C_1$

next,

$a_1 \neq 1$, $a_2 = 1$ iff $x \in (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$

iff $x \notin C_2$.

Thus, $a_{i_0} = 1$ for some i_0 iff $x \notin C_{i_0}$

Now, let $x \in C = \bigcap C_n$ & $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$. Suppose some of $a_i = 1$, then $x \notin C_i \Rightarrow x \notin C$.
$\Rightarrow$ all the $a_i \in \{0,2\}$.

That is, $C \subseteq \{x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i},\ a_i \in \{0,2\}\}$. —(*)

On the other hand, let $x \notin C$, then $x \notin C_{i_0}$ for some i_0. This implies $a_{i_0} = 1$.

That is, $x \notin$ RHS of (*).

Thus, $C = \{x \in [0,1] : x = \sum \frac{a_i}{3^i};\ a_i = 0, 2\}$.

This implies Cantor set loses only one decimal index from $\{0,1,2\}$. Can it though some light about non-countability of Cantor set? (31)

(ix) Representation is unique:

For every $x \in C$, $\exists !$ seqn (a_n) from $\{0,2\}$ such that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \quad \text{——— (1)}$$

Suppose $x = \sum_{i=1}^{\infty} \frac{b_i}{3^i}$; $b_i \in \{0,2\}$. ——— (2)

Then claim $a_i = b_i, \forall i$.

If not, let i_0 be the smallest integer st $a_{i_0} \neq b_{i_0}$. Then

$$a_i = b_i \;;\; i = 1, 2, \ldots i_0 - 1.$$

Now, w.l.g. we can take $i_0 = 1$.

That is, $a_1 \neq b_1 \Rightarrow a_1 = 0$ & $b_1 = 2$ (or otherwise

From (1), $x \in [0, \frac{1}{3}]$ and from (2), $x \in [2/3, 1]$, which is absurd.

Exercise. Conclude without assuming $b_0 = 1$.

Cantor set is uncountable:

Define $f : C \to [0,1] = \{ x = \sum_{i=1}^{\infty} \frac{b_i}{2^i} : b_i \in \{0,1\} \}$ by $f(x) = f\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) = \sum_{i=1}^{\infty} \frac{(a_i/2)}{2^i}$, then $b_i = a_i/2 \in \{0,1\}$ & $f(x) \in [0,1]$.

Since each $x \in C$ has a unique rep^n, the map f is well defined.

f is not one-one: binary rep^n is not unique

$f(\frac{1}{3}) = f((0.022\ldots)_3) = (0.011\ldots)_2 = (0.1)_2 = \frac{1}{2}$.

& $f(2/3) = f((0.2)_3) = (0.1)_2 = \frac{1}{2}$.

$\Rightarrow f(\frac{1}{3}) = f(2/3)$.

Ex. Show that $f(x) = f(y)$ iff x, y are end pts of one of the deleted open interval.

<u>f is an onto map:</u>

Here $f: C \longrightarrow [0,1] \ni y$.

s.t. $f(x) = y = \sum_{i=1}^{\infty} a_i 2^{-i}$, let

$x = \sum \frac{2a_i}{3^i}$, then $f(x) = y$ holds

Hence C is an uncountable set.

<u>f is monotone increasing:</u>

Let $x, y \in C$ & $x < y$. Since ternary rep^n of C is unique, $\exists$ the least positive integer $n \in \mathbb{N}$ s.t.

$a_n < b_n$. Hence $a_i = b_i$; $i = 1, 2, .., n-1$.
Thus, while comparing $f(x)$ & $f(y)$, we can ignore the 1st $n-1$ terms. (34)
Therefore, wlog, we can assume $n = 1$.

That is, $a_1 < b_1 \Rightarrow a_1 = 0$, $b_1 = 2$.

$$\left.\begin{array}{l} \therefore f(x) \le \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1}{2} \\ \& \ f(y) = \frac{1}{2} + \frac{b_2/2}{2^2} + \frac{b_3/2}{2^3} + \cdots \ge \frac{1}{2} \end{array}\right\} \Rightarrow f(x) \le f(y).$$

Notice that $f(1/3) = f(2/3) = \frac{1}{2}$. Hence,
~~by keeping f constant on each deleted~~
we can extend f to $[0,1]$ by keeping it constant on the deleted intervals.

Thus, $\tilde{f} : [0,1] \longrightarrow [0,1]$ is defined by $\tilde{f}|_C = f$ & $\tilde{f}([0,1] \setminus C) = \{\alpha_i\}$,
where α_i is the common value of f at the end pt of deleted interval.

Thus, $\tilde{f} : [0,1] \to [0,1]$ is a monotone increasing onto function, Hence $\tilde{f}$ is continuous (why?). (we see it later).

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Now, define $g : [0,1] \to [0,2]$ by

$$g(x) = \tilde{f}(x) + x.$$

Then g is a strictly monotone increasing and onto function.

If $x < y$, $\therefore g(x) = \tilde{f}(x) + x \leq \tilde{f}(y) + x < \tilde{f}(y) + y$

ie $g(x) < g(y)$.

Hence, $g(0) = 0$ & $g(1) = 2$.

($\because g(1) = f(1) + 1 = f\left(\sum \frac{2}{3^i}\right) + 1 = 2$).

Since g is cont on $[0,1]$, by IVT,

$$g([0,1]) = [0,2].$$

Ex. Show that g^{-1} is monotone ~~continuous~~ and ~~monotone~~ continuous.

Limit and Continuity:

Let f be a real valued function, which is defined in an open nbd of a pt a, and may not be necessarily at a.

A number L is called left limit of at a if for each $\epsilon > 0$, $\exists\ \delta > 0$ s.t

for $x \in (a-\delta, a) \Rightarrow |f(x) - L| < \epsilon$

or simply, we write $L = \lim_{x \to a^-} f(x) = f(x-)$.

Similarly, right limit of f for $\epsilon > 0$, $\exists\ \delta > 0$

s.t. $x \in (a, a+\delta) \Rightarrow |f(x) - M| < \epsilon$

or $M = \lim_{x \to a^+} f(x) = f(x+)$.

Moreover, if f is defined in nbd of a and a ~~also~~, then f is said to be continuous at a if $\forall\ \epsilon > 0$, $\exists\ \delta > 0$

such that

$x \in (a-\delta, a+\delta) \Rightarrow |f(x) - f(a)| < \epsilon$

or $f(x^-) = f(x) = f(x^+)$.

In case, when $f(x^-)$ & $f(x^+)$ exists and unequal, we say f has jump dis-continuity at a.

<u>Monotone function:</u>

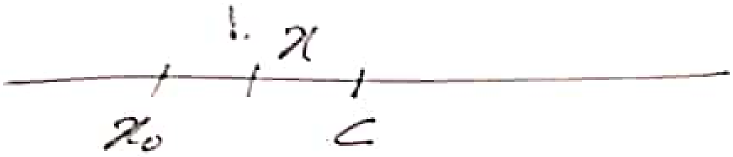
we shall see that a monotone function is continuous except on a countable set and, and it is also known that such functions are very closed to differentiable function. we skip here the later one property.

<u>Theorem:</u> let $f: (a,b) \rightarrow \mathbb{R}$ be a monotone function, then for $c \in (a,b)$; $f(c+)$ & $f(c^-)$ both ~~exists~~ exist.

proof: let f be an increasing function.

Then

$$\left.\begin{aligned} f(c-) &= \sup_{a<x<c} f(x) = L \le f(c) \\ \& \quad f(c+) &= \inf_{c<x<b} f(x) = M \ge f(c) \end{aligned}\right\} (*)$$



For $\epsilon>0$, $\exists\, x_0 \in (a,c)$ s.t. $f(x_0) > L-\epsilon$.
Let $\delta = c - x_0$, then for $x \in (c-\delta, c)$,

$$L+\epsilon > f(x) \ge f(x_0) > L-\epsilon \quad (\because f \uparrow)$$

i.e. for $x \in (c-\delta, c) \Rightarrow |f(x)-L| < \epsilon$.

Hence $f(c^-) = \sup_{a<x<c} f(x) = L$.

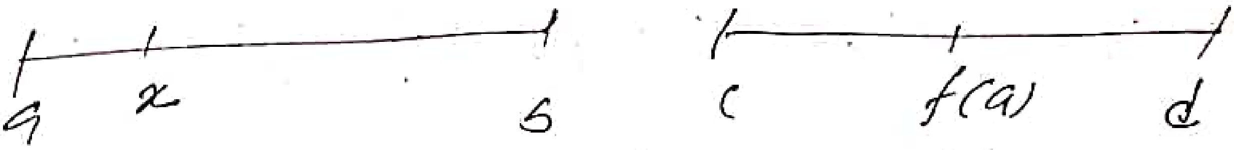
Similarly, $f(c+) = \inf_{c<x<b} f(x) = M$.

Notice from (*) that if $c, d \in (a,b)$ & $c<d$, then $f(c+) \le f(d^-)$.

Hence either $(f(c^-), f(c^+))$ and $(f(d^-), f(d^+))$ both coincide or disjoint. Choose rational r_c & r_d from the above intervals. Then these intervals have one-one correspondence with the set of rationals. Hence, the set of discontinuity of a monotone function is almost countable.

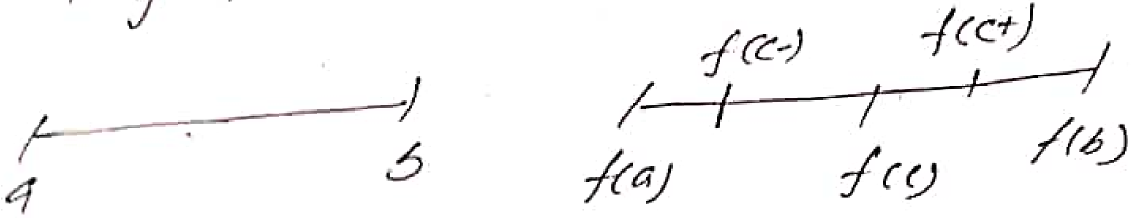
Ex. If $f : [a,b] \to [c,d]$ is monotone and onto, then f is continuous.

Let f be an $\uparrow$ function. Then $f(a) = c$ and $f(b) = d$.



If $f(a) > c$, then for $y \in [c, f(a))$, $\nexists$ any $x \in [a,b]$ s.t. $f(x) = y$. If so, then $f(x) = y < f(a) \Rightarrow x < a$ ($\because f$ is $\uparrow$).

Further, if possible, let $f(c-) < f(c)$.



Then $y \in (f(c-), f(c))$ has no pre-image.

On contrary, if $\exists$ $x_0 \in (a, c)$ s.t. $f(x_0) = y$.

Then $L = \sup\limits_{a<x<c} f(x) = f(c^-) < y = f(x_0) < f(c)$ ~~$< f(x_0) < f(c)$~~,

which contradicts the fact that L is sup on (a, c). Thus, $f(c-) = f(c) = f(c^+)$.

Hence, f is continuous.

Ex. If $f : (a,b) \longrightarrow (c,d)$ is monotone and onto, then f is continuous.

(Proof is similar to the above case).

~~Obser~~ Observe that if f is monotone onto then f need not be one-one.

For example, Cantor function

$f : [0,1] \to [0,1]$ is monotone & onto but not one-one. (41)

However, if $f : (a,b) \to (c,d)$ is strictly monotone and onto, then $f^{-1} : (c,d) \to (a,b)$ is continuous, because, in this case, f^{-1} is also strictly monotone. For this, if $f \uparrow s$, then for $y_1 < y_2 \Rightarrow f^{-1}(y_1) < f^{-1}(y_2)$.

If not, then for $y_1 = f(x_1)$ & $y_2 = f(x_2)$, it follows that $x_1 \geq x_2$ ($\because f^{-1}(y_1) \geq f^{-1}(y_2)$), but then $f(x_1) = y_1 < y_2 = f(x_2)$ is ~~absurd~~ a contradiction to the fact that f is strictly increasing.

Notice that $f : [c,d] \xrightarrow{\text{onto}} (e,f)$ need not be continuous if f is monotone, else $f([a,b])$ is compact.

Finally, if $f : \mathbb{R} \to \mathbb{R}$ is one-one & onto, then f & f^{-1} both are continuous. (42)

Ex. If I be an interval in $\mathbb{R}$, and $f : I \to \mathbb{R}$ be a monotone function, then $E_\alpha = \{x \in I : f(x) > \alpha\} = I'$ or Φ, where I' is an interval.

$= f^{-1}\{(\alpha, \infty)\} =$ Interval

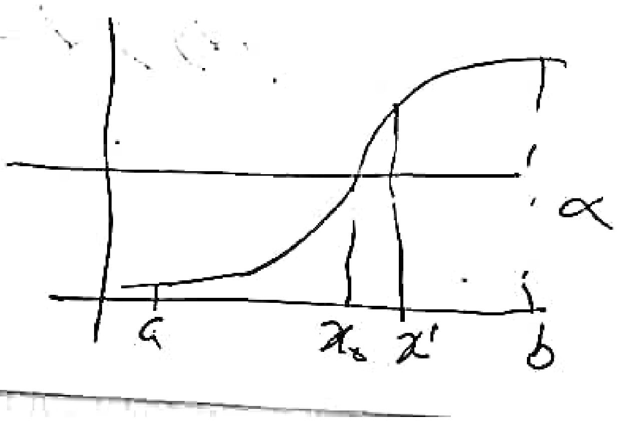
If f is bounded then $\exists M$ $f^{-1}(M, \infty) = \Phi$.

Let f be an $\uparrow$ function.

If $x' \in E_\alpha$, then for $x' < x \leq b \Rightarrow$
$f(b) \geq f(x) \geq f(x') > \alpha \Rightarrow [x', b] \subset E_\alpha$.

Let $x_0 = \inf\{x \in I : f(x) > \alpha\} = \inf E_\alpha$.

(i) If $x_0 = a$, then for $x \in I$, $\exists$ $x_1 \in E_\alpha$ s.t. $x_1 < x$ & $f(x) \geq f(x_1) > \alpha \Rightarrow x \in E_\alpha$
$\Rightarrow I = E_\alpha$.



(ii) If $a < x_0 \le b$, then for $x > x_0$, $\exists\, x_1 \in E_\alpha$ such that $x_0 < x_1 < x$ ~~&~~ $f(x) \ge f(x_1) > \alpha$.

$\Rightarrow (x_0, b] \subset E_\alpha$.

(iii) If $x < x_0$, then $f(x) \le \alpha \Rightarrow x \notin E_\alpha$.

$\Rightarrow (x_0, b] \subseteq E_\alpha \subseteq [x_0, b]$.

This proves the claim that E_α is an interval.

<u>Construction of monotone function:</u>

Let D be a countable set in $\mathbb{R}$, then we can construct a monotone ↑ function which is discontinuous only ~~at~~ on D.

Let $D = \{x_1, x_2, \ldots\}$ & $0 < t_n < 1$ be seqn s.t. $\sum_{n=1}^{\infty} c_n < \infty$. Let us define

$f(x) = \sum_{x_n < x} c_n$, where the sum is

On the set $\{n : x_n \le x\} = A_x$ (say) (44)
and $f(x) = 0$ if the set $A_x = \emptyset$.

If $x < y$, then

$$f(y) = \sum_{x_n \le y} \epsilon_n = \sum_{x_n \le x} \epsilon_n + \sum_{x < x_n \le y} \epsilon_n \geq f(x).$$

Note that for $x = x_k < y$, we get

$$f(y) = f(x_k) + \sum_{x_k < x_n \le y} \epsilon_n$$

Then $f(x_k^+) = f(x_k) + \lim_{y \to x_k^+} \sum_{x_k < x_n \le y} \epsilon_n = f(x_k)$,
since $\sum_{n=N}^{\infty} \epsilon_n \to 0$ as $N \to \infty$.

And when $x < x_k = y \Rightarrow$

$$f(x_k) = f(x) + \sum_{x < x_n \le x_k} \epsilon_n \geq f(x) + \epsilon_k.$$

(iii) Then $\lim_{x \to x_k^-} f(x) = f(x_k) - \lim_{x \to x_k^-} \sum_{x < x_n \le x_k} \epsilon_n$

i.e. $f(x_k^-) = f(x_k) - \epsilon_k$.

Thus, $f(x_k^+) - f(x_k^-) = \epsilon_k$.

The prove. of f is continuous at each pt of $\mathbb{R} \setminus D$ is similar to the above

Let $x \in \mathbb{R} \setminus D$. Then $x \neq x_n$ for any n.

For $x < y$, $f(y) = f(x) + \sum\limits_{x < x_n \leq y} \epsilon_n$.

When $y \to x^+$, then $\sum\limits_{x < x_n \leq y} \epsilon_n \to 0$ $\left(\because \{n : x < x_n \leq y\} \to \Phi \right)$

If $y < x$, then

$$f(x) = f(y) + \sum_{y < x_n \leq x} \epsilon_n$$

Hence $f(x) = \lim\limits_{y \to x^-} f(y) + \lim\limits_{y \to x^-} \sum\limits_{y < x_n < x} \epsilon_n$

$= f(x^-) + 0$ $\left(\because \{n : y < x_n < x\} \to \Phi \right)$

Ex. Let $D = \mathbb{Z}$, then

$$f(x) = \sum_{n \leq x} \epsilon_n.$$

For $x \in (0,1)$, $f(x) = \sum\limits_{n \leq 0} \epsilon_n = c$.

$\Rightarrow$ Constant on each open interval $(n, n+1)$.

Ex. Let D = be the end pts of deleted open intervals in the construction of Cantor set. Find appropriate seq^n $0 < b_n < 1$ to define Cantor function via (46)

$$f(x) = \sum_{x_n < x} b_n, \quad x_n \in D.$$

Ex. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = x + \sum_{n=0}^{n_x} 2^{-n}, \quad n_x = \left[\frac{1}{1-x}\right] \text{ if } x < 1.$$

and $f(1) = 3$.

Show that f is strictly increasing and discontinuous on $\{1 - \frac{1}{k} : k \in \mathbb{N}\}$.