Department of Mathematics

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Operator Theory in Hilbert Spaces

Lecture Notes

MA641

Spectral theory, compact operators, and Hilbert space methods

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Graduate lecture notes

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Contents

Preface

These lecture notes were prepared for the course *MA641: Operator Theory in Hilbert Spaces*. They originate in the 2020 course file and have been revised into a coherent graduate-level text suitable for lectures, guided reading, and examination preparation. The guiding aim of the revision has been to preserve the mathematical substance of the original notes while improving the logical organization, notational consistency, proof architecture, exposition, and overall typographic presentation.

The notes are intended for a first graduate course in operator theory on Hilbert spaces. They begin with the geometry of Hilbert spaces and the Riesz representation theorem, proceed to bounded operators and the foundational principles of functional analysis, and then develop orthogonal projections, adjoints, canonical operator classes, spectrum and resolvent, invariant and reducing subspaces, compact operators, singular values, and the spectral theorem for bounded selfadjoint and normal operators. A final chapter of exercises is included for consolidation, revision, and seminar-style problem solving.

This edition has also been aligned explicitly with the standard graduate syllabus for a first course in operator theory on Hilbert spaces: Hilbert spaces and orthonormal bases; weak convergence; bounded operators and the operator algebra $\mathcal{B}(H)$; adjoints; orthogonal projections; isometries and unitary operators; finite-rank, compact, Hilbert–Schmidt, and Schatten class operators; selfadjoint and normal operators; spectral theory, invariant and reducing subspaces; the spectral theorem for compact operators; polar and singular value decompositions; and the bounded selfadjoint and normal spectral theorem in the language of functional calculus and projection-valued measures.

The present revision sharpens the manuscript in three further respects. First, the theorem–proof flow has been tightened so that definitions, model examples, and structural propositions appear as close as possible to the points at which they are needed. Second, remarks have been added to clarify the scope of each major result, the role of the hypotheses, and the points at which finite-dimensional intuition must be modified in the infinite-dimensional setting. Third, the exercise chapter has been reorganized into a graded collection of diagnostic, developmental, and synthesis problems so that the notes can support both formal teaching and independent study.

Learning objectives.

- Develop the geometric and topological foundations of Hilbert space theory.
- Understand bounded linear operators, adjoints, and the three fundamental principles of functional analysis.
- Study orthogonal projections and the principal operator classes arising on Hilbert spaces.
- Master the notions of spectrum, resolvent, and spectral radius.

- Analyze compact operators, Fredholm-type phenomena, and spectral decompositions.
- Formulate the spectral theorem in both compact and general bounded settings.

Key ideas.

- Repeated material from the source notes has been consolidated into a single streamlined exposition.
- The chapter sequence has been adjusted so that conceptual dependencies appear in a more natural order.
- Statements and proofs have been revised for greater precision, readability, and journal-style mathematical presentation.

Introduction

Operator theory on Hilbert spaces is one of the principal points of interaction among linear algebra, geometry, and functional analysis. In finite dimensions, linear maps are analyzed through matrices, invariant subspaces, eigenvalues, and diagonal forms. In infinite dimensions, the same themes remain central, but they are reshaped by analytic phenomena with no genuine finite-dimensional analogue: completeness, weak convergence, compactness, boundedness, adjoints, and the subtle distinction between algebraic and spectral behavior. The subject is therefore best understood not as a departure from matrix theory, but as its rigorous extension to a setting in which topology and analysis become inseparable from algebra.

The decisive geometric ingredient is the Hilbert space structure itself. Inner products make it possible to speak of orthogonality, orthogonal projection, Fourier coefficients, and best approximation by closed subspaces. The Riesz representation theorem then identifies continuous linear functionals with inner products against fixed vectors, thereby explaining why Hilbert spaces admit a particularly transparent adjoint calculus. Once bounded operators and their adjoints enter the picture, one arrives naturally at the central classes of the theory—selfadjoint, unitary, normal, compact, Hilbert–Schmidt, and Schatten class operators—each of which encodes a different interaction between geometry and algebra.

A second governing theme is spectral analysis. In infinite-dimensional spaces, the spectrum replaces the classical list of eigenvalues, while the resolvent records invertibility through an analytic operator-valued function. For compact selfadjoint operators one still obtains an orthonormal basis of eigenvectors and a genuine diagonalization theorem. For general bounded selfadjoint and normal operators, however, diagonalization is replaced by the spectral theorem in the language of functional calculus and projection-valued measures. One of the pedagogical aims of these notes is to explain precisely how this transition occurs and why compact operators provide the conceptual bridge between matrix theory and the full bounded spectral theorem.

The notes are written with three complementary objectives in view. The first is structural: results are arranged so that the logical dependence of the later spectral material on the earlier geometric and functional-analytic foundations remains visible. The second is pedagogical: whenever possible, general theorems are accompanied by model operators—diagonal operators, shifts, projections, and multiplication operators—so that abstract statements may be tested against concrete examples. The third is stylistic: the exposition aims at a mathematically formal yet readable register, with proofs written to support both classroom presentation and self-study.

The notes are organized as follows.

1. *Foundations of Hilbert Space Theory*: inner products, orthogonality, projection, orthonormal systems, Parseval's identity, Gram–Schmidt orthogonalization, and weak convergence.

2. *Bounded Operators and Adjoint Operators*: continuity, boundedness, operator norm, extensions, the open mapping theorem, the inverse mapping theorem, the closed graph theorem, and the basic theory of adjoints.
3. *Orthogonal Projections and Canonical Operator Classes*: orthogonal projections, isometries, unitary operators, selfadjoint and normal operators, and preliminary compact-type classes.
4. *Spectrum, Resolvent, and Invariant Subspaces*: resolvent identities, the spectral radius, decomposition of the spectrum, spectral behavior under adjoints, invariant and reducing subspaces, and block-operator forms.
5. *Compact Operators, Fredholm Theory, Singular Values, and Schatten Classes*: compactness, finite-rank approximation, Fredholm-type alternatives, compact selfadjoint and compact normal spectral theory, polar decomposition, singular values, and Schatten norms.
6. *Spectral Theorem for Bounded Selfadjoint and Normal Operators*: continuous and bounded Borel functional calculi, projection-valued measures, and the general bounded spectral theorem.
7. *Problem Sets*: exercises arranged in thematic order to support revision, tutorials, and cumulative study.

Throughout, the default scalar field is $\mathbb{C}$ unless the real case is explicitly mentioned, and inner products are taken to be linear in the first variable and conjugate-linear in the second. Standard background from linear algebra and introductory functional analysis is assumed, but the notes have been organized so that the central arguments can be followed with minimal dependence on external references.

Syllabus map

The following table is meant to display the internal architecture of the notes at a glance. It may be used as a guide for first reading, selective revision, or examination preparation: each syllabus theme is paired with the chapter in which its main ideas are developed and with the principal structural conclusion established there.

Syllabus theme	**Primary location in the notes**	**Main structural outcome**
Hilbert spaces, orthonormal bases, weak convergence	Chapters 1 and the opening of Chapter 2	geometric projection, Fourier expansion, and weak compactness tools
Bounded operators, adjoints, and operator algebra	Chapter 2	$\mathcal{B}(H)$ as a Banach $*$-algebra with adjoint and norm control
Projections, isometries, compact and Hilbert–Schmidt operators	Chapter 3 and the opening of Chapter 5	canonical operator classes and approximation by finite-rank models
Spectra, invariant/reducing subspaces	Chapter 4	resolvent calculus and decomposition of operators by invariant geometry
Compact spectral theory, polar decomposition, singular values, Schatten classes	Chapter 5	discrete spectral expansion and singular-value methods
Bounded selfadjoint and normal spectral theorems	Chapter 6	functional calculus and projection-valued measure formulations

Guide to the Reader

Course navigation.

- Each chapter can be read at two levels: first for the principal statements, examples, and structural conclusions; then for the finer proof architecture and supplementary remarks.
- For a one-semester course, a natural core reading path is Chapters 1–4, the compact-operator portion of Chapter 5, and the statement-oriented parts of Chapter 6.
- The exercise chapter is organized progressively: diagnostic problems, theorem-building exercises, and synthesis or challenge problems suitable for seminars, qualifying-exam preparation, or take-home assignments.
- Readers using the notes for self-study should test the main implications in the adjoint, projection, compactness, and spectral chapters against model examples such as diagonal operators, shifts, and multiplication operators.

Chapter 1

Foundations of Hilbert Space Theory

This chapter develops the geometric and analytic core of Hilbert-space theory. We introduce orthogonality, projections, orthonormal expansions, the Riesz representation theorem, and weak convergence, emphasizing the structural principles that distinguish Hilbert spaces from general Banach spaces and that recur throughout the rest of the notes.

Learning objectives.

- Pass fluently between geometric language (orthogonality, projection, best approximation) and analytic language (norm, completeness, duality).
- Understand how orthonormal systems encode vectors by Fourier coefficients.
- Use weak convergence as a compactness surrogate in Hilbert-space arguments.

Key ideas.

- Closed subspaces are detected by orthogonal projection and by annihilation under orthogonal complements.
- Orthonormal bases linearize many nonlinear geometric questions by reducing them to ℓ^2 calculations.
- Weak convergence preserves bounded linear information but not metric size.

1.1 Inner product spaces, orthogonality, and projections

Hilbert-space theory begins with the observation that an inner product supplies both a metric and a geometry. Once orthogonality is available, closed subspaces can be studied through best approximation and orthogonal projection, and many later operator-theoretic statements may be read as geometric facts in disguise.

Definition 1.1 (Inner product)**.** Let X be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ is an *inner product* if, for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$,

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry);

(iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ (linearity in the first variable).

The pair $(X, \langle \cdot, \cdot \rangle)$ is called an *inner product space.*

Definition 1.2 (Induced norm and Hilbert space)**.** On an inner product space define

$$\|x\| := \langle x, x \rangle^{1/2}, \qquad x \in X.$$

A *Hilbert space* is an inner product space that is complete with respect to the metric induced by this norm.

Proposition 1.3 (Cauchy–Schwarz and triangle inequality). *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space with induced norm $\|\cdot\|$. Then for all $x, y \in X$,*

$$|\langle x, y\rangle| \le \|x\|\,\|y\| \qquad \textit{(Cauchy–Schwarz)},$$

and consequently $\|x+y\| \le \|x\| + \|y\|$. In particular, $\|\cdot\|$ is a norm on X.

Proof. If $y = 0$ the claim is trivial. For $y \neq 0$ and $t \in \mathbb{F}$ we have $0 \le \|x - ty\|^2 = \langle x - ty, x - ty\rangle$, hence

$$0 \le \|x\|^2 - t\,\langle y, x\rangle - \bar{t}\,\langle x, y\rangle + |t|^2\|y\|^2.$$

Choosing $t = \langle x, y\rangle/\|y\|^2$ gives $0 \le \|x\|^2 - |\langle x, y\rangle|^2/\|y\|^2$, i.e. $|\langle x, y\rangle| \le \|x\|\,\|y\|$. Then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\Re\langle x, y\rangle \le (\|x\| + \|y\|)^2,$$

which yields the triangle inequality. □

Proposition 1.4 (Parallelogram law and polarization). *In every inner product space,*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \qquad (x, y \in X). \tag{1.1}$$

Conversely, if a normed space $(X, \|\cdot\|)$ satisfies (1.1)*, then the inner product is recovered from the norm by the polarization identity:*

$$\langle x, y\rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) \quad (\mathbb{F} = \mathbb{R}),$$

and

$$\langle x, y\rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2 + i\big(\|x+iy\|^2 - \|x-iy\|^2\big)\Big) \quad (\mathbb{F} = \mathbb{C}).$$

Proof. If $\|\cdot\|$ is induced by an inner product, then expanding $\langle x \pm y, x \pm y\rangle$ gives (1.1). Conversely, assume (1.1) holds. Define $\langle x, y\rangle$ by the corresponding polarization identity. A routine verification (using (1.1)) shows that $\langle\cdot,\cdot\rangle$ is sesquilinear, conjugate symmetric, positive definite, and that $\|x\|^2 = \langle x, x\rangle$. □

Definition 1.5 (Orthogonality and orthogonal complement). Let X be an inner product space. Vectors $x, y \in X$ are *orthogonal*, written $x \perp y$, if $\langle x, y\rangle = 0$. For a subset $S \subset X$ define

$$S^\perp := \{x \in X : \langle x, s\rangle = 0\ \forall s \in S\}.$$

If M is a linear subspace, then $M^\perp$ is a linear subspace, and it is always closed (in the norm topology).

Proposition 1.6. *Let X be an inner product space and $M \subset X$ a linear subspace. Then the following assertions hold.*

(a) $M \subset (M^\perp)^\perp$.

(b) If X is complete (i.e. a Hilbert space), then $(M^\perp)^\perp = \overline{M}$.

Proof. (a) If $m \in M$ and $x \in M^\perp$, then $\langle x, m\rangle = 0$, hence $\langle m, x\rangle = 0$ by conjugate symmetry; thus $m \in (M^\perp)^\perp$. (b) The inclusion $\overline{M} \subset (M^\perp)^\perp$ follows from continuity of $x \mapsto \langle x, y\rangle$ for fixed y. For the reverse inclusion, let $x \in (M^\perp)^\perp$ and consider the closed subspace $\overline{M}$. By Theorem 1.7 below, there is a decomposition $x = u + v$ with $u \in \overline{M}$ and $v \in \overline{M}^\perp = M^\perp$. But $x \in (M^\perp)^\perp$ implies $0 = \langle x, v\rangle = \langle u, v\rangle + \langle v, v\rangle = \|v\|^2$, hence $v = 0$ and $x = u \in \overline{M}$. □

Theorem 1.7 (Orthogonal decomposition and projection theorem)**.** *Let H be a Hilbert space and let $M \subset H$ be a closed linear subspace. Then, for every $x \in H$, there exist unique vectors $u \in M$ and $v \in M^\perp$ such that*

$$x = u + v.$$

Equivalently, $H = M \oplus M^\perp$. Moreover:

(a) The vector u is the unique minimizer of $y \mapsto \|x - y\|$ over M, and $\operatorname{dist}(x, M) = \|x - u\|$.

(b) The map $P_M : H \to M$ defined by $P_M x := u$ is a bounded linear projection, called the orthogonal projection *onto M. If $M \neq \{0\}$, then $\|P_M\| = 1$; moreover, $P_M^2 = P_M$ and $P_M^* = P_M$.*

(c) Writing $Q_M := I - P_M$, one has $Q_M(H) \subset M^\perp$, $Q_M^2 = Q_M$, and $\|Q_M\| = 1$ whenever $M \neq H$.

Proof. Fix $x \in H$ and set $d := \inf_{y \in M}\|x - y\|$. Choose a sequence $(y_n) \subset M$ such that $\|x - y_n\| \to d$. Using the parallelogram law (1.1) and the fact that $(y_n + y_m)/2 \in M$, we obtain

$$\|y_n - y_m\|^2 \leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4d^2 \longrightarrow 0 \quad (n, m \to \infty).$$

Thus (y_n) is a Cauchy sequence. Since M is closed in H, there exists $u \in M$ with $y_n \to u$. By continuity, $\|x - u\| = d$, so u is a best approximation to x from M.

Set $v := x - u$. We show that $v \in M^\perp$. Let $m \in M$. For every real t, the point $u + tm$ lies in M; hence the minimality of u implies

$$\|v\|^2 = \|x - u\|^2 \leq \|x - (u + tm)\|^2 = \|v - tm\|^2 = \|v\|^2 - 2t\,\Re\langle v, m\rangle + t^2\|m\|^2.$$

Since this holds for all $t \in \mathbb{R}$, we must have $\Re\langle v, m\rangle = 0$. In the complex case, replacing m by im also gives $\Im\langle v, m\rangle = 0$. Therefore $\langle v, m\rangle = 0$ for every $m \in M$, so $v \in M^\perp$.

To prove uniqueness, suppose

$$x = u_1 + v_1 = u_2 + v_2, \qquad u_1, u_2 \in M, \quad v_1, v_2 \in M^\perp.$$

Then $u_1 - u_2 = v_2 - v_1 \in M \cap M^\perp = \{0\}$, so $u_1 = u_2$ and $v_1 = v_2$.

Define $P_M x := u$. Linearity follows from the uniqueness of the decomposition. Since $x = P_M x + Q_M x$ with $\langle P_M x, Q_M x\rangle = 0$, we have

$$\|P_M x\| \le \|P_M x + Q_M x\| = \|x\|,$$

so $\|P_M\| \le 1$. If $M \ne \{0\}$ and $0 \ne m \in M$, then $P_M m = m$, and therefore $\|P_M\| = 1$. The identities $P_M^2 = P_M$ and $Q_M^2 = Q_M$ are immediate. Finally, $x - P_M x \in M^\perp$ for every $x \in H$, so for all $x \in H$ and $m \in M$,

$$\langle P_M x, m\rangle = \langle x, m\rangle.$$

Taking $m = P_M y$ yields $\langle P_M x, y\rangle = \langle P_M x, P_M y\rangle = \langle x, P_M y\rangle$, and hence $P_M^* = P_M$. $\square$

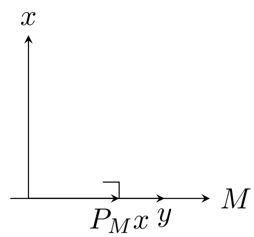

Figure 1.1: Orthogonal projection: $P_M x \in M$ is the unique point of M closest to x.

1.2 Riesz representation theorem

The following theorem identifies the continuous dual of a Hilbert space with the space itself.

Theorem 1.8 (Riesz representation)**.** *Let H be a Hilbert space. For each $y \in H$ define $f_y : H \to \mathbb{F}$ by*

$$f_y(x) = \langle x, y\rangle, \qquad x \in H,$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then $f_y \in H^*$ and $\|f_y\| = \|y\|$. Conversely, for every $f \in H^*$ there exists a unique vector $y \in H$ such that $f = f_y$.

Proof. Fix $y \in H$. The Cauchy–Schwarz inequality gives

$$|f_y(x)| = |\langle x, y\rangle| \le \|x\|\,\|y\|,$$

so f_y is continuous and $\|f_y\| \le \|y\|$. If $y \ne 0$, choosing $x = y/\|y\|$ yields $|f_y(x)| = \|y\|$; hence $\|f_y\| = \|y\|$.

For the converse, let $0 \ne f \in H^*$ and set $M := \ker f$, which is a closed subspace of H. Choose $x_0 \in H$ with $f(x_0) = 1$, and write the orthogonal decomposition

$$x_0 = m_0 + z, \qquad m_0 \in M, \quad z \in M^\perp.$$

Then $f(z) = f(x_0 - m_0) = 1$, so in particular $z \neq 0$. For any $x \in H$ we have $x - f(x)z \in \ker f = M$, and therefore $x - f(x)z \perp z$. Taking inner products with z gives

$$0 = \langle x - f(x)z, z\rangle = \langle x, z\rangle - f(x)\|z\|^2.$$

Hence

$$f(x) = \frac{\langle x, z\rangle}{\|z\|^2} = \langle x, y\rangle, \qquad y := \frac{z}{\|z\|^2}.$$

Thus $f = f_y$. Uniqueness is immediate: if $\langle x, y_1\rangle = \langle x, y_2\rangle$ for all $x \in H$, then taking $x = y_1 - y_2$ gives $\|y_1 - y_2\|^2 = 0$, so $y_1 = y_2$. □

Corollary 1.9. *If M is a closed subspace of H, then*

$$M = (M^\perp)^\perp = \bigcap_{y \in M^\perp} \ker f_y.$$

In particular, M is the kernel of a single *continuous linear functional if and only if* $\operatorname{codim} M = 1$.

Proof. If $x \in M$, then $\langle x, y\rangle = 0$ for all $y \in M^\perp$, so $M \subseteq (M^\perp)^\perp$. Conversely, if $x \in (M^\perp)^\perp$, then $x \perp M^\perp$, hence the orthogonal decomposition $H = M \oplus M^\perp$ forces $x \in M$. The intersection-of-kernels identity is just a reformulation of $x \perp M^\perp$. □

Definition 1.10 (Orthonormal sets and orthonormal bases)**.** A subset $E \subset H$ is called *orthonormal* if every $e \in E$ satisfies $\|e\| = 1$ and

$$\langle e, f\rangle = 0 \qquad (e, f \in E,\ e \neq f).$$

An orthonormal set E is called an *orthonormal basis* of H if

$$\overline{\operatorname{span} E} = H.$$

Equivalently, E is an orthonormal basis if and only if $x \perp E$ implies $x = 0$.

Proposition 1.11 (Existence of orthonormal bases)**.** *Every Hilbert space admits an orthonormal basis.*

Proof. Let $\mathcal{F}$ be the family of all orthonormal subsets of H, ordered by inclusion. If $\mathcal{C} \subset \mathcal{F}$ is a chain, then $\bigcup_{E \in \mathcal{C}} E$ is again orthonormal, so it is an upper bound for $\mathcal{C}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element, say E_0.

We claim that E_0 is complete. If not, then there exists $0 \neq x \in H$ with $x \perp E_0$. After normalization, the set

$$E_0 \cup \left\{ \frac{x}{\|x\|} \right\}$$

is orthonormal, contradicting maximality. Hence $\overline{\operatorname{span} E_0} = H$, so E_0 is an orthonormal basis. □

Lemma 1.12 (Finite orthogonal expansion and finite Bessel inequality)**.** *Let $\{e_1, \dots, e_n\}$ be an orthonormal set in H, and define*

$$S_n x := \sum_{j=1}^{n} \langle x, e_j \rangle e_j.$$

Then, for every $x \in H$,

$$x - S_n x \perp e_k \qquad (k = 1, \dots, n), \tag{1.2}$$

$$\|S_n x\|^2 = \sum_{j=1}^{n} |\langle x, e_j \rangle|^2, \tag{1.3}$$

and

$$\|x - S_n x\|^2 = \|x\|^2 - \sum_{j=1}^{n} |\langle x, e_j \rangle|^2. \tag{1.4}$$

In particular,

$$\sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \le \|x\|^2.$$

Proof. For each $k \in \{1, \dots, n\}$,

$$\langle x - S_n x, e_k \rangle = \langle x, e_k \rangle - \sum_{j=1}^{n} \langle x, e_j \rangle \langle e_j, e_k \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0,$$

which proves (1.2). Next,

$$\|S_n x\|^2 = \langle S_n x, S_n x \rangle = \sum_{j,k=1}^{n} \langle x, e_j \rangle \overline{\langle x, e_k \rangle} \langle e_j, e_k \rangle = \sum_{j=1}^{n} |\langle x, e_j \rangle|^2,$$

so (1.3) holds. Finally, the orthogonality of $x - S_n x$ and $S_n x$ gives the Pythagorean identity

$$\|x\|^2 = \|x - S_n x\|^2 + \|S_n x\|^2,$$

which, together with (1.3), yields (1.4). □

Lemma 1.13. *Let E be an orthonormal set in a Hilbert space H. Then for each $x \in H$, the set*

$$E_x := \{e \in E : \langle x, e \rangle \neq 0\}$$

is countable.

Proof. For each $m \in \mathbb{N}$ let

$$E_m := \left\{ e \in E : |\langle x, e \rangle|^2 > \frac{1}{m} \right\}.$$

If $e_1, \dots, e_r \in E_m$ are distinct, then Lemma 1.12 gives

$$\frac{r}{m} < \sum_{j=1}^{r} |\langle x, e_j \rangle|^2 \leq \|x\|^2,$$

so $r < m\|x\|^2$. Hence each E_m is finite. Since every $e \in E_x$ satisfies $|\langle x, e \rangle|^2 > 1/m$ for some m, we conclude

$$E_x = \bigcup_{m=1}^{\infty} E_m,$$

which is a countable union of finite sets. □

Theorem 1.14 (Bessel inequality)**.** *Let $E = \{e_\alpha\}_{\alpha \in I}$ be an orthonormal set in H. Then for every $x \in H$,*

$$\sum_{\alpha \in I} |\langle x, e_\alpha \rangle|^2 \leq \|x\|^2. \tag{1.5}$$

The sum is well defined because, by Lemma 1.13, at most countably many terms are nonzero.

Proof. Enumerate the countable set $E_x = \{e \in E : \langle x, e \rangle \neq 0\}$ as $\{e_n\}_{n \geq 1}$. For each $k \in \mathbb{N}$, Lemma 1.12 yields

$$\sum_{n=1}^{k} |\langle x, e_n \rangle|^2 \leq \|x\|^2.$$

Letting $k \to \infty$ proves (1.5). □

1.3 Parseval's identity

Once an orthonormal basis is fixed, Hilbert-space geometry can be encoded by scalar coefficients. Parseval's identity makes this principle precise by showing that norm and inner-product information are completely captured by Fourier coefficients, thereby reducing many geometric questions to calculations in ℓ^2.

Theorem 1.15 (Parseval's identity)**.** *Let $E = \{e_\alpha\}_{\alpha \in I}$ be an orthonormal basis of a Hilbert space H. Then, for every $x \in H$,*

$$x = \sum_{\alpha \in I} \langle x, e_\alpha \rangle e_\alpha, \tag{1.6}$$

and

$$\|x\|^2 = \sum_{\alpha \in I} |\langle x, e_\alpha \rangle|^2. \tag{1.7}$$

Only countably many terms in these expressions are nonzero.

Proof. Fix $x \in H$, and enumerate the countable set $E_x := \{e_\alpha \in E : \langle x, e_\alpha \rangle \neq 0\}$ as $\{e_n\}_{n \geq 1}$. Define

$$y_n := \sum_{j=1}^{n} \langle x, e_j \rangle e_j.$$

If $m > n$, then orthonormality gives

$$\|y_m - y_n\|^2 = \sum_{j=n+1}^{m} |\langle x, e_j\rangle|^2.$$

By Bessel's inequality, the right-hand side tends to 0 as $m, n \to \infty$, so (y_n) is a Cauchy sequence in H. Let $y \in H$ denote its limit.

For every $\alpha \in I$, continuity of the inner product yields

$$\langle y, e_\alpha\rangle = \lim_{n\to\infty} \langle y_n, e_\alpha\rangle = \langle x, e_\alpha\rangle.$$

Hence $\langle x - y, e_\alpha\rangle = 0$ for all $\alpha \in I$. Since E is complete, this implies $x - y = 0$, and therefore (1.6) holds.

Finally,

$$\|x\|^2 = \lim_{n\to\infty} \|y_n\|^2 = \lim_{n\to\infty} \sum_{j=1}^{n} |\langle x, e_j\rangle|^2 = \sum_{\alpha\in I} |\langle x, e_\alpha\rangle|^2,$$

which is exactly (1.7). □

1.4 Gram–Schmidt process

The Gram–Schmidt process is the basic constructive mechanism that turns linear independence into orthonormality. It explains how orthonormal bases arise in finite dimensions and provides the first systematic procedure for producing the coordinate systems on which Hilbert-space analysis depends.

Theorem 1.16 (Gram–Schmidt orthonormalization)**.** *Let $x_1, \dots, x_n$ be linearly independent vectors in a Hilbert space H. Define*

$$u_1 := x_1, \qquad u_k := x_k - \sum_{j=1}^{k-1} \langle x_k, g_j\rangle g_j \quad (k \geq 2),$$

and set

$$g_k := \frac{u_k}{\|u_k\|} \qquad (k = 1, \dots, n).$$

Then $\{g_1, \dots, g_n\}$ is an orthonormal set and, for every $k \leq n$,

$$\operatorname{span}\{x_1, \dots, x_k\} = \operatorname{span}\{g_1, \dots, g_k\}.$$

Proof. We proceed by induction on k. Since $x_1 \neq 0$, the vector $g_1 = x_1/\|x_1\|$ is well defined and has norm one. Assume that $g_1, \dots, g_{k-1}$ have already been constructed and form an orthonormal set. Because

$$\operatorname{span}\{x_1, \dots, x_{k-1}\} = \operatorname{span}\{g_1, \dots, g_{k-1}\},$$

the linear independence of $x_1, \dots, x_n$ implies that $x_k \notin \operatorname{span}\{g_1, \dots, g_{k-1}\}$. Hence $u_k \neq 0$. Moreover, for each $j = 1, \dots, k-1$,

$$\langle u_k, g_j \rangle = \langle x_k, g_j \rangle - \sum_{i=1}^{k-1} \langle x_k, g_i \rangle \langle g_i, g_j \rangle = 0,$$

so $g_k = u_k / \|u_k\|$ is orthogonal to $g_1, \dots, g_{k-1}$ and has norm one. Thus $g_1, \dots, g_k$ is orthonormal.

By construction,

$$x_k = u_k + \sum_{j=1}^{k-1} \langle x_k, g_j \rangle g_j \in \operatorname{span}\{g_1, \dots, g_k\},$$

so $\operatorname{span}\{x_1, \dots, x_k\} \subseteq \operatorname{span}\{g_1, \dots, g_k\}$. The reverse inclusion is immediate from the definitions of u_k and g_k. This proves the claim for every $k \leq n$. □

Example 1.17 (Hermite functions)**.** Applying the Gram–Schmidt process to the sequence $1, t, t^2, \dots$ in the Gaussian-weighted setting leads to the Hermite functions

$$h_n(t) := c_n H_n(t) e^{-t^2/2}, \qquad n \geq 0,$$

where H_n denotes the nth Hermite polynomial and c_n is the normalizing constant. The family $(h_n)_{n \geq 0}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Sketch of completeness. Suppose $g \in L^2(\mathbb{R})$ is orthogonal to every h_n. Equivalently,

$$\int_{\mathbb{R}} g(t) e^{-t^2/2} t^n \, dt = 0 \qquad (n \geq 0).$$

Define

$$G(z) := \int_{\mathbb{R}} g(t) e^{-t^2/2} e^{itz} \, dt, \qquad z \in \mathbb{C}.$$

Then G is entire, and all derivatives of G at 0 vanish because the moments above are zero. Hence $G \equiv 0$. By uniqueness of the Fourier transform, this implies $g = 0$. Therefore the Hermite functions are complete. □

Theorem 1.18. *Every infinite-dimensional separable Hilbert space is isometrically isomorphic to ℓ^2.*

Proof. Let $\{e_n\}_{n \geq 1}$ be a countable orthonormal basis of H, and define

$$F : H \to \ell^2, \qquad F(x) := (\langle x, e_1 \rangle, \langle x, e_2 \rangle, \dots).$$

By Bessel's inequality, $F(x) \in \ell^2$ for every $x \in H$. Parseval's identity yields

$$\|F(x)\|_{\ell^2}^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|x\|^2,$$

so F is an isometry.

It remains to prove surjectivity. Let $y = (y_n) \in \ell^2$. Then the series $\sum_{n=1}^{\infty} y_n e_n$ converges in H because

$$\left\| \sum_{n=m}^{N} y_n e_n \right\|^2 = \sum_{n=m}^{N} |y_n|^2 \xrightarrow[m,N\to\infty]{} 0.$$

Define

$$x := \sum_{n=1}^{\infty} y_n e_n.$$

Orthogonality gives $F(x) = (y_1, y_2, \dots) = y$. Thus F is onto, hence an isometric isomorphism of H onto ℓ^2. □

1.5 Weak convergence and weak compactness

Strong convergence is often too rigid for infinite-dimensional arguments, especially when compactness is unavailable. Weak convergence provides a more flexible mode of convergence that preserves linear information, and weak compactness supplies the substitute compactness principle that underlies many later extraction arguments.

Definition 1.19 (Weak convergence)**.** Let H be a Hilbert space. A sequence (x_n) in H is said to *converge weakly* to $x \in H$, written

$$x_n \rightharpoonup x,$$

if

$$f(x_n) \to f(x) \qquad \text{for every } f \in H^*.$$

Proposition 1.20 (Hilbert-space characterization of weak convergence)**.** *Let H be a Hilbert space and (x_n) a sequence in H. Then*

$$x_n \rightharpoonup x \quad \Longleftrightarrow \quad \langle x_n, y\rangle \to \langle x, y\rangle \text{ for every } y \in H.$$

Proof. If $x_n \rightharpoonup x$, then the conclusion holds by applying the definition to the functionals $f_y(z) = \langle z, y\rangle$. Conversely, if $\langle x_n, y\rangle \to \langle x, y\rangle$ for every $y \in H$, then for each $f \in H^*$ the Riesz representation theorem yields $f(z) = \langle z, y_f\rangle$ for a unique $y_f \in H$. Hence $f(x_n) \to f(x)$ for every $f \in H^*$, which is precisely $x_n \rightharpoonup x$. □

Example 1.21 (Orthonormal sequences are weakly null)**.** If (e_n) is an orthonormal sequence in H, then $e_n \rightharpoonup 0$.

Proof. Fix $y \in H$. By Bessel's inequality,

$$\sum_{n=1}^{\infty} |\langle y, e_n\rangle|^2 \le \|y\|^2.$$

Hence the scalar sequence $\langle y, e_n\rangle$ tends to 0. By theorem 1.20, this means $e_n \rightharpoonup 0$. □

Theorem 1.22 (Weakly convergent sequences are bounded)**.** *If $x_n \rightharpoonup x$ in a Hilbert space H, then the sequence (x_n) is bounded in norm.*

Proof. For each n, define a functional $F_n \in H^*$ by

$$F_n(y) := \langle x_n, y \rangle.$$

If $x_n \rightharpoonup x$, then for every fixed $y \in H$ the scalar sequence $(F_n(y))$ converges, and therefore is bounded. By the Uniform Boundedness Principle, $\sup_n \|F_n\| < \infty$. The Riesz representation theorem yields $\|F_n\| = \|x_n\|$ for every n, so $\sup_n \|x_n\| < \infty$. □

Proposition 1.23 (Weak convergence plus convergence of norms)**.** *Let (x_n) be a sequence in a Hilbert space H and let $x \in H$. Then*

$$x_n \to x \text{ in norm} \quad \Longleftrightarrow \quad x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\|.$$

Proof. Norm convergence clearly implies weak convergence and convergence of norms. Conversely, assume $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\Re\langle x_n, x \rangle.$$

By weak convergence, $\langle x_n, x \rangle \to \langle x, x \rangle = \|x\|^2$, and by hypothesis $\|x_n\|^2 \to \|x\|^2$. Therefore $\|x_n - x\|^2 \to 0$, so $x_n \to x$ in norm. □

Theorem 1.24 (Weak compactness of the unit ball)**.** *The closed unit ball*

$$\overline{B}_H := \{x \in H : \|x\| \leq 1\}$$

of a Hilbert space H is compact in the weak topology.

Comment. This follows from the Banach–Alaoglu theorem together with the reflexivity of Hilbert spaces. □

Remark 1.25. In an infinite-dimensional Hilbert space the closed unit ball is not norm-compact. Indeed, any orthonormal sequence lies in the unit sphere, but by theorem 1.21 it converges only weakly, not in norm. This contrast between norm compactness and weak compactness is one of the central geometric features of Hilbert space.

Example 1.26 (Weak but not norm convergence)**.** In ℓ^2, let $e_n = (0, \dots, 0, 1, 0, \dots)$ be the standard orthonormal basis. Then for every $x = (x_n) \in \ell^2$ one has

$$\langle x, e_n \rangle = x_n \to 0,$$

so $e_n \rightharpoonup 0$ weakly. However, $\|e_n\| = 1$ for all n, hence (e_n) does not converge to 0 in norm.

Remark 1.27. This is the basic model showing that weak compactness and norm compactness are fundamentally different. It is also the prototype behind many compactness tests in later chapters: compact operators convert bounded weakly convergent sequences into norm-convergent sequences.

Chapter 2

Bounded Operators and Adjoint Operators

We introduce bounded linear operators between normed and Hilbert spaces and develop the operator norm as well as the fundamental equivalence between continuity and boundedness. The chapter also contains the classical principles of Banach–Steinhaus, open mapping, inverse mapping, and closed graph, together with the Hilbert-space notion of adjoint and the elementary spectral restrictions for selfadjoint and unitary operators.

Learning objectives.

- Recognize $\mathcal{B}(H)$ simultaneously as a normed space, an algebra, and a $*$-algebra.
- Interpret the adjoint as the Hilbert-space shadow of duality.
- See where the open mapping, inverse mapping, closed graph, and uniform boundedness theorems enter operator theory.

Key ideas.

- Boundedness is the correct topological replacement for matrix finiteness in infinite dimensions.
- Adjoints convert range questions into kernel questions and are therefore structural rather than merely formal.
- The major principles of functional analysis are repeatedly used to turn pointwise information into uniform operator bounds.

2.1 Continuous linear transformations

Operator theory begins with the simplest question one can ask about a linear map: when does linearity interact well with the ambient topology? For normed spaces, boundedness and continuity turn out to be equivalent, and this equivalence allows one to move freely between algebraic and analytic descriptions of linear transformations.

Definition 2.1 (Bounded linear map)**.** A linear map $T : X \to Y$ is said to be *bounded* if there exists $M \geq 0$ such that

$$\|Tx\|_Y \leq M\|x\|_X \qquad (x \in X).$$

Proposition 2.2. *For a linear map $T : X \to Y$, the following are equivalent:*

(i) T is continuous on X;

(ii) T is continuous at 0;

(iii) T is bounded.

Proof. The implication (i)⇒(ii) is immediate. Assume (ii). Then there exists $\delta > 0$ such that

$$\|x\|_X < \delta \quad \Longrightarrow \quad \|Tx\|_Y < 1.$$

If $y \neq 0$, set $x := \frac{\delta}{2\|y\|_X} y$. Then $\|x\|_X = \delta/2 < \delta$, so

$$\left\| T\left(\frac{\delta}{2\|y\|_X} y \right) \right\|_Y < 1.$$

By linearity,

$$\|Ty\|_Y < \frac{2}{\delta}\|y\|_X,$$

so T is bounded. Finally, if T is bounded, say $\|Tx\|_Y \leq M\|x\|_X$, then $x_n \to x$ implies

$$\|Tx_n - Tx\|_Y = \|T(x_n - x)\|_Y \leq M\|x_n - x\|_X \to 0.$$

Hence (iii)⇒(i). □

Definition 2.3 (Operator norm)**.** If $T \in \mathcal{B}(X, Y)$, its *operator norm* is

$$\|T\| := \inf\{M \geq 0 : \|Tx\|_Y \leq M\|x\|_X \text{ for all } x \in X\}.$$

Lemma 2.4. *For $T \in \mathcal{B}(X, Y)$,*

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\| \leq 1} \|Tx\|.$$

Proof. Set

$$\alpha := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Then $\|Tx\| \leq \alpha\|x\|$ for every $x \neq 0$, and trivially also for $x = 0$. Hence $\|T\| \leq \alpha$. Conversely, by definition of the operator norm,

$$\frac{\|Tx\|}{\|x\|} \leq \|T\| \qquad (x \neq 0),$$

so $\alpha \leq \|T\|$. Thus $\alpha = \|T\|$.

If $\|x\| = 1$, then $\|Tx\| \leq \|T\|$, hence $\sup_{\|x\|=1} \|Tx\| \leq \|T\|$. On the other hand, for any $x \neq 0$, writing $u = x/\|x\|$ gives

$$\frac{\|Tx\|}{\|x\|} = \|Tu\| \leq \sup_{\|u\|=1} \|Tu\|.$$

Taking the supremum over $x \neq 0$ yields $\|T\| \leq \sup_{\|x\|=1} \|Tx\|$. The equality with $\sup_{\|x\| \leq 1} \|Tx\|$ is immediate. □

Proposition 2.5. *If Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space with respect to the operator norm.*

Proof. Let (T_n) be a Cauchy sequence in $\mathcal{B}(X,Y)$. For each fixed $x \in X$,

$$\|T_n x - T_m x\| \le \|T_n - T_m\| \, \|x\|,$$

so $(T_n x)$ is Cauchy in Y. Since Y is Banach, there exists $Tx \in Y$ such that $T_n x \to Tx$. This defines a map $T : X \to Y$. Because pointwise limits of linear maps are linear, T is linear.

Since (T_n) is Cauchy in norm, it is bounded: choose $M > 0$ such that $\|T_n\| \le M$ for all n. Then

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \le M\|x\|,$$

so $T \in \mathcal{B}(X,Y)$. Finally, if $\varepsilon > 0$, choose N such that $\|T_n - T_m\| < \varepsilon$ whenever $m, n \ge N$. For $n \ge N$ and $\|x\| \le 1$, letting $m \to \infty$ gives

$$\|(T_n - T)x\| = \lim_{m\to\infty} \|(T_n - T_m)x\| \le \varepsilon.$$

Taking the supremum over the unit ball yields $\|T_n - T\| \le \varepsilon$ for all $n \ge N$. Hence $T_n \to T$ in $\mathcal{B}(X,Y)$. □

2.2 Algebra of bounded operators on Hilbert spaces

Once bounded operators are admitted as the natural morphisms of Hilbert spaces, one is led immediately to their algebraic structure. Addition, composition, scalar multiplication, and inversion all interact with the operator norm, and the resulting operator algebra is the framework in which spectral and structural questions will be posed.

Proposition 2.6 (Basic algebraic identities)**.** *Let X, Y, and Z be normed spaces.*

(a) If $S \in \mathcal{B}(Y,Z)$ and $T \in \mathcal{B}(X,Y)$, then $ST \in \mathcal{B}(X,Z)$ and

$$\|ST\| \le \|S\| \, \|T\|.$$

(b) If X is a normed space, then $\mathcal{B}(X)$ is a normed algebra with identity I satisfying $\|I\| = 1$.

(c) If X is Banach, then $\mathcal{B}(X)$ is a Banach algebra.

Proof. For $x \in X$,

$$\|STx\| \le \|S\| \, \|Tx\| \le \|S\| \, \|T\| \, \|x\|,$$

which proves (a). The algebraic properties in (b) are immediate, and the norm estimate in (a) shows that multiplication is jointly continuous. Part (c) follows immediately from Theorem 2.5. □

Remark 2.7. Thus, for a Hilbert space H, the space $\mathcal{B}(H)$ is a unital Banach algebra. After the adjoint is introduced below, it becomes a unital Banach $*$-algebra and serves as the natural ambient algebra for operator theory on H.

2.3 Adjoint operators and the $*$-structure on $\mathcal{B}(H)$

The adjoint operation packages the geometry of the inner product into operator language. It allows orthogonality, symmetry, positivity, and normality to be expressed algebraically, and it endows $\mathcal{B}(H)$ with the involutive structure that governs much of Hilbert-space operator theory.

Theorem 2.8 (Existence and uniqueness of the adjoint)**.** *For every $T \in \mathcal{B}(H, K)$ there exists a unique bounded operator $T^* \in \mathcal{B}(K, H)$ such that*

$$\langle Tx, y\rangle_K = \langle x, T^*y\rangle_H \qquad (x \in H,\ y \in K).$$

Moreover, $\|T^*\| = \|T\|$.

Proof. Fix $y \in K$. The map $x \mapsto \langle Tx, y\rangle_K$ is a bounded linear functional on H, so by the Riesz representation theorem there exists a unique vector $z_y \in H$ such that

$$\langle Tx, y\rangle_K = \langle x, z_y\rangle_H \qquad (x \in H).$$

Define $T^*y := z_y$. The uniqueness in the Riesz theorem shows that T^* is well defined. A direct verification shows that T^* is linear. Also,

$$\|T^*y\| = \sup_{\|x\|=1} |\langle x, T^*y\rangle| = \sup_{\|x\|=1} |\langle Tx, y\rangle| \leq \|T\|\,\|y\|,$$

so T^* is bounded and $\|T^*\| \leq \|T\|$. Applying the same argument to T^* gives $(T^*)^* = T$ and hence $\|T\| \leq \|T^*\|$. Therefore $\|T^*\| = \|T\|$. $\square$

Proposition 2.9 (Basic identities for adjoints)**.** *Let $R, S \in \mathcal{B}(K, L)$, $T \in \mathcal{B}(H, K)$, and $\alpha, \beta \in \mathbb{F}$. Then the following assertions hold.*

(a) $(\alpha S + \beta T)^* = \overline{\alpha}\, S^* + \overline{\beta}\, T^*$;

(b) $(ST)^* = T^*S^*$;

(c) $(T^*)^* = T$;

(d) $\operatorname{Ker} T^* = (\operatorname{Ran} T)^\perp$ *and* $\overline{\operatorname{Ran} T} = (\operatorname{Ker} T^*)^\perp$;

(e) $\operatorname{Ker} T = (\operatorname{Ran} T^*)^\perp$ *and* $\overline{\operatorname{Ran} T^*} = (\operatorname{Ker} T)^\perp$.

Proof. Parts (a)–(c) follow directly from the defining identity of the adjoint and uniqueness. For (d),

$$y \in \operatorname{Ker} T^* \iff \langle Tx, y\rangle = 0 \text{ for every } x \in H \iff y \perp \operatorname{Ran} T.$$

Taking orthogonal complements gives $\overline{\operatorname{Ran} T} = (\operatorname{Ker} T^*)^\perp$. Part (e) follows by applying (d) to T^*. □

Definition 2.10 (Selfadjoint, positive, isometric, unitary, and normal operators)**.** For $T \in \mathcal{B}(H)$:

- T is *selfadjoint* if $T = T^*$;
- T is *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in H$;
- T is an *isometry* if $\|Tx\| = \|x\|$ for all $x \in H$;
- T is *unitary* if $T^*T = TT^* = I$;
- T is *normal* if $T^*T = TT^*$.

Proposition 2.11 (Elementary characterizations)**.** *Let $T \in \mathcal{B}(H)$. Then the following assertions hold.*

(a) T is an isometry if and only if $T^*T = I$.

(b) T is unitary if and only if T is a surjective isometry.

(c) T is selfadjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for every $x \in H$.

(d) If T is normal, then $\|Tx\| = \|T^*x\|$ for every $x \in H$.

Proof. For (a), $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle$, so $\|Tx\| = \|x\|$ for all x if and only if $\langle (T^*T - I)x, x\rangle = 0$ for all x, equivalently $T^*T = I$. Part (b) is immediate from (a). For (c), one implication is obvious; the converse follows from the polarization identity applied to the sesquilinear form $(x, y) \mapsto \langle Tx, y\rangle$. For (d),

$$\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \|T^*x\|^2.$$

□

Proposition 2.12 (First spectral restrictions)**.** *If T is selfadjoint, then $\sigma(T) \subset \mathbb{R}$. If U is unitary, then $\sigma(U) \subset \{z \in \mathbb{C} : |z| = 1\}$.*

Proof. Assume first that $T = T^*$ and let $\lambda = a + ib \in \mathbb{C}$ with $b \neq 0$. For any $x \in H$,

$$\Im\langle (\lambda I - T)x, x\rangle = \Im\big(\lambda\|x\|^2 - \langle Tx, x\rangle\big) = b\,\|x\|^2,$$

because $\langle Tx, x\rangle \in \mathbb{R}$ for selfadjoint T. Hence

$$|b|\,\|x\|^2 \leq |\langle (\lambda I - T)x, x\rangle| \leq \|(\lambda I - T)x\|\,\|x\|.$$

Therefore

$$\|(\lambda I - T)x\| \geq |b|\,\|x\| \qquad (x \in H).$$

So $\lambda I - T$ is injective with closed range. If its range were proper, choose $0 \neq y$ orthogonal to the range. Then

$$0 = \langle (\lambda I - T)x, y\rangle = \langle x, (\overline{\lambda} I - T)y\rangle \qquad (x \in H),$$

which implies $(\overline{\lambda}I - T)y = 0$. Applying the same inequality to $\overline{\lambda}$ gives $y = 0$, a contradiction. Thus $\lambda I - T$ is surjective, hence invertible. Therefore $\lambda \in \rho(T)$, and so $\sigma(T) \subset \mathbb{R}$.

Next, let U be unitary. If $|\lambda| \neq 1$, then

$$\|Ux - \lambda x\|^2 = \|Ux\|^2 + \|\lambda x\|^2 - 2\Re\langle Ux, \lambda x\rangle = \|x\|^2 + |\lambda|^2\|x\|^2 - 2\Re\big(\lambda\langle Ux, x\rangle\big).$$

Since $|\langle Ux, x\rangle| \le \|Ux\|\,\|x\| = \|x\|^2$, we conclude

$$\|Ux - \lambda x\|^2 \ge (1 - |\lambda|)^2\|x\|^2.$$

Hence $U - \lambda I$ is injective with closed range. If its range were proper, choose $0 \neq y$ orthogonal to the range. Then

$$0 = \langle (U - \lambda I)x, y\rangle = \langle x, (U^* - \overline{\lambda}I)y\rangle \qquad (x \in H),$$

so $(U^* - \overline{\lambda}I)y = 0$. But U^* is also unitary, and the same estimate yields $y = 0$, a contradiction. Therefore $U - \lambda I$ is surjective and hence invertible. Thus $\lambda \in \rho(U)$ whenever $|\lambda| \neq 1$, proving the claim. □

Example 2.13 (A diagonal operator on ℓ^p)**.** For $1 \le p \le \infty$, define

$$T(x_1, x_2, x_3, \dots) = \Big(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, \dots\Big).$$

Then $T \in \mathcal{B}(\ell^p, \ell^p)$ and $\|T\| = 1$.

Proof. For $1 \le p < \infty$,

$$\|Tx\|_p^p = \sum_{n=1}^{\infty} \left|\frac{x_n}{n}\right|^p \le \sum_{n=1}^{\infty} |x_n|^p = \|x\|_p^p,$$

and similarly $\|Tx\|_\infty \le \|x\|_\infty$ when $p = \infty$. Thus $\|T\| \le 1$. Equality follows by evaluating T at $e_1 = (1, 0, 0, \dots)$. □

Example 2.14 (The Volterra operator on continuous functions)**.** Define $T : C([0,1]) \to C([0,1])$ by

$$(Tf)(x) = \int_0^x f(t)\,dt.$$

Then $T \in \mathcal{B}(C([0,1]), C([0,1]))$ and $\|T\| = 1$.

Proof. For $x \in [0,1]$,

$$|(Tf)(x)| \le \int_0^x |f(t)|\,dt \le \|f\|_\infty,$$

so $\|Tf\|_\infty \le \|f\|_\infty$ and hence $\|T\| \le 1$. Taking $f \equiv 1$ gives $(Tf)(x) = x$, so $\|Tf\|_\infty = 1 = \|f\|_\infty$. □

Example 2.15 (Integral operators on function spaces)**.** Let $\varphi : [0,1] \times [0,1] \to \mathbb{C}$ be continuous, and define

$$(Tf)(x) = \int_0^1 \varphi(x,t) f(t)\, dt.$$

Then T defines a bounded operator on $C([0,1])$, with

$$\|T\| \le \sup_{x\in[0,1]} \int_0^1 |\varphi(x,t)|\, dt.$$

Moreover, the same formula defines a bounded operator $T : L^2([0,1]) \to L^2([0,1])$, and

$$\|T\| \le \left(\int_0^1 \int_0^1 |\varphi(x,t)|^2\, dt\, dx\right)^{1/2}.$$

Proof. For $f \in C([0,1])$,

$$|(Tf)(x)| \le \|f\|_\infty \int_0^1 |\varphi(x,t)|\, dt,$$

which gives the stated bound after taking the supremum in x.

For $f \in L^2([0,1])$, the Cauchy–Schwarz inequality gives

$$|(Tf)(x)| \le \left(\int_0^1 |\varphi(x,t)|^2\, dt\right)^{1/2} \|f\|_2.$$

Squaring and integrating in x yields the L^2 estimate. □

Example 2.16 (Cesàro averaging operator)**.** Let $BC([0,\infty))$ denote the space of bounded continuous functions on $[0,\infty)$, equipped with the supremum norm. Define

$$(Tf)(t) = \begin{cases} \frac{1}{t}\int_0^t f(s)\, ds, & t > 0, \\ f(0), & t = 0. \end{cases}$$

Then $T \in \mathcal{B}(BC([0,\infty)), BC([0,\infty)))$ and $\|T\| = 1$.

Proof. For $t > 0$,

$$|(Tf)(t)| \le \frac{1}{t}\int_0^t |f(s)|\, ds \le \|f\|_\infty,$$

and the same bound holds at $t = 0$. Hence $\|Tf\|_\infty \le \|f\|_\infty$. Continuity at $t = 0$ follows from

$$\lim_{t\downarrow 0} \frac{1}{t}\int_0^t f(s)\, ds = f(0),$$

by continuity of f. Equality $\|T\| = 1$ follows from $T\mathbf{1} = \mathbf{1}$. □

Example 2.17 (Differentiation)**.** The differentiation operator need not be bounded unless the domain is equipped with a stronger norm.

(i) The map $D : C^1([0,1]) \to C([0,1])$, $Df = f'$, is bounded when $C^1([0,1])$ is endowed with the norm
$$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty.$$
In fact, $\|Df\|_\infty \le \|f\|_{C^1}$, so $\|D\| \le 1$, and equality holds.

(ii) If one instead regards differentiation as a map from $(C^1([0,1]), \|\cdot\|_\infty)$ to $(C([0,1]), \|\cdot\|_\infty)$, then it is unbounded. Indeed, for $f_n(t) = t^n$ we have $\|f_n\|_\infty = 1$ but $\|f_n'\|_\infty = n \to \infty$.

Example 2.18 (The unilateral shift and its adjoint)**.** On $\ell^2(\mathbb{N})$ define
$$S(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$
A direct computation shows that
$$S^*(x_1, x_2, \dots) = (x_2, x_3, \dots).$$
Thus $S^*S = I$, whereas $SS^* = I - P_{\mathrm{span}\{e_1\}}$. In particular, S is an isometry but not a unitary operator. This example will reappear when we discuss invariant subspaces and spectral behaviour.

2.4 Extension of uniformly continuous functions

Completion arguments are useful only if one can extend the maps already defined on a dense set. Uniform continuity is precisely the hypothesis that makes such extension possible without ambiguity, and the result proved here will later be adapted to linear operators defined initially on a convenient core.

Proposition 2.19. *Let A be a dense subset of a metric space (M, d), and let $f : A \to \mathbb{R}$ be uniformly continuous. Then there exists a unique uniformly continuous extension $\widehat{f} : \overline{A} \to \mathbb{R}$ such that $\widehat{f}|_A = f$.*

Proof. Fix $x \in \overline{A}$. Choose a sequence $(x_n) \subset A$ with $x_n \to x$. Since (x_n) is Cauchy and f is uniformly continuous, $(f(x_n))$ is Cauchy in $\mathbb{R}$, hence convergent. Define
$$\widehat{f}(x) := \lim_{n\to\infty} f(x_n).$$
A standard argument shows that this value does not depend on the approximating sequence, so $\widehat{f}$ is well defined.

To prove uniform continuity, let $\varepsilon > 0$, and choose $\delta > 0$ such that
$$d(u, v) < \delta \quad \Longrightarrow \quad |f(u) - f(v)| < \varepsilon \qquad (u, v \in A).$$

Take $x, y \in \overline{A}$ with $d(x, y) < \delta/3$, and choose sequences $x_n, y_n \in A$ converging to x and y, respectively. For large n we have $d(x_n, x) < \delta/3$ and $d(y_n, y) < \delta/3$, hence $d(x_n, y_n) < \delta$. Therefore $|f(x_n) - f(y_n)| < \varepsilon$, and letting $n \to \infty$ yields

$$|\widehat{f}(x) - \widehat{f}(y)| \leq \varepsilon.$$

Thus $\widehat{f}$ is uniformly continuous.

Uniqueness follows because any two continuous extensions agreeing on the dense set A must agree on $\overline{A}$. □

2.5 Extension theorem

In analysis one frequently defines an operator first on a dense subspace of simple or smooth vectors and only afterwards asks whether it extends to the whole space. The extension theorem gives the precise criterion for doing so while preserving linearity and norm control.

Theorem 2.20 (Extension from a dense subspace)**.** *Let M be a dense subspace of a normed space X, and let Y be a Banach space. If $T : M \to Y$ is bounded and linear, then there exists a unique bounded linear extension $\widehat{T} : X \to Y$ such that*

$$\widehat{T}|_M = T \qquad \text{and} \qquad \|\widehat{T}\| = \|T\|.$$

Proof. For $x \in X$, choose a sequence $(x_n) \subset M$ with $x_n \to x$. Since T is bounded,

$$\|Tx_n - Tx_m\| \leq \|T\| \, \|x_n - x_m\|,$$

so (Tx_n) is Cauchy in Y. As Y is Banach, (Tx_n) converges. Define

$$\widehat{T}(x) := \lim_{n\to\infty} Tx_n.$$

The limit is independent of the approximating sequence, so $\widehat{T}$ is well defined. Linearity follows from linearity of limits.

Moreover,

$$\|\widehat{T}(x)\| = \lim_{n\to\infty} \|Tx_n\| \leq \|T\| \lim_{n\to\infty} \|x_n\| = \|T\| \, \|x\|,$$

so $\widehat{T}$ is bounded and $\|\widehat{T}\| \leq \|T\|$. Since $\widehat{T}$ extends T, we also have $\|T\| \leq \|\widehat{T}\|$. Hence $\|\widehat{T}\| = \|T\|$. Uniqueness follows from continuity and density of M. □

Example 2.21. Every linear map from a finite-dimensional normed space X into a normed space Y is bounded.

Proof. Choose a basis $e_1, \dots, e_n$ of X. For $x = \sum_{j=1}^n a_j e_j$,

$$\|Tx\| \le \sum_{j=1}^n |a_j| \, \|Te_j\|.$$

Since all norms on X are equivalent, there exists $C > 0$ such that $\sum_{j=1}^n |a_j| \le C\|x\|$. Therefore

$$\|Tx\| \le C\Big(\max_{1\le j\le n} \|Te_j\|\Big)\|x\|,$$

which proves boundedness. □

2.6 Open mapping theorem

Completeness has decisive consequences for surjective linear maps between Banach spaces. The open mapping theorem shows that surjectivity is automatically quantitative: a bounded onto operator does not merely hit every point, but does so with uniform control on neighbourhoods.

Theorem 2.22 (Open mapping theorem)**.** *Let X and Y be Banach spaces and let $T \in \mathcal{B}(X, Y)$ be surjective. Then T is an open map: for every open set $U \subset X$, the image $T(U)$ is open in Y.*

Proof. It suffices to prove that $T(B_X(0,1))$ contains a ball about 0; then scaling and translation yield openness for arbitrary open sets.

Since T is surjective,

$$Y = \bigcup_{n=1}^{\infty} T(B_X(0,n)) = \bigcup_{n=1}^{\infty} n\, T(B_X(0,1)).$$

Set $E := \overline{T(B_X(0,1))}$. Then $Y = \bigcup_{n\ge1} nE$. Each nE is closed, so by the Baire category theorem there exist $n_0 \in \mathbb{N}$ and $y_0 \in Y$ and $r > 0$ such that

$$B_Y(y_0, r) \subset n_0 E.$$

By translation invariance of balls,

$$B_Y(0,r) \subset n_0E - n_0E \subset n_0\overline{T(B_X)} - n_0\overline{T(B_X)} \subset \overline{T(B_X(0,2n_0))}.$$

After rescaling X (replace $B_X(0,1)$ by $B_X(0,2n_0)$), we may assume that

$$B_Y(0,1) \subset \overline{T(B_X(0,1))}. \tag{2.1}$$

Claim. Under (2.1), one has $B_Y(0, \frac{1}{2}) \subset T(B_X(0,1))$.

Fix $y \in Y$ with $\|y\| < \frac{1}{2}$. By (2.1) there exists $x_1 \in B_X(0,1)$ such that $\|y - Tx_1\| < 2^{-2}$. Inductively, having chosen $x_1, \dots, x_k$ with $\|x_j\| \le 2^{-(j-1)}$ and

$$\|y - T(x_1 + \cdots + x_k)\| < 2^{-(k+1)},$$

apply (2.1) to the residual $y_k := 2^k\big(y - T(x_1 + \cdots + x_k)\big)$ (which satisfies $\|y_k\| < \frac{1}{2} < 1$) to choose $u_{k+1} \in B_X(0,1)$ with $\|y_k - Tu_{k+1}\| < 2^{-2}$. Put $x_{k+1} := 2^{-k}u_{k+1}$. Then $\|x_{k+1}\| \le 2^{-k}$ and

$$\|y - T(x_1 + \cdots + x_{k+1})\| < 2^{-(k+2)}.$$

Thus the series $\sum_{j\ge1} x_j$ converges in X (since $\sum \|x_j\| \le \sum 2^{-(j-1)} = 2$), and letting $x := \sum_{j\ge1} x_j$ we have $\|x\| \le 2$ and $Tx = y$ by continuity of T. Scaling x by a factor $1/2$ gives $y \in T(B_X(0,1))$, establishing the claim.

Returning to the original scaling, we conclude that $T(B_X(0,1))$ contains a nontrivial open ball, hence T is open. □

2.7 Inverse mapping theorem

Bijectivity by itself is not enough in infinite-dimensional analysis; one must also know that the inverse remains bounded. The inverse mapping theorem provides exactly this conclusion for Banach spaces and is one of the principal reasons completeness enters the subject at every stage.

Corollary 2.23 (Inverse mapping theorem)**.** *Let X and Y be Banach spaces and let $T \in \mathcal{B}(X,Y)$ be bijective. Then $T^{-1} : Y \to X$ is bounded.*

Proof. By Theorem 2.22, T is an open map. Hence $T(B_X(0,1))$ contains a ball $B_Y(0,r)$ for some $r > 0$. If $y \in Y$ and $y \ne 0$, write $y = \|y\| \frac{y}{\|y\|}$ and note that $\frac{r}{\|y\|}y \in B_Y(0,r)$, so $\frac{r}{\|y\|}y = Tx$ for some x with $\|x\| \le 1$. Therefore

$$\|T^{-1}y\| = \|\tfrac{\|y\|}{r}x\| \le \tfrac{1}{r}\|y\|.$$

Thus $\|T^{-1}\| \le 1/r$. □

2.8 Closed graph theorem

Direct norm estimates are not always the most efficient route to boundedness. The closed graph theorem replaces them with a geometric condition on the graph of the operator and thereby yields a remarkably robust criterion for continuity in Banach space settings.

Theorem 2.24 (Closed graph theorem)**.** *Let X and Y be Banach spaces and let $T : X \to Y$ be linear. Then T is bounded if and only if its graph*

$$\mathcal{G}(T) = \{(x, Tx) : x \in X\} \subset X \times Y$$

is closed (with respect to the product norm $\|(x,y)\| := \|x\| + \|y\|$).

Proof. If T is bounded and $(x_n, Tx_n) \to (x, y)$ in $X \times Y$, then $x_n \to x$ and $Tx_n \to y$. By continuity of T, we also have $Tx_n \to Tx$, hence $y = Tx$. Therefore $(x, y) \in \mathcal{G}(T)$, so the graph is closed.

Conversely, assume that $\mathcal{G}(T)$ is closed. Since $X \times Y$ is Banach, $\mathcal{G}(T)$ is Banach as a closed subspace. Consider the projection

$$P : \mathcal{G}(T) \to X, \qquad P(x, Tx) = x.$$

This map is linear, bijective, and bounded because

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|.$$

By the inverse mapping theorem, $P^{-1} : X \to \mathcal{G}(T)$ is bounded. Therefore there exists $C > 0$ such that

$$\|x\| + \|Tx\| = \|P^{-1}x\| \leq C\|x\| \qquad (x \in X).$$

Hence $\|Tx\| \leq (C - 1)\|x\|$, so T is bounded. □

Remark 2.25. Completeness is essential in Theorems 2.22–2.24. For instance, the identity map

$$I : (\ell^1, \|\cdot\|_1) \to (\ell^1, \|\cdot\|_\infty)$$

is bounded, but its inverse is not bounded, because the ℓ^1 norm cannot be controlled by the ℓ^∞ norm on ℓ^1.

Example 2.26 (Coordinatewise multiplication on ℓ^p)**.** Let (a_n) be a scalar sequence with the property that for every $x = (x_n) \in \ell^p$, the sequence $(a_n x_n)$ again belongs to ℓ^p. Define

$$T : \ell^p \to \ell^p, \qquad T(x_n) = (a_n x_n).$$

Then T is bounded.

Proof. We verify that the graph of T is closed. Suppose $x^{(k)} \to x$ in ℓ^p and $Tx^{(k)} \to y$ in ℓ^p. Then for each coordinate n,

$$x_n^{(k)} \to x_n \qquad \text{and} \qquad a_n x_n^{(k)} \to y_n.$$

Hence $y_n = a_n x_n$ for every n, so $y = Tx$. Therefore the graph of T is closed, and the closed graph theorem implies that T is bounded. □

Example 2.27 (Multiplication operators on $L^1(\mathbb{R})$)**.** If $\varphi \in L^\infty(\mathbb{R})$ and $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is defined by

$$T(f) = \varphi f,$$

then T is bounded and

$$\|T\| = \|\varphi\|_\infty.$$

Conversely, if φ is measurable and $\varphi f \in L^1(\mathbb{R})$ for every $f \in L^1(\mathbb{R})$, then the same formula defines a bounded operator on $L^1(\mathbb{R})$; in particular, necessarily $\varphi \in L^\infty(\mathbb{R})$.

Proof. If $\varphi \in L^\infty$, then

$$\|T(f)\|_1 = \int_{\mathbb{R}} |\varphi(x) f(x)|\, dx \le \|\varphi\|_\infty \|f\|_1,$$

so $\|T\| \le \|\varphi\|_\infty$. To obtain the reverse inequality, fix $\varepsilon > 0$ and consider

$$E_\varepsilon := \{x \in \mathbb{R} : |\varphi(x)| > \|\varphi\|_\infty - \varepsilon\}.$$

This set has positive measure. Choose a measurable subset $F \subset E_\varepsilon$ with $0 < m(F) < \infty$, and set

$$f_\varepsilon := \frac{\chi_F}{m(F)}.$$

Then $\|f_\varepsilon\|_1 = 1$, and

$$\|T f_\varepsilon\|_1 = \frac{1}{m(F)} \int_F |\varphi(x)|\, dx \ge \|\varphi\|_\infty - \varepsilon.$$

Hence $\|T\| \ge \|\varphi\|_\infty - \varepsilon$, and letting $\varepsilon \downarrow 0$ gives $\|T\| = \|\varphi\|_\infty$.

For the converse, assume merely that $\varphi f \in L^1(\mathbb{R})$ for every $f \in L^1(\mathbb{R})$. Define $T(f) = \varphi f$. If $f_n \to f$ in L^1 and $T f_n \to g$ in L^1, then by passing to a subsequence we may assume that $f_n(x) \to f(x)$ and $T f_n(x) \to g(x)$ almost everywhere. Thus $g(x) = \varphi(x) f(x)$ almost everywhere, so the graph of T is closed. By the closed graph theorem, T is bounded. Applying the first part now shows that necessarily $\varphi \in L^\infty(\mathbb{R})$ and $\|T\| = \|\varphi\|_\infty$. □

Remark 2.28. The same argument works on $L^p(\mathbb{R})$ for every $1 \le p < \infty$: whenever multiplication by a measurable function φ sends $L^p(\mathbb{R})$ into itself for every input, the resulting operator is bounded, and necessarily $\varphi \in L^\infty(\mathbb{R})$.

2.8.1 Banach–Steinhaus theorem (uniform boundedness principle)

A family of continuous linear maps can be pointwise bounded without being uniformly bounded on the whole unit ball if the underlying space is incomplete. Completeness is precisely the ingredient that rules out this pathology.

Theorem 2.29 (Banach–Steinhaus theorem)**.** *Let X be a Banach space, let Y be a normed space, and let $\{T_i : i \in I\} \subset \mathcal{B}(X, Y)$. Define*

$$E := \Big\{x \in X : \sup_{i \in I} \|T_i x\| = \infty\Big\}.$$

Then exactly one of the following alternatives holds:

(i) $\sup_{i\in I} \|T_i\| < \infty$;

(ii) *E is a dense G_δ subset of X.*

Proof. For each $n \in \mathbb{N}$, let
$$E_n := \Big\{x \in X : \sup_{i\in I} \|T_i x\| > n\Big\}.$$
Then
$$X \setminus E_n = \bigcap_{i\in I} \{x \in X : \|T_i x\| \le n\},$$
which is closed because each T_i is continuous. Hence every E_n is open, and
$$E = \bigcap_{n=1}^{\infty} E_n.$$

Assume first that each E_n is dense in X. By the Baire category theorem, $E = \bigcap_{n\ge1} E_n$ is a dense G_δ set, so alternative (ii) holds.

Now assume that some E_n is not dense. Then there exists a ball $B(x_0, r)$ such that
$$B(x_0, r) \cap E_n = \varnothing.$$
Equivalently,
$$\sup_{i\in I} \|T_i x\| \le n \qquad (x \in B(x_0, r)).$$
If $\|x\| < r$, then $x + x_0 \in B(x_0, r)$, and therefore
$$\|T_i x\| \le \|T_i(x + x_0)\| + \|T_i x_0\| \le 2n \qquad (i \in I).$$
Next, let $y \in X$ be arbitrary, $y \ne 0$, and set $x = \frac{r}{2\|y\|} y$. Then $\|x\| = r/2 < r$, so
$$\left\| T_i \left(\frac{r}{2\|y\|} y \right) \right\| \le 2n.$$
Hence
$$\|T_i y\| \le \frac{4n}{r} \|y\| \qquad (i \in I),$$
which implies $\|T_i\| \le 4n/r$ for all $i \in I$. Thus alternative (i) holds. □

Corollary 2.30 (Uniform boundedness principle)**.** *Let $\{T_i\}_{i\in I} \subset \mathcal{B}(X, Y)$, where X is Banach, and suppose that for each $x \in X$,*
$$\sup_{i\in I} \|T_i x\| < \infty.$$
Then
$$\sup_{i\in I} \|T_i\| < \infty.$$

Proof. Under the pointwise boundedness hypothesis, the exceptional set E in Theorem 2.29 is empty. Hence alternative (ii) cannot occur, and so alternative (i) must hold. $\square$

Remark 2.31. Completeness of X is essential. Consider the vector space $P(\mathbb{R})$ of all real polynomials, equipped with the norm

$$\|p\|_1 := \sup_{j\geq 0} |a_j|, \qquad p(x) = a_0 + a_1 x + \cdots + a_d x^d.$$

For $n \in \mathbb{N}$ define

$$T_n(p) := \sum_{j=0}^{n} a_j.$$

For each fixed polynomial p, the sequence $\{T_n(p)\}$ is eventually constant, so the family $\{T_n\}$ is pointwise bounded. On the other hand, if $p_n(x) = 1 + x + \cdots + x^n$, then $\|p_n\|_1 = 1$ and $T_n(p_n) = n + 1$, so $\|T_n\| = n + 1 \to \infty$. Thus uniform boundedness fails on this incomplete normed space.

Corollary 2.32 (Pointwise limits of bounded operators)**.** *Let X be a Banach space, let Y be a normed space, and let $(T_n) \subset \mathcal{B}(X, Y)$. Suppose that for every $x \in X$ the limit*

$$T(x) := \lim_{n\to\infty} T_n(x)$$

exists in Y. Then $T \in \mathcal{B}(X, Y)$ and

$$\|T\| \leq \limsup_{n\to\infty} \|T_n\|.$$

In particular, whenever $\|T_n\| \to L$, then $\|T\| \leq L$.

Proof. For each $x \in X$, the sequence $\{T_n x\}$ converges and is therefore bounded. By the uniform boundedness principle, there exists $M > 0$ such that $\|T_n\| \leq M$ for all n. Passing to the limit in

$$\|T_n x\| \leq M\|x\|$$

gives $\|Tx\| \leq M\|x\|$, so T is bounded.

Moreover, for every $x \in X$,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \Big(\limsup_{n\to\infty} \|T_n\|\Big)\|x\|.$$

Taking the supremum over $\|x\| \leq 1$ yields the stated bound. $\square$

2.8.2 Application: divergence of Fourier series for some continuous functions

For $f \in C([-\pi, \pi])$, its Fourier coefficients are defined by

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt, \qquad k \in \mathbb{Z},$$

and the nth symmetric partial sum of the Fourier series is

$$S_n(f)(x) := \sum_{k=-n}^{n} \widehat{f}(k)e^{ikx}.$$

Using the Dirichlet kernel

$$D_n(t) := \sum_{k=-n}^{n} e^{ikt} = \frac{\sin((n+\frac{1}{2})t)}{\sin(t/2)} \qquad (t \notin 2\pi\mathbb{Z}),$$

we may write

$$S_n(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)D_n(x-s)\,ds.$$

We next use the uniform boundedness principle to show that the Fourier series of a continuous function need not converge even at a single point.

Proposition 2.33. *For $f \in C([-\pi,\pi])$, define*

$$T_n(f) := S_n(f)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(t)\,dt.$$

Then $T_n \in \mathcal{B}(C([-\pi,\pi]),\mathbb{C})$ and

$$\|T_n\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt.$$

Proof. The upper bound is immediate:

$$|T_n(f)| \le \|f\|_\infty \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt,$$

so

$$\|T_n\| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt.$$

For the reverse inequality, approximate the sign of D_n by continuous functions. For $\delta > 0$ define

$$f_{n,\delta}(t) := \frac{D_n(t)}{|D_n(t)|+\delta}.$$

Then $f_{n,\delta} \in C([-\pi,\pi])$ and $\|f_{n,\delta}\|_\infty \le 1$. Moreover,

$$f_{n,\delta}(t)D_n(t) = \frac{D_n(t)^2}{|D_n(t)|+\delta} \longrightarrow |D_n(t)| \qquad (\delta \downarrow 0),$$

and the integrands are dominated by $|D_n| \in L^1([-\pi,\pi])$. By dominated convergence,

$$\lim_{\delta\downarrow 0} T_n(f_{n,\delta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt.$$

Since $\|f_{n,\delta}\|_\infty \le 1$, this shows that $\|T_n\|$ is at least the right-hand side. Therefore equality holds. $\square$

Lemma 2.34 (Growth of the L^1 norm of the Dirichlet kernel)**.** *There exists an absolute constant $c > 0$ such that, for all $n \ge 1$,*

$$\int_{-\pi}^{\pi} |D_n(t)|\,dt \ge c\log(n+1).$$

In particular,

$$\|D_n\|_{L^1([-\pi,\pi])} \to \infty.$$

Proof. Recall that

$$D_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin(t/2)}.$$

For $m = 0, 1, \dots, n-1$ set

$$t_m := \frac{(2m+1)\pi}{2n+1}, \qquad \delta := \frac{\pi}{4n+2}, \qquad I_m := [t_m - \delta,\, t_m + \delta] \subset (0, \pi).$$

The intervals I_m are pairwise disjoint. If $t \in I_m$, then

$$\Big|(n+\tfrac{1}{2})t - (n+\tfrac{1}{2})t_m\Big| \le (n+\tfrac{1}{2})\delta = \frac{\pi}{4},$$

so

$$\Big|\sin((n+\tfrac{1}{2})t)\Big| \ge \sin(\pi/4) = \frac{\sqrt{2}}{2}.$$

Also, for $t \in (0,\pi)$ we have $\sin(t/2) \le t/2$, hence $1/|\sin(t/2)| \ge 2/t$. Therefore

$$|D_n(t)| \ge \frac{\sqrt{2}}{t} \qquad (t \in I_m).$$

Integrating over these disjoint intervals gives

$$\int_0^{\pi} |D_n(t)|\,dt \ge \sqrt{2}\sum_{m=0}^{n-1}\int_{t_m-\delta}^{t_m+\delta}\frac{dt}{t} \ge \sqrt{2}\sum_{m=0}^{n-1}\frac{2\delta}{t_m+\delta}.$$

A direct computation shows that $\frac{2\delta}{t_m+\delta} = \frac{2}{4m+3}$. Hence

$$\int_0^{\pi} |D_n(t)|\,dt \ge \sqrt{2}\sum_{m=0}^{n-1}\frac{2}{4m+3} \ge \frac{\sqrt{2}}{2}\sum_{m=1}^{n}\frac{1}{m} \ge \frac{\sqrt{2}}{2}\log(n+1).$$

Doubling the estimate proves the claim. $\square$

Theorem 2.35 (A dense set of continuous functions with divergent Fourier series)**.** *There exists a dense G_δ set $G \subset C([-\pi, \pi])$ such that for every $f \in G$,*

$$\sup_{n \geq 1} |S_n(f)(0)| = \infty.$$

In particular, the Fourier series of f diverges at 0*.*

Proof. By Proposition 2.33 and Lemma 2.34,

$$\|T_n\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt \xrightarrow[n \to \infty]{} \infty.$$

Thus the family $\{T_n\}_{n \geq 1} \subset \mathcal{B}(C([-\pi, \pi]), \mathbb{C})$ is not uniformly bounded. Since $C([-\pi, \pi])$ is Banach, Theorem 2.29 implies that the set

$$G := \left\{ f \in C([-\pi, \pi]) : \sup_{n \geq 1} |T_n(f)| = \infty \right\}$$

is a dense G_δ subset of $C([-\pi, \pi])$. But $T_n(f) = S_n(f)(0)$, so the asserted divergence follows. □

Summary: the three fundamental principles

In this chapter we have seen how completeness (via the Baire category theorem) forces strong operator-theoretic consequences. The *uniform boundedness principle* controls operator norms from pointwise bounds; the *open mapping theorem* upgrades surjectivity to quantitative openness; and the *closed graph theorem* turns graph-closedness into boundedness. These results will be used repeatedly in duality and spectral theory.

Chapter 3

Orthogonal Projections and Canonical Operator Classes

This chapter studies the operator classes that arise naturally from Hilbert-space geometry. We discuss orthogonal projections, isometries, unitary operators, finite-rank, compact, and Hilbert–Schmidt operators, as well as the first structural properties of selfadjoint and normal operators. The aim is to organize the main classes that repeatedly appear in spectral theory and in decomposition arguments.

Learning objectives.

- Characterize orthogonal projections algebraically and geometrically.
- Distinguish isometries from unitary operators and finite-rank operators from compact ones.
- Place selfadjoint and normal operators in the hierarchy of canonical Hilbert-space operator classes.

Key ideas.

- Projection identities encode orthogonal decomposition.
- Isometries preserve geometry but need not be onto.
- Compactness is an approximation property and normality is a commutation property.

3.1 Orthogonal projections

Orthogonal projection is the basic Hilbert-space mechanism for turning geometry into analysis. It encodes best approximation by closed subspaces, yields canonical decompositions, and later reappears in functional calculus, spectral theory, and the study of invariant subspaces.

Theorem 3.1 (Characterization of orthogonal projections)**.** *For $P \in \mathcal{B}(H)$, the following are equivalent:*

(i) P *is an orthogonal projection;*

(ii) *there exists a closed subspace* $M \subset H$ *such that* $P = P_M$*, the orthogonal projection onto* M*;*

(iii) $P^2 = P$ *and* $\operatorname{Ran}(P) \perp \operatorname{Ker}(P)$.

In this case,

$$\operatorname{Ran}(P) = M, \qquad \operatorname{Ker}(P) = M^{\perp}, \qquad H = \operatorname{Ran}(P) \oplus \operatorname{Ker}(P).$$

Proof. The implication (ii)⇒(i) was established in Theorem 1.7. If (i) holds and $M = \mathrm{Ran}(P)$, then for every $x \in H$ we may write

$$x = Px + (I - P)x,$$

where $Px \in M$. Moreover, for every $y \in H$,

$$\langle Px, (I - P)y \rangle = \langle x, P(I - P)y \rangle = \langle x, (P - P^2)y \rangle = 0,$$

so $(I - P)y \in M^\perp$. Thus x has the orthogonal decomposition relative to M, which means $P = P_M$ and hence (i)⇒(ii). Finally, (ii) clearly implies (iii), and if (iii) holds then for $x = u + v$ with $u \in \mathrm{Ran}(P)$ and $v \in \mathrm{Ker}(P)$ one has $Px = u$, so P is the orthogonal projection onto $\mathrm{Ran}(P)$. □

Proposition 3.2 (Norm and minimization properties)**.** *Let $P = P_M$ be the orthogonal projection onto a closed subspace $M \subset H$. Then the following assertions hold.*

(a) $\|x - Px\| = \mathrm{dist}(x, M)$ *for every* $x \in H$*;*

(b) $\|P\| = 1$ *if* $M \neq \{0\}$ *and* $\|I - P\| = 1$ *if* $M \neq H$*;*

(c) a closed subspace M reduces $T \in \mathcal{B}(H)$ if and only if $TP = PT$.

Proof. Parts (a) and (b) are immediate from Theorem 1.7. For (c), $TP = PT$ implies both $\mathrm{Ran}(P) = M$ and $\mathrm{Ker}(P) = M^\perp$ are invariant under T. Conversely, if M reduces T, then both M and $M^\perp$ are invariant, and checking on the orthogonal decomposition $H = M \oplus M^\perp$ shows that T commutes with P. □

3.2 Isometries and unitary operators

Isometries and unitary operators are the linear symmetries of Hilbert-space geometry. They preserve lengths and inner products, and they serve both as structural model operators and as concrete realizations of shifts, Fourier transforms, and other symmetry phenomena.

Proposition 3.3 (Equivalent descriptions)**.** *Let $U \in \mathcal{B}(H)$. Then the following assertions hold.*

(a) U is an isometry if and only if $U^*U = I$;

(b) U is unitary if and only if $U^*U = UU^* = I$;

(c) U is unitary if and only if it is surjective and preserves the inner product:

$$\langle Ux, Uy \rangle = \langle x, y \rangle \qquad (x, y \in H).$$

Proof. Part (a) has already been noted in Theorem 2.11. Part (b) is the definition of unitarity together with surjectivity of isometries. For (c), the polarization identity shows that preservation of the norm is equivalent to preservation of the inner product, and surjectivity then gives the inverse relation $U^{-1} = U^*$. □

Remark 3.4. Unitary operators are the Hilbert-space analogues of orthogonal matrices in Euclidean geometry. They preserve distances, angles, orthogonality, orthonormal bases, and spectral data up to unitary equivalence.

3.3 Finite-rank, compact, and Hilbert–Schmidt operators

This section introduces the first hierarchy of operator classes that go beyond mere boundedness. Finite-rank operators are the closest analogues of matrices, compact operators preserve much of finite-dimensional spectral behaviour, and Hilbert–Schmidt operators form a quantitatively stronger class that already points toward trace and singular-value methods.

Definition 3.5. Let H and K be Hilbert spaces.

- $T \in \mathcal{B}(H, K)$ has *finite rank* if $\mathrm{Ran}(T)$ is finite-dimensional.
- T is *compact* if it maps bounded subsets of H to relatively compact subsets of K.
- T is *Hilbert–Schmidt* if for one (hence every) orthonormal basis (e_n) of H,

$$\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty.$$

Proposition 3.6 (Finite-rank operators)**.** *An operator $T \in \mathcal{B}(H, K)$ has finite rank if and only if there exist vectors $u_1, \dots, u_m \in H$ and $v_1, \dots, v_m \in K$ such that*

$$Tx = \sum_{j=1}^{m} \langle x, u_j \rangle v_j \qquad (x \in H).$$

Proof. If T has finite rank, choose a basis $v_1, \dots, v_m$ of $\mathrm{Ran}(T)$. Then $Tx = \sum_{j=1}^{m} \alpha_j(x) v_j$ for suitable linear functionals α_j on H. These functionals are bounded because $\mathrm{Ran}(T)$ is finite-dimensional, so by the Riesz representation theorem $\alpha_j(x) = \langle x, u_j \rangle$ for some $u_j \in H$. The converse is immediate. □

Proposition 3.7 (Compact operators as an ideal)**.** *Let H, K, and L be Hilbert spaces.*

(a) Every finite-rank operator is compact.

(b) The class $\mathcal{K}(H, K)$ of compact operators is a closed linear subspace of $\mathcal{B}(H, K)$.

(c) If $S \in \mathcal{B}(K, L)$ and $T \in \mathcal{K}(H, K)$, then $ST \in \mathcal{K}(H, L)$; similarly, if $R \in \mathcal{B}(G, H)$ then $TR \in \mathcal{K}(G, K)$.

Proof. Part (a) follows because bounded sets in finite-dimensional spaces are relatively compact. The linearity in (b) is immediate, and closedness is proved by a standard $\varepsilon/3$ argument using uniform convergence on the unit ball. Part (c) follows because continuous images of relatively compact sets are relatively compact. □

Theorem 3.8 (Hilbert–Schmidt operators)**.** *Let $T \in \mathcal{B}(H, K)$. If (e_n) and (f_m) are orthonormal bases of H and K, respectively, then the following are equivalent:*

(i) T is Hilbert–Schmidt;

(ii) $\sum_n \|Te_n\|^2 < \infty$;

(iii) $\sum_{m,n} |\langle Te_n, f_m\rangle|^2 < \infty$.

Moreover, the quantity

$$\|T\|_{HS}^2 := \sum_n \|Te_n\|^2 = \sum_{m,n} |\langle Te_n, f_m\rangle|^2$$

is independent of the chosen orthonormal bases, and

$$\|T\| \leq \|T\|_{HS}.$$

Every Hilbert–Schmidt operator is compact.

Proof. Parseval's identity shows that

$$\|Te_n\|^2 = \sum_m |\langle Te_n, f_m\rangle|^2,$$

and summing over n yields the equivalence of (ii) and (iii). Basis-independence follows from the same identity. The estimate $\|T\| \leq \|T\|_{HS}$ is immediate from

$$\|Tx\|^2 = \left\|\sum_n \langle x, e_n\rangle Te_n\right\|^2 \leq \left(\sum_n |\langle x, e_n\rangle|^2\right)\left(\sum_n \|Te_n\|^2\right) = \|x\|^2\|T\|_{HS}^2$$

by Cauchy–Schwarz and Parseval. Finally, if P_N denotes the orthogonal projection onto $\operatorname{span}\{e_1, \ldots, e_N\}$, then

$$\|T - TP_N\|_{HS}^2 = \sum_{n>N} \|Te_n\|^2 \longrightarrow 0.$$

Thus T is a norm-limit of finite-rank operators and hence compact. □

Corollary 3.9 (Adjoints and the Hilbert–Schmidt norm)**.** *If $T \in HS(H,K)$, then $T^* \in HS(K,H)$ and*

$$\|T^*\|_{HS} = \|T\|_{HS}.$$

Proof. Let (e_n) be an orthonormal basis of H and (f_m) an orthonormal basis of K. By Theorem 3.8,

$$\|T\|_{HS}^2 = \sum_{m,n} |\langle Te_n, f_m\rangle|^2.$$

Since $\langle Te_n, f_m\rangle = \langle e_n, T^*f_m\rangle$, Parseval's identity gives

$$\|T\|_{HS}^2 = \sum_{m,n} |\langle e_n, T^*f_m\rangle|^2 = \sum_m \|T^*f_m\|^2 = \|T^*\|_{HS}^2.$$

In particular, T^* is Hilbert–Schmidt. □

Theorem 3.10 (The Hilbert–Schmidt class is a Hilbert space)**.** *Let $HS(H,K)$ denote the Hilbert–Schmidt operators from H to K. For any orthonormal basis (e_n) of H, define*

$$\langle A, B\rangle_{HS} := \sum_{n=1}^{\infty} \langle Ae_n, Be_n\rangle, \qquad A, B \in HS(H,K).$$

Then the series converges absolutely, the value is independent of the chosen orthonormal basis, and

$$\|A\|_{HS}^2 = \langle A, A\rangle_{HS}.$$

With this inner product, $HS(H,K)$ is a Hilbert space.

Proof. Fix orthonormal bases (e_n) of H and (f_m) of K. By Parseval's identity,

$$\langle Ae_n, Be_n\rangle = \sum_m \langle Ae_n, f_m\rangle\, \overline{\langle Be_n, f_m\rangle}.$$

Hence

$$\sum_n |\langle Ae_n, Be_n\rangle| \le \sum_{m,n} |\langle Ae_n, f_m\rangle|\, |\langle Be_n, f_m\rangle| \le \|A\|_{HS}\, \|B\|_{HS}$$

by Cauchy–Schwarz on $\ell^2(\mathbb{N}\times\mathbb{N})$. Thus the series defining $\langle A, B\rangle_{HS}$ converges absolutely. Moreover,

$$\langle A, B\rangle_{HS} = \sum_{m,n} \langle Ae_n, f_m\rangle\, \overline{\langle Be_n, f_m\rangle},$$

which is symmetric in the choice of orthonormal basis and therefore basis-independent. The identity $\|A\|_{HS}^2 = \langle A, A\rangle_{HS}$ is immediate.

To prove completeness, let (A_j) be Cauchy in the Hilbert–Schmidt norm. Then for each fixed basis vector e_n we have

$$\|A_j e_n - A_k e_n\| \le \|A_j - A_k\|_{HS},$$

so $(A_j e_n)_j$ is Cauchy in K. Write $Ae_n := \lim_j A_j e_n$. Using Fatou's lemma for series,

$$\sum_n \|Ae_n\|^2 \le \liminf_{j\to\infty} \sum_n \|A_j e_n\|^2 < \infty,$$

so A is Hilbert–Schmidt. Finally,

$$\|A_j - A\|_{HS}^2 = \sum_n \|A_j e_n - Ae_n\|^2 \longrightarrow 0$$

by dominated convergence, because (A_j) is Cauchy and the tails are uniformly square-summable. Hence $HS(H,K)$ is complete. ☐

Example 3.11 (Hilbert–Schmidt integral operators)**.** Let (X, μ) be a σ-finite measure space and let $k \in L^2(X \times X, \mu \times \mu)$. Define

$$(Kf)(x) := \int_X k(x, y) f(y)\, d\mu(y), \qquad f \in L^2(X, \mu).$$

Then K is a bounded Hilbert–Schmidt operator on $L^2(X, \mu)$ and

$$\|K\|_{HS} = \|k\|_{L^2(X \times X)}.$$

In particular, every such integral operator is compact.

Proof. For almost every x, the section $k_x(y) := k(x, y)$ belongs to $L^2(X, \mu)$, and by Cauchy–Schwarz,

$$|(Kf)(x)| = |\langle k_x, \overline{f} \rangle_{L^2}| \le \|k_x\|_{L^2} \|f\|_{L^2}.$$

Thus Kf is defined for almost every x. Let (e_n) be any orthonormal basis of $L^2(X, \mu)$. Then Parseval's identity in the y-variable gives, for almost every x,

$$\sum_n |(Ke_n)(x)|^2 = \sum_n |\langle k_x, \overline{e_n} \rangle|^2 = \|k_x\|^2_{L^2(X)}.$$

Integrating over x and using Tonelli's theorem,

$$\sum_n \|Ke_n\|^2_{L^2(X)} = \int_X \|k_x\|^2_{L^2(X)}\, d\mu(x) = \int_{X \times X} |k(x, y)|^2\, d(\mu \times \mu)(x, y) = \|k\|^2_{L^2(X \times X)}.$$

Hence K is Hilbert–Schmidt with the stated norm formula, and therefore compact by Theorem 3.8. $\square$

3.4 Selfadjoint and normal operators

Selfadjoint and normal operators occupy a privileged place because the inner product strongly constrains their algebraic behaviour. They are the natural starting point for spectral theory, and many of the later theorems should be read as refinements of the elementary identities assembled here.

Proposition 3.12 (Basic properties of selfadjoint and normal operators)**.** *Let $T \in \mathcal{B}(H)$.*

(a) If $T = T^*$, then $\langle Tx, x \rangle \in \mathbb{R}$ for all $x \in H$, and every eigenvalue of T is real.

(b) If T is normal and $Tx = \lambda x$, then $T^* x = \overline{\lambda} x$.

(c) Eigenvectors corresponding to distinct eigenvalues of a normal operator are orthogonal.

(d) If T is positive, then T is selfadjoint and $\sigma(T) \subset [0, \infty)$.

Proof. Part (a) is immediate. If $Tx = \lambda x$ and T is normal, then

$$\|(T^* - \overline{\lambda} I)x\|^2 = \|T^* x\|^2 - \lambda \langle T^* x, x \rangle - \overline{\lambda} \langle x, T^* x \rangle + |\lambda|^2 \|x\|^2.$$

Using normality and $Tx = \lambda x$, one checks that this equals

$$\|Tx\|^2 - 2|\lambda|^2\|x\|^2 + |\lambda|^2\|x\|^2 = 0,$$

which proves (b). For (c), if $Tx = \lambda x$ and $Ty = \mu y$ with $\lambda \neq \mu$, then by (b),

$$\lambda\langle x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \overline{\mu}\langle x, y\rangle.$$

Interchanging the roles of x and y gives the same relation with λ and μ swapped, and therefore $\langle x, y\rangle = 0$. Part (d) is standard: positivity implies

$$\langle (T - T^*)x, x\rangle = 0 \qquad (x \in H),$$

hence $T = T^*$ by polarization, and the spectral inclusion follows from the same resolvent estimate used for selfadjoint operators. □

Remark 3.13. The full force of normality is revealed only in the spectral theorem. At this stage, the most important lesson is that the adjoint symmetry $TT^* = T^*T$ is precisely the condition under which orthogonality of eigenspaces and unitary diagonalization become possible.

Chapter 4

Spectrum, Resolvent, and Invariant Subspaces of Bounded Operators

This chapter develops the basic spectral theory of bounded operators on Hilbert spaces. After introducing the resolvent set and spectrum and establishing their foundational analytic properties, we prove the spectral radius formula, study the decomposition of the spectrum and its behaviour under adjoints, and then pass to invariant and reducing subspaces, where spectral information interacts with operator decomposition and block-matrix form.

Learning objectives.

- Define spectrum and resolvent in a way that is stable under perturbation and compatible with functional calculus.
- Use resolvent identities and analyticity to deduce global information about bounded operators.
- Distinguish point, continuous, residual, and approximate point spectrum in model examples.
- Relate invariant and reducing subspaces to operator decompositions and block-operator representations.

Key ideas.

- The spectrum records failure of invertibility rather than only failure of injectivity.
- Resolvent estimates are the analytic doorway into spectral theory.
- Invariant-subspace geometry turns spectral information into structural decompositions of operators.
- Diagonal and shift models should always be kept in mind when testing general statements.

Common pitfalls.

- Do not identify spectrum with eigenvalues outside compact or finite-dimensional settings.
- Do not assume that injective or surjective operators are automatically invertible on infinite-dimensional spaces.

Example 4.1 (Diagonal operators as a spectral model)**.** Let $D_\alpha : \ell^2 \to \ell^2$ be defined by

$$D_\alpha(x_n) = (\alpha_n x_n), \qquad (\alpha_n) \in \ell^\infty.$$

Then $D_\alpha \in \mathcal{B}(\ell^2)$, and for $\lambda \notin \overline{\{\alpha_n : n \in \mathbb{N}\}}$ one has

$$(\lambda I - D_\alpha)^{-1}(x_n) = \Big(\frac{x_n}{\lambda - \alpha_n}\Big).$$

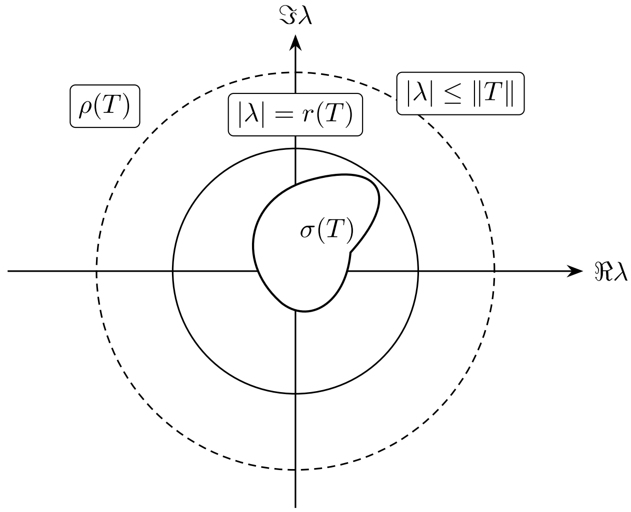

Figure 4.1: A schematic picture of the resolvent set and the spectrum of a bounded operator. The spectrum is compact and lies in the closed disk of radius $\|T\|$.

Hence

$$\sigma(D_\alpha) = \overline{\{\alpha_n : n \in \mathbb{N}\}}.$$

Moreover, D_α is selfadjoint if and only if each $\alpha_n \in \mathbb{R}$, positive if and only if $\alpha_n \ge 0$ for all n, and unitary if and only if $|\alpha_n| = 1$ for every n.

4.1 Resolvent and spectrum

The spectrum of an operator records the failure of invertibility and is therefore subtler than the set of eigenvalues alone. Introducing the resolvent alongside the spectrum makes it possible to combine operator theory with complex analysis, a viewpoint that drives much of modern spectral theory.

Lemma 4.2 (Neumann series). *If $\|S\| < 1$ for some $S \in \mathcal{B}(X)$, then $I - S$ is invertible and*

$$(I - S)^{-1} = \sum_{n=0}^{\infty} S^n,$$

with convergence in operator norm.

Proof. Since $\|S\| < 1$, the series $\sum_{n=0}^{\infty} \|S^n\|$ converges, so the operator series

$$A := \sum_{n=0}^{\infty} S^n$$

converges in $\mathcal{B}(X)$. For each N,

$$(I-S)\sum_{n=0}^{N} S^n = I - S^{N+1} = \sum_{n=0}^{N} S^n (I-S).$$

Letting $N \to \infty$ and using $\|S^{N+1}\| \le \|S\|^{N+1} \to 0$, we conclude

$$(I-S)A = A(I-S) = I.$$

Thus $A = (I-S)^{-1}$. □

Proposition 4.3. *The resolvent set $\rho(T)$ is open and the resolvent map*

$$R_T(\lambda) := (\lambda I - T)^{-1}$$

is holomorphic on $\rho(T)$. Consequently, $\sigma(T)$ is closed.

Proof. Let $\lambda_0 \in \rho(T)$. Then

$$\lambda I - T = (\lambda_0 I - T)\Big(I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}\Big).$$

If

$$|\lambda - \lambda_0| < \|(\lambda_0 I - T)^{-1}\|^{-1},$$

then

$$\|(\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}\| < 1.$$

By theorem 4.2, the factor in parentheses is invertible, so $\lambda \in \rho(T)$. Hence $\rho(T)$ is open.

Moreover, in that same neighborhood,

$$R_T(\lambda) = \Big(I - (\lambda_0 - \lambda)R_T(\lambda_0)\Big)^{-1} R_T(\lambda_0) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R_T(\lambda_0)^{n+1},$$

an operator-norm convergent power series in λ. Thus R_T is holomorphic on $\rho(T)$. Since $\sigma(T) = \mathbb{C} \setminus \rho(T)$, the spectrum is closed. □

Theorem 4.4 (Non-emptiness of the spectrum)**.** *If $X \neq \{0\}$ is complex and $T \in \mathcal{B}(X)$, then $\sigma(T) \neq \emptyset$.*

Proof. Assume, for contradiction, that $\sigma(T) = \emptyset$. Then $\rho(T) = \mathbb{C}$, so the resolvent

$$R_T(\lambda) = (\lambda I - T)^{-1}$$

is an entire operator-valued function.

Fix $x \in X$ and $f \in X^*$, and define the scalar entire function

$$g(\lambda) := f(R_T(\lambda)x).$$

For $|\lambda| > \|T\|$, we may write

$$\lambda I - T = \lambda\Big(I - \lambda^{-1}T\Big),$$

hence by the Neumann series,

$$R_T(\lambda) = \lambda^{-1}\sum_{n=0}^{\infty} \lambda^{-n}T^n.$$

Therefore

$$\|R_T(\lambda)\| \le \frac{1}{|\lambda|}\sum_{n=0}^{\infty}\Big(\frac{\|T\|}{|\lambda|}\Big)^n = \frac{1}{|\lambda| - \|T\|} \qquad (|\lambda| > \|T\|),$$

so g is bounded outside a large disk. Being entire, g is bounded on the whole plane and hence constant by Liouville's theorem. Since $R_T(\lambda) \to 0$ in norm as $|\lambda| \to \infty$, we have $g(\lambda) \to 0$, so $g \equiv 0$.

Because this holds for every $f \in X^*$, the Hahn–Banach theorem implies $R_T(\lambda)x = 0$ for every λ and every $x \in X$, a contradiction to invertibility. Hence $\sigma(T) \neq \emptyset$. □

4.2 Spectral radius

Among all possible spectral data, the size of the spectrum is measured most economically by the spectral radius. Its formula in terms of operator norms of powers is one of the first deep links between algebraic iteration and analytic control.

Theorem 4.5 (Spectral radius formula)**.**

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\ge 1} \|T^n\|^{1/n}.$$

Proof. Set

$$a_n := \|T^n\|^{1/n} \qquad (n \ge 1).$$

Since $\|T^{m+n}\| \le \|T^m\|\,\|T^n\|$, the sequence $b_n := \log\|T^n\|$ is subadditive. By Fekete's lemma,

$$\lim_{n\to\infty} \frac{b_n}{n} = \inf_{n\ge 1} \frac{b_n}{n},$$

which is equivalent to

$$\lim_{n\to\infty} a_n = \inf_{n\ge 1} a_n =: L.$$

We first prove $r(T) \le L$. If $|\lambda| > L$, choose n such that $\|T^n\| < |\lambda|^n$. Then

$$\|T^n/\lambda^n\| < 1,$$

so by the Neumann series,

$$I - \lambda^{-n} T^n$$

is invertible. Since

$$I - \lambda^{-n} T^n = \prod_{k=0}^{n-1} \Big(I - \omega_k \lambda^{-1} T \Big),$$

where $\omega_k = e^{2\pi i k/n}$, each factor must be invertible. In particular, $I - \lambda^{-1} T$ is invertible, i.e. $\lambda \in \rho(T)$. Thus $|\lambda| > r(T)$, proving $r(T) \le L$.

For the reverse inequality, let $|\lambda| > r(T)$. Then $\lambda \in \rho(T)$ and

$$R_T(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n$$

converges in operator norm (as the Taylor expansion of the resolvent at infinity). Hence

$$\sup_n \|\lambda^{-n} T^n\| < \infty,$$

which implies

$$\|T^n\|^{1/n} \le |\lambda| \, C^{1/n}$$

for some constant $C > 0$. Letting $n \to \infty$ yields $L \le |\lambda|$. Since this holds for every $|\lambda| > r(T)$, we conclude $L \le r(T)$.

Combining the two inequalities gives $L = r(T)$. □

Further spectral structure

4.3 Motivating examples from infinite-dimensional spectral theory

Abstract definitions acquire real force only when compared with model operators. The examples in this section illustrate how dramatically infinite-dimensional spectra can differ from finite-dimensional intuition and prepare the way for the more systematic theory that follows.

Example 4.6 (Right shift on ℓ^2)**.** The right shift operator $R : \ell^2 \to \ell^2$ given by

$$R(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

is one-to-one but not onto, and $\|R\| = 1$. Similarly, the left shift is onto but not one-to-one. Hence neither is invertible. This already shows why spectral theory in infinite dimensions is subtler.

Example 4.7 (A compact operator without eigenvalues)**.** Define $T : \ell^2 \to \ell^2$ by

$$T(x_1, x_2, x_3, \dots) = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right).$$

Then T is compact, but it has no eigenvalue.

Example 4.8 (A non-compact selfadjoint operator without eigenvalues)**.** Let $T : L^2[0,1] \to L^2[0,1]$ be given by

$$(Tf)(t) = tf(t).$$

Then T is selfadjoint and non-compact, and it has no eigenvalue.

Example 4.9 (The Laplacian on $L^2(\mathbb{R})$ as an unbounded selfadjoint model)**.** Let A be the operator on $L^2(\mathbb{R})$ defined by

$$Af = -f'', \qquad \mathcal{D}(A) = H^2(\mathbb{R}).$$

Then A is a standard selfadjoint unbounded operator, it is not compact, and it has no eigenvalues. Indeed, if $Af = \lambda f$, then taking Fourier transforms yields

$$(4\pi^2\xi^2 - \lambda)\widehat{f}(\xi) = 0.$$

Hence $\widehat{f}$ is supported in the finite set

$$\{\xi \in \mathbb{R} : 4\pi^2\xi^2 = \lambda\},$$

which has Lebesgue measure zero. Since $\widehat{f} \in L^2(\mathbb{R})$, it follows that $\widehat{f} = 0$ almost everywhere and therefore $f = 0$.

This example lies outside the bounded-operator framework of the present notes, but it is a useful reminder that even selfadjoint operators need not possess eigenvalues once one leaves the compact setting. By contrast, on a bounded domain $\Omega \subset \mathbb{R}^n$, the Dirichlet Laplacian on $L^2(\Omega)$ has purely discrete spectrum.

4.4 Further resolvent identities and Banach-valued analyticity

The resolvent is not merely a convenient inverse; it is an analytic operator-valued function. Resolvent identities and analyticity give powerful control over how spectra behave under perturbation and provide the analytic mechanism behind several global spectral consequences.

Theorem 4.10 (Liouville for Banach-valued analytic functions)**.** *Let $f : \mathbb{C} \to X$ be entire and assume that*

$$\sup_{z\in\mathbb{C}} \|f(z)\| < \infty.$$

Then f is constant on $\mathbb{C}$.

Proof. Let $h \in X^*$ and define $g(z) = h(f(z))$. Then g is a bounded scalar-valued entire function. By the usual Liouville theorem, g is constant. Hence

$$h(f(z) - f(0)) = 0 \qquad \text{for all } h \in X^*.$$

Since X^* separates points, $f(z) = f(0)$ for all z. □

4.4.1 Resolvent identity and compactness of the spectrum

If $\lambda, \mu \in \rho(T)$, then

$$\begin{aligned} R(\lambda) - R(\mu) &= R(\lambda)(\lambda - \mu)I\, R(\mu) \\ &= (\lambda - \mu)R(\lambda)R(\mu). \end{aligned}$$

Thus

$$R(\lambda)R(\mu) = R(\mu)R(\lambda).$$

Moreover,

$$T - \lambda I = (T - \mu I)\big(I - (\lambda - \mu)R(\mu)\big).$$

If $|\lambda - \mu|\,\|R(\mu)\| < 1$, then $I - (\lambda - \mu)R(\mu)$ is invertible, so $\lambda \in \rho(T)$. Hence $\rho(T)$ is open.

Also, for such λ,

$$R(\lambda) = R(\mu)\big(I - (\lambda - \mu)R(\mu)\big)^{-1} = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\mu)^{n+1},$$

so $R(\lambda)$ is analytic on $\rho(T)$.

Since $\rho(T)$ is open, $\sigma(T)$ is closed. Since also $\sigma(T) \subset \overline{B_{\|T\|}(0)}$, it follows that $\sigma(T)$ is compact.

Finally, $\sigma(T)$ is nonempty. Indeed, if $\sigma(T) = \varnothing$, then $\rho(T) = \mathbb{C}$, so $R(\lambda)$ would be an entire operator-valued function. For $|\lambda| > \|T\|$ we have the Neumann series estimate

$$R(\lambda) = -(\lambda I - T)^{-1} = -\sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}},$$

whence

$$\|R(\lambda)\| \le \sum_{k=0}^{\infty} \frac{\|T\|^k}{|\lambda|^{k+1}} = \frac{1}{|\lambda| - \|T\|} \to 0 \quad (|\lambda| \to \infty).$$

So $R(\lambda)$ would be bounded and entire; by Liouville it would be constant, which is impossible. Therefore $\sigma(T) \neq \varnothing$.

4.5 Further remarks on the spectral radius

After the basic spectral radius formula is established, a number of related estimates and consequences become available. These supplementary observations sharpen intuition about the relation between spectral size, powers of an operator, and asymptotic norm growth.

Lemma 4.11. *If $S, T \in \mathcal{B}(X)$ and $ST = TS$, then ST is invertible if and only if both S and T are invertible.*

Proof. If both S and T are invertible, then ST is invertible with inverse $T^{-1}S^{-1}$. Conversely, assume that ST is invertible and let $U = (ST)^{-1}$. Then

$$(US)T = U(ST) = I \qquad \text{and} \qquad T(SU) = (TS)U = (ST)U = I,$$

so T has both a left inverse and a right inverse. Hence these coincide, and T is invertible. Similarly,

$$(UT)S = U(TS) = U(ST) = I \qquad \text{and} \qquad S(TU) = (ST)U = I,$$

so S is invertible as well. □

Next we identify the spectrum of T^n.

Lemma 4.12. *Let X be a complex Banach space and $T \in \mathcal{B}(X)$. Then*

$$\sigma(T^n) = \{\mu^n : \mu \in \sigma(T)\}.$$

Proof. For $\lambda \in \mathbb{C}$ write

$$z^n - \lambda = (z - \lambda_1) \cdots (z - \lambda_n),$$

where $\lambda_1, \dots, \lambda_n$ are the n-th roots of λ. Hence

$$T^n - \lambda I = (T - \lambda_1 I) \cdots (T - \lambda_n I).$$

Since these factors commute, $T^n - \lambda I$ is invertible if and only if each $T - \lambda_j I$ is invertible. Therefore

$$\lambda \in \sigma(T^n) \iff \lambda_j \in \sigma(T) \text{ for some } j \iff \lambda = \mu^n \text{ for some } \mu \in \sigma(T).$$

□

Theorem 4.13 (Gelfand's spectral radius formula)**.** *Let X be a complex Banach space and $T \in \mathcal{B}(X)$. Then*

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\geq 1} \|T^n\|^{1/n}.$$

Proof. By the previous lemma,

$$r(T^n) = r(T)^n.$$

Taking suprema on the spectrum gives

$$r(T)^n = r(T^n) \leq \|T^n\|,$$

so

$$r(T) \leq \|T^n\|^{1/n} \quad \text{for all } n.$$

Hence

$$r(T) \le \liminf_{n\to\infty} \|T^n\|^{1/n}.$$

Now if $|\lambda| > r(T)$, then $\lambda \in \rho(T)$ and

$$R(\lambda) = (\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

converges absolutely in $\mathcal{B}(X)$. Therefore for some $C > 0$,

$$\frac{\|T^n\|}{|\lambda|^{n+1}} \le C \quad \text{for all } n,$$

which implies

$$\|T^n\|^{1/n} \le C^{1/n} |\lambda|^{1+1/n}.$$

Passing to the limit superior gives

$$\limsup_{n\to\infty} \|T^n\|^{1/n} \le |\lambda|.$$

Since this holds for every $|\lambda| > r(T)$,

$$\limsup_{n\to\infty} \|T^n\|^{1/n} \le r(T).$$

Together with (1), this proves the formula. The identity with the infimum follows from the submultiplicativity of the sequence $\|T^n\|$. □

Example 4.14 (Volterra operator)**.** Let $T \in \mathcal{B}(C[0,1])$ be defined by

$$(Tf)(x) = \int_0^x f(t)\, dt.$$

Then $r(T) = 0$. Indeed, $\|T^n\| \le 1/n!$, so $\|T^n\|^{1/n} \to 0$.

Remark 4.15. The complex spectral radius formula is inherently a complex statement. It need not hold in the same form over real Banach spaces; for example, rotations on $\mathbb{R}^2$ illustrate the failure of a purely real formulation.

4.6 Decomposition of the spectrum

In infinite dimensions, non-invertibility can occur for several distinct reasons. Decomposing the spectrum into point, continuous, residual, and approximate point parts clarifies these mechanisms and provides a vocabulary for later examples and theorems.

(i) If $T - \lambda I$ is not one-to-one, then λ belongs to the *point spectrum*, denoted $\sigma_p(T)$. Equivalently, λ is an eigenvalue of T, and

$$\operatorname{Ker}(T - \lambda I)$$

is the eigenspace corresponding to λ.

(ii) If $T - \lambda I$ is one-to-one and $(T - \lambda I)X$ is a proper dense subspace of X, then λ belongs to the *continuous spectrum*, denoted $\sigma_c(T)$.

(iii) If $T - \lambda I$ is one-to-one but $(T - \lambda I)X$ is not dense in X, then λ belongs to the *residual spectrum*, denoted $\sigma_r(T)$.

Thus

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Example 4.16. Define $T : \ell^2 \to \ell^2$ by

$$T(x_1, x_2, \dots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots \right).$$

Then $T(\ell^2)$ contains c_{00}, hence $T(\ell^2)$ is dense in ℓ^2, but it is a proper subset. This gives an example of continuous spectrum.

There is another useful characterization of invertibility.

Theorem 4.17. *Let X and Y be Banach spaces and let $T \in \mathcal{B}(X, Y)$. Then T is invertible if and only if*

(i) $T(X) = Y$, and

(ii) there exists $k > 0$ such that

$$\|Tx\| \geq k\|x\| \qquad \textit{for all } x \in X.$$

Proof. If T^{-1} exists and is bounded, then T is onto and

$$\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\| \, \|Tx\|,$$

so $\|Tx\| \geq \|T^{-1}\|^{-1}\|x\|$.

Conversely, suppose $T(X) = Y$ and $\|Tx\| \geq k\|x\|$. Then T is one-to-one. To prove that T^{-1} is bounded, it suffices to show that $T(X)$ is closed. Let $Tx_n \to y$ in Y. Then

$$\|x_n - x_m\| \leq k^{-1}\|Tx_n - Tx_m\|,$$

so (x_n) is Cauchy in X. Since X is complete, $x_n \to x$ for some $x \in X$, and then $Tx_n \to Tx = y$. Thus $y \in T(X)$, so $T(X)$ is closed. Since $T(X) = Y$, T is a continuous bijection between Banach spaces, hence invertible by the inverse mapping theorem. ∎

This motivates two more spectral notions.

- If $\overline{(T-\lambda I)X} \neq X$, then λ belongs to the *compression spectrum*, denoted $\sigma_{\text{com}}(T)$.
- If $T-\lambda I$ is not bounded below, then λ belongs to the *approximate spectrum*, denoted $\sigma_{\text{app}}(T)$.

Equivalently,

$$\lambda \in \sigma_{\text{app}}(T) \iff \exists x_n \in X,\ \|x_n\| = 1,\ (T-\lambda I)x_n \to 0.$$

From the previous theorem we deduce

$$\sigma(T) = \sigma_{\text{app}}(T) \cup \sigma_{\text{com}}(T).$$

Moreover, the following relations hold:

$$\begin{aligned}\sigma_r(T) &= \sigma_{\text{com}}(T) \setminus \sigma_p(T),\\ \sigma_c(T) &= \sigma(T) \setminus \big(\sigma_{\text{com}}(T) \cup \sigma_p(T)\big).\end{aligned}$$

So, for convenience, the various pieces of the spectrum may be listed as follows:

(i) $\sigma_p(T)$: $T-\lambda I$ is not one-to-one;

(ii) $\sigma_c(T)$: $T-\lambda I$ is one-to-one and $\overline{(T-\lambda I)X} = X$;

(iii) $\sigma_r(T)$: $T-\lambda I$ is one-to-one but $\overline{(T-\lambda I)X} \neq X$;

(iv) $\sigma_{\text{com}}(T)$: $(T-\lambda I)X$ is not dense in X;

(v) $\sigma_{\text{app}}(T)$: $T-\lambda I$ is not bounded below.

4.7 Spectrum of the adjoint operator

Because the adjoint reflects the Hilbert-space geometry, one naturally expects a close relation between the spectra of T and T^*. The precise comparison is subtle enough to be informative and will later be indispensable in the study of normal and compact operators.

Theorem 4.18. *Let X and Y be Banach spaces and let $T \in \mathcal{B}(X,Y)$. Then T is invertible if and only if T^* is invertible.*

Proof. Suppose T^* is invertible. If $Tx = 0$, then for every $g \in Y^*$,

$$T^*(g)(x) = g(Tx) = 0.$$

If $x \neq 0$, pick $f \in X^*$ with $f(x) = \|x\|$. Since T^* is onto, there exists $g \in Y^*$ such that $T^*(g) = f$, and then

$$0 = T^*(g)(x) = f(x) = \|x\|,$$

a contradiction. Thus T is one-to-one.

Now

$$\operatorname{Ker} T^* = (\operatorname{Ran} T)^{\perp}.$$

Since T^* is one-to-one, $(\operatorname{Ran} T)^{\perp} = \{0\}$, so by Hahn-Banach $\overline{\operatorname{Ran} T} = Y$. We claim that $\operatorname{Ran} T$ is closed. Let $Tx_n \to y$. Since T^* is invertible, so is $(T^*)^{-1} =: S^*$. Then, for $f \in X^*$ with $\|f\| = 1$,

$$\begin{aligned} |f(x_n - x_m)| &= |T^* S^* f(x_n - x_m)| = |S^* f(Tx_n - Tx_m)| \\ &\leq \|S^*\| \, \|Tx_n - Tx_m\|. \end{aligned}$$

Taking the supremum over $\|f\| = 1$ shows that (x_n) is Cauchy, hence converges to some $x \in X$. Then $Tx_n \to Tx = y$, so $y \in \operatorname{Ran} T$. Thus $\operatorname{Ran} T$ is closed. Since it is also dense, $\operatorname{Ran} T = Y$.

Hence T is a continuous bijection and therefore invertible.

Conversely, if T is invertible then T^* is clearly invertible, with inverse $(T^{-1})^*$. □

Remark 4.19. On a Banach space,

$$(T - \lambda I)^* = T^* - \lambda I.$$

On a Hilbert space, with the usual convention for the adjoint,

$$(T - \lambda I)^* = T^* - \overline{\lambda} I.$$

The adjoint relates compression and point spectrum.

Theorem 4.20. *Let $T \in \mathcal{B}(X)$. Then*

$$\sigma_{\text{com}}(T) \subset \sigma_p(T^*) \qquad \text{and} \qquad \sigma_p(T) \subset \sigma_{\text{com}}(T^*).$$

Proof. Suppose $\lambda \in \sigma_{\text{com}}(T)$. Then $\overline{(T - \lambda I)X} \neq X$. Set

$$M = \overline{(T - \lambda I)X}.$$

Then $M \neq X$, so by Hahn-Banach there exists $0 \neq f \in X^*$ such that $f|_M = 0$. In particular,

$$f((T - \lambda I)x) = 0 \qquad \text{for all } x \in X,$$

which means

$$(T^* - \lambda I)f = 0.$$

Hence $\lambda \in \sigma_p(T^*)$.

Now suppose $\lambda \in \sigma_p(T)$, so there exists $0 \neq x \in X$ such that $(T - \lambda I)x = 0$. If $\overline{\operatorname{Ran}(T^* - \lambda I)} = X^*$, then every functional vanishing on $\operatorname{Ran}(T^* - \lambda I)$ must be zero. But for every $g \in \operatorname{Ran}(T^* - \lambda I)$

we have $g(x) = 0$, so the evaluation functional at x would vanish on a dense subspace and hence on all of X^*, forcing $x = 0$, a contradiction. Therefore

$$\overline{\operatorname{Ran}(T^* - \lambda I)} \neq X^*,$$

so $\lambda \in \sigma_{\mathrm{com}}(T^*)$. □

4.8 Spectrum of compact operators

Compact operators exhibit one of the clearest remnants of finite-dimensional spectral theory. Apart from the possible accumulation point at 0, their nonzero spectrum is forced into a discrete pattern, and this rigidity is the cornerstone of the compact spectral theorem proved later.

Lemma 4.21. *A finite-dimensional, or finite-codimensional, closed subspace of a Banach space is complemented.*

Proof. Let $M = \operatorname{span}\{e_1, \dots, e_n\}$. Define functionals $\phi_i : M \to \mathbb{C}$ by $\phi_i(e_j) = \delta_{ij}$. By Hahn-Banach, each ϕ_i extends to a continuous linear functional on X. If

$$N = \bigcap_{j=1}^{n} \operatorname{Ker}(\phi_j),$$

then $X = M \oplus N$. The finite-codimensional case is similar. □

The main structural theorem is the following.

Theorem 4.22. *Let X be a Banach space and let T be a compact operator on X. Then for every $\lambda \neq 0$,*

$$\dim \operatorname{Ker}(T - \lambda I) < \infty \qquad \text{and} \qquad \operatorname{Ran}(T - \lambda I) \text{ is closed.}$$

Proof. Since $\lambda \neq 0$, it suffices to consider $\lambda = 1$.

Let $N = \operatorname{Ker}(I - T)$. Since T is compact, the restriction

$$T|_N : N \to X$$

is compact. But $T|_N = I|_N$, so the unit ball of N is compact. Hence N is finite-dimensional.

Since N is closed and finite-dimensional, it is complemented: write

$$X = N \oplus M,$$

where M is closed. Let $S = (I - T)|_M$. We claim that S is bounded below on M: there exists $k > 0$ such that

$$\|Sx\| \geq k\|x\| \qquad (x \in M).$$

If not, there would exist $x_n \in M$ with $\|x_n\| = 1$ and $Sx_n \to 0$. Since T is compact, passing to a subsequence we may assume $Tx_n \to x_0$. But

$$x_n = (I - T)x_n + Tx_n = Sx_n + Tx_n \to x_0.$$

Because M is closed, $x_0 \in M$. Also $Sx_n \to 0$, so $Sx_0 = 0$, i.e.

$$x_0 \in \operatorname{Ker}(I - T) = N.$$

Thus $x_0 \in M \cap N = \{0\}$, contradicting $\|x_n\| = 1$.

Therefore S is bounded below on M, so $\operatorname{Ran}(I - T)$ is closed by the earlier theorem. □

This yields the Fredholm alternative.

Theorem 4.23 (Fredholm alternative)**.** *Let X be a Banach space and $T \in \mathcal{B}_0(X)$ be compact. Then for every $\lambda \neq 0$,*

$$\operatorname{Ker}(T - \lambda I) = \{0\} \iff \operatorname{Ran}(T - \lambda I) = X.$$

Equivalently, for $\lambda \neq 0$, the equation

$$Tx - \lambda x = y$$

has a solution for every $y \in X$ if and only if the homogeneous equation

$$Tx - \lambda x = 0$$

has only the trivial solution.

Remark 4.24. The Fredholm alternative need not hold for $\lambda = 0$. For example, let $X = C[0, 1]$ and define

$$(Tf)(x) = \int_0^x f(t)\, dt.$$

Then T is compact, $\operatorname{Ker}(T) = \{0\}$, and

$$\operatorname{Ran}(T) = \{g \in C^1[0, 1] : g(0) = 0\},$$

which is not all of X.

We shall also need the following lemma.

Lemma 4.25. *If $T \in \mathcal{B}_0(X)$, then* $\operatorname{Ran}(I - T)$ *has finite-dimensional complement.*

Proof. Write $S = I - T$. Let

$$N_k = \operatorname{Ker}(S^k), \qquad M_k = S^k(X).$$

Then

$$N_0 \subset N_1 \subset N_2 \subset \cdots, \qquad M_0 \supset M_1 \supset M_2 \supset \cdots.$$

Since $S^k = I - T_k$ for a compact operator T_k, the previous theorem shows that each N_k is finite-dimensional. If the chain (M_k) did not stabilize, then by Riesz' lemma one could construct $y_n \in M_n$ with $\|y_n\| = 1$ and

$$\operatorname{dist}(y_n, M_{n+1}) > \frac{1}{2}.$$

A standard argument then yields a contradiction to compactness of T. Hence there exists p such that

$$M_p = M_{p+1} = M_{p+2} = \cdots.$$

Similarly, the chain (N_k) stabilizes. One then proves that

$$X = N_p \oplus M_p.$$

Hence $M_p = \operatorname{Ran}(S^p)$ has finite codimension, and therefore so does $\operatorname{Ran}(S) = \operatorname{Ran}(I - T)$. □

Proof of the Fredholm alternative. Again write $S = I - T$. By the preceding lemma, $\operatorname{Ran}(S)$ has finite-dimensional complement. Thus, if $\operatorname{Ker}(S) = \{0\}$ then the stabilization argument for the chains (N_k) and (M_k) shows that

$$M_1 = M_0 = X,$$

so S is onto. The converse is immediate. □

We now obtain the basic description of the spectrum of a compact operator on an infinite-dimensional space.

Proposition 4.26. *Let X be an infinite-dimensional Banach space and let $T \in \mathcal{B}_0(X)$ be compact. Then*

$$\sigma(T) = \sigma_p(T) \cup \{0\}.$$

Proof. Since T is compact on an infinite-dimensional space, it cannot be invertible; hence $0 \in \sigma(T)$. Let $\lambda \neq 0$. If $\lambda \notin \sigma_p(T)$, then $T - \lambda I$ is one-to-one. By the Fredholm alternative it is onto, hence invertible. So $\lambda \notin \sigma(T)$. Therefore every nonzero spectral value is an eigenvalue. □

Theorem 4.27. *Let X be a Banach space and let $T \in \mathcal{B}_0(X)$ be compact. Then $\sigma_p(T)$ is countable, and its only possible limit point is* 0*.*

Proof. Fix $\varepsilon > 0$ and consider

$$\sigma_{p,\varepsilon}(T) = \{\lambda \in \sigma_p(T) : |\lambda| \geq \varepsilon\}.$$

It suffices to show that this set is finite. Suppose not. Then choose distinct eigenvalues $\lambda_n \in \sigma_{p,\varepsilon}(T)$ and corresponding eigenvectors x_n with $\|x_n\| = 1$. Let

$$M_n = \operatorname{span}\{x_1, \dots, x_n\}.$$

By Riesz' lemma, for each $n \geq 2$ there exists $y_n \in M_n$ such that

$$\operatorname{dist}(y_n, M_{n-1}) > \frac{1}{2}.$$

Writing $y_n = \sum_{j=1}^{n} a_j x_j$, one gets

$$Ty_n - \lambda_n y_n \in M_{n-1}.$$

Hence for $n > m$,

$$Ty_n - Ty_m = \lambda_n y_n + z$$

for some $z \in M_{n-1}$, so

$$\|Ty_n - Ty_m\| \geq |\lambda_n| \operatorname{dist}(y_n, M_{n-1}) > \frac{\varepsilon}{2}.$$

Thus (Ty_n) has no convergent subsequence, contradicting compactness of T. Therefore $\sigma_{p,\varepsilon}(T)$ is finite.

Since

$$\sigma_p(T) = \bigcup_{n=1}^{\infty} \{\lambda \in \sigma_p(T) : |\lambda| \geq 1/n\},$$

it follows that $\sigma_p(T)$ is countable. Any nonzero accumulation point would belong to one of these finite sets, which is impossible; hence 0 is the only possible limit point. □

4.9 Example: the multiplication operator on $L^2[0, 1]$

Multiplication operators provide a class of examples in which spectral information can be computed directly from the underlying function. They are therefore ideal for testing general definitions and for anticipating the measure-theoretic form of the spectral theorem.

Learning objectives.

- Interpret invariant and reducing subspaces as decomposition mechanisms rather than merely as subsets preserved by an operator.
- Translate geometric invariance into block-operator form.
- Understand why adjoints, normality, and compactness naturally produce reducing subspaces.

Key ideas.

- Invariant subspaces support restrictions, while reducing subspaces support orthogonal direct-sum decomposition.

- Commutation with the orthogonal projection onto a subspace is the operator-theoretic signature of reduction.

4.10 Invariant subspaces

Invariant subspaces are the most basic mechanism for reducing an operator to simpler pieces. They identify those closed subspaces on which the action of the operator remains internal, and thus they provide the starting point for triangular forms, decompositions, and spectral reduction.

Definition 4.28. Let $T \in \mathcal{B}(H)$ and let $M \subset H$ be a closed subspace.

- M is *invariant* for T if $T(M) \subset M$.
- M is *reducing* for T if both M and $M^\perp$ are invariant for T.

Remark 4.29. Invariant subspaces are the natural domains on which one can restrict an operator. Reducing subspaces are stronger: they permit an orthogonal decomposition of the operator into independent pieces.

Proposition 4.30 (Equivalent form of invariance)**.** *A closed subspace $M \subset H$ is invariant under $T \in \mathcal{B}(H)$ if and only if the restriction $T|_M : M \to M$ is a bounded operator.*

Proof. This is immediate from the definition, since boundedness of the restriction follows from boundedness of T. □

4.11 Reducing subspaces

A reducing subspace is stronger than an invariant subspace: it is compatible with the orthogonal decomposition of the ambient Hilbert space as well as with the operator itself. Such subspaces are especially important because they permit an operator to split into independent components that can be analysed separately.

Proposition 4.31 (Closed invariant subspaces of selfadjoint operators are reducing)**.** *Let $T = T^* \in \mathcal{B}(H)$ and let $M \subset H$ be a closed invariant subspace for T. Then $M^\perp$ is invariant for T, and therefore M reduces T.*

Proof. Let $x \in M^\perp$ and $y \in M$. Since M is invariant and $T = T^*$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, Ty\rangle = 0.$$

Thus $Tx \perp M$, so $Tx \in M^\perp$. □

Remark 4.32. For general operators, invariance of M need not imply invariance of $M^\perp$. The preceding proposition is therefore a genuinely Hilbert-space phenomenon tied to adjoints and symmetry.

Proposition 4.33 (Equivalent characterizations of reducing subspaces)**.** *Let $M \subset H$ be a closed subspace and let P_M denote the orthogonal projection onto M. For $T \in \mathcal{B}(H)$ the following are equivalent:*

(i) *M reduces T;*

(ii) *$P_M T = T P_M$;*

(iii) *both M and $M^\perp$ are invariant for T;*

(iv) *with respect to $H = M \oplus M^\perp$, the operator T has block diagonal form*

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}.$$

Proof. The equivalence of (i), (ii), and (iii) follows from Theorem 3.2. Statement (iv) is simply the matrix form of the condition that both M and $M^\perp$ are invariant. □

Proposition 4.34 (Selfadjoint case)**.** *If $T = T^*$ and M is invariant for T, then M automatically reduces T.*

Proof. Let $y \in M^\perp$ and $m \in M$. Then

$$\langle Ty, m\rangle = \langle y, Tm\rangle = 0$$

because $Tm \in M$. Hence $Ty \in M^\perp$, so $M^\perp$ is invariant. □

4.12 Block matrix form

Once an invariant or reducing decomposition is available, the operator can be written in block form. This matrix viewpoint is indispensable because it translates geometric information about subspaces into concrete algebraic relations among operator entries.

Proposition 4.35 (Triangular and diagonal forms)**.** *With respect to $H = M \oplus M^\perp$:*

(a) *M is invariant for T if and only if*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix};$$

(b) *M reduces T if and only if*

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}.$$

Proof. This is an immediate reformulation of the block definitions. □

4.13 Examples

The distinction between invariance and reduction is best appreciated through explicit models. The examples collected here show how the abstract definitions operate for shifts, multiplication operators, and other canonical constructions, and they make clear that these notions are genuinely different.

Example 4.36 (Unilateral shift)**.** Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the unilateral shift, $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. For each $m \geq 1$, the closed span of $\{e_n : n \geq m\}$ is invariant for S. However, this subspace is not reducing for S because its orthogonal complement is not invariant.

Example 4.37 (Diagonal operators)**.** If T is diagonal with respect to an orthonormal basis (e_n), then the closed span of any subfamily of the basis vectors reduces T. This is the basic model behind spectral decompositions.

Example 4.38 (Eigenspaces of normal operators)**.** If T is normal and λ is an eigenvalue of T, then the eigenspace $\operatorname{Ker}(T - \lambda I)$ reduces T.

Proof. Let $M = \operatorname{Ker}(T-\lambda I)$. Then M is invariant. If $y \in M^\perp$ and $x \in M$, then by Theorem 3.12,

$$\langle Ty, x\rangle = \langle y, T^* x\rangle = \langle y, \overline{\lambda} x\rangle = 0.$$

Hence $Ty \in M^\perp$, so M reduces T. □

Theorem 4.39 (Invariant subspaces for compact normal operators)**.** *Let $T \in \mathcal{B}(H)$ be compact and normal, and let $M \subseteq H$ be a closed subspace invariant under T. Then M is automatically a reducing subspace for T.*

Proof. By theorem 5.16, there is an orthonormal basis of H consisting of eigenvectors of T. For each nonzero eigenvalue λ, let $E_\lambda = \operatorname{Ker}(T - \lambda I)$, and let $E_0 = \operatorname{Ker} T$. Then

$$H = \bigoplus_{\lambda \in \sigma_p(T)} E_\lambda,$$

orthogonally.

Because M is invariant under T, it is invariant under every polynomial in T. Fix a nonzero eigenvalue λ. Since the nonzero spectrum of a compact operator has no accumulation point away from 0, one can choose a polynomial p such that $p(\lambda) = 1$ and $p(\mu) = 0$ for every other eigenvalue μ appearing in a fixed finite spectral truncation. Applying these polynomials and passing to limits shows that the orthogonal projection onto E_λ leaves M invariant. The same is trivial for $E_0 = \operatorname{Ker} T$.

Hence each spectral subspace E_λ decomposes as

$$E_\lambda = (E_\lambda \cap M) \oplus (E_\lambda \cap M^\perp),$$

and therefore

$$M = \overline{\bigoplus_{\lambda} (E_\lambda \cap M)}.$$

Since T acts on each E_λ as multiplication by λ, both M and $M^\perp$ are invariant under T. Thus M reduces T. $\square$

Remark 4.40. For general bounded operators on an infinite-dimensional separable Hilbert space, the invariant subspace problem is deep and subtle. Compact operators form one of the classical classes for which nontrivial invariant subspaces always exist, and in the compact normal case invariant subspaces automatically reduce the operator.

CHAPTER 5

Compact Operators, Fredholm Theory, Singular Values, and Schatten Classes

This chapter develops the structure theory of compact operators on Hilbert spaces. After establishing basic closure properties and the Fredholm alternative, we prove the spectral theorem for compact selfadjoint operators, extend the discussion to compact normal operators, and conclude with polar decomposition, singular values, and Schatten classes as quantitative refinements of compact spectral theory.

Learning objectives.

- Treat compact operators as norm-limits of finite-dimensional models.
- Understand Fredholm-type phenomena for compact perturbations of the identity.
- Organize spectral information through eigenvalues, singular values, and Schatten norms.

Key ideas.

- Compact operators behave like infinite matrices whose mass concentrates in finite-dimensional corners.
- Singular values measure size independently of phase and are therefore the natural bridge to Schatten classes.
- Compact spectral theory is the discrete model behind the general spectral theorem.

5.1 Compact operators

Compactness is the infinite-dimensional analogue of being nearly finite-dimensional. A compact operator sends the unit ball to a set that can be approximated by finitely many small balls, and for this reason compact operators retain many of the structural features of matrices while still acting on infinite-dimensional spaces.

Proposition 5.1. *Finite-rank operators are compact. The collection $\mathcal{K}(X, Y)$ of compact operators is a closed subspace of $\mathcal{B}(X, Y)$, and if $K \in \mathcal{K}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$, $R \in \mathcal{B}(W, X)$, then $SKR \in \mathcal{K}(W, Z)$.*

Proof. We first consider the case in which $F : X \to Y$ has finite-dimensional range. Then $F(B_X)$ is a bounded subset of the finite-dimensional space $\operatorname{Ran}(F)$. Since bounded subsets

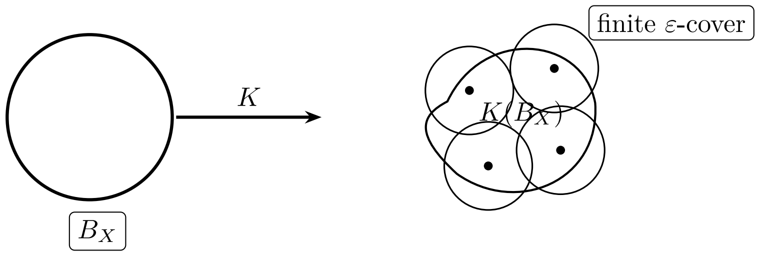

Figure 5.1: A compact operator sends the unit ball to a totally bounded set; equivalently, $K(B_X)$ admits finite ε-covers for every $\varepsilon > 0$.

of finite-dimensional normed spaces are relatively compact, it follows that $F(B_X)$ is relatively compact in Y; hence F is compact.

If K_1, K_2 are compact and $\alpha, \beta \in F$, then

$$(\alpha K_1 + \beta K_2)(B_X) \subset \alpha K_1(B_X) + \beta K_2(B_X).$$

The right-hand side is relatively compact because sums and scalar multiples of relatively compact sets are relatively compact. Hence $\alpha K_1 + \beta K_2$ is compact.

Next, let $K_n \to K$ in operator norm with each K_n compact. Fix $\varepsilon > 0$. Choose n so large that

$$\|K - K_n\| < \varepsilon/3.$$

Since $K_n(B_X)$ is relatively compact, it can be covered by finitely many balls of radius $\varepsilon/3$. Then $K(B_X)$ is covered by the same centers with radius ε. Thus $K(B_X)$ is totally bounded, hence relatively compact; so K is compact.

Finally, if K is compact, then $R(B_W) \subset \|R\|\, B_X$, so $K(R(B_W))$ is relatively compact in Y, and applying the bounded map S preserves relative compactness. Therefore SKR is compact. □

5.1.1 Completely continuous operators

Definition 5.2. Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is called *completely continuous* if whenever $x_n \rightharpoonup x$ weakly in X, one has

$$Tx_n \to Tx \qquad \text{in norm in } Y.$$

Equivalently, T sends weakly convergent sequences to norm-convergent sequences.

Proposition 5.3. *Every compact operator between Banach spaces is completely continuous.*

Proof. Let $T : X \to Y$ be compact and suppose $x_n \rightharpoonup x$ in X. Then (x_n) is bounded, hence (Tx_n) has relatively compact range. To prove $Tx_n \to Tx$ in norm, it is enough to show that every subsequence of (Tx_n) has a further subsequence converging to Tx.

Let (Tx_{n_k}) be any subsequence. Since T is compact, it has a norm-convergent further subsequence, say

$$Tx_{n_{k_j}} \to y \qquad \text{in } Y.$$

For every $f \in Y^*$ we have

$$f(Tx_{n_{k_j}}) = (f \circ T)(x_{n_{k_j}}) \longrightarrow (f \circ T)(x) = f(Tx),$$

because $f \circ T \in X^*$ and $x_{n_{k_j}} \rightharpoonup x$. Thus $Tx_{n_{k_j}} \rightharpoonup Tx$ weakly. Since a norm-convergent sequence has the same weak limit as its norm limit, we must have $y = Tx$. Therefore every subsequence of (Tx_n) has a further subsequence converging in norm to Tx, which forces the whole sequence to converge in norm to Tx. □

Theorem 5.4. *Let X and Y be Banach spaces, and assume that X is reflexive. Then every completely continuous operator $T : X \to Y$ is compact.*

Proof. Let (x_n) be a bounded sequence in X. Since X is reflexive, the closed unit ball is weakly compact; hence, after passing to a subsequence, we may assume that $x_n \rightharpoonup x$ weakly for some $x \in X$. Because T is completely continuous,

$$Tx_n \to Tx \qquad \text{in norm.}$$

Thus every bounded sequence has an image subsequence converging in norm, which is precisely the compactness of T. □

Corollary 5.5. *If H and K are Hilbert spaces, then for a bounded operator $T : H \to K$ the following are equivalent:*

(i) *T is compact;*

(ii) *T is completely continuous.*

Proof. Hilbert spaces are reflexive. Hence theorems 5.3 and 5.4 apply. □

Remark 5.6. The implication "compact ⇒ completely continuous" is valid on arbitrary Banach spaces, but the converse may fail when the domain is not reflexive. For MA641, the Hilbert-space case is the relevant one, and there compactness and complete continuity coincide.

5.1.2 Finite-rank approximation and compact adjoints

Lemma 5.7. *Let M be a Hilbert space, let $K \subset M$ be compact, and let (P_n) be a sequence of bounded operators on M such that $\sup_n \|P_n\| < \infty$ and $P_n x \to x$ for every $x \in M$. Then the convergence is uniform on K:*

$$\sup_{x \in K} \|P_n x - x\| \to 0.$$

In particular, whenever P_n are orthogonal projections onto an increasing sequence of finite-dimensional subspaces whose union is dense in M, then

$$\sup_{x \in K} \|(I - P_n)x\| \to 0.$$

Proof. Fix $\varepsilon > 0$. Choose $C > 0$ such that $\|P_n\| \le C$ for all n. Since K is compact, there exist points $x^{(1)}, \dots, x^{(m)} \in K$ such that

$$K \subset \bigcup_{j=1}^{m} B\Big(x^{(j)}, \frac{\varepsilon}{3(C+1)}\Big).$$

Since $P_n x^{(j)} \to x^{(j)}$ for each j, there exists N such that for $n \ge N$,

$$\|P_n x^{(j)} - x^{(j)}\| < \varepsilon/3, \qquad j = 1, \dots, m.$$

Next, let $x \in K$ and choose j with

$$\|x - x^{(j)}\| < \frac{\varepsilon}{3(C+1)}.$$

Then, for $n \ge N$,

$$\begin{aligned}
\|P_n x - x\| &\le \|P_n(x - x^{(j)})\| + \|P_n x^{(j)} - x^{(j)}\| + \|x^{(j)} - x\| \\
&\le C\|x - x^{(j)}\| + \varepsilon/3 + \|x - x^{(j)}\| \\
&< \varepsilon.
\end{aligned}$$

Taking the supremum over $x \in K$ proves the claim. □

Theorem 5.8. *Let H and K be Hilbert spaces and let $T : H \to K$ be compact. Then there exists a sequence of finite-rank operators (F_n) such that*

$$\|T - F_n\| \to 0.$$

Consequently, finite-rank operators are norm-dense in $\mathcal{K}(H, K)$.

Proof. Let $M := \overline{T(H)} \subset K$. Since

$$T(H) = \bigcup_{m=1}^{\infty} m\, T(B_H),$$

and $\overline{T(B_H)}$ is compact, the space M is separable. Choose an orthonormal basis (e_n) of M, and let P_n be the orthogonal projection onto $\operatorname{span}\{e_1, \dots, e_n\}$. Then $P_n y \to y$ for every $y \in M$.

Apply theorem 5.7 to the compact set $\overline{T(B_H)} \subset M$. We obtain

$$\sup_{\|x\|\le 1} \|(I - P_n)Tx\| = \sup_{y \in \overline{T(B_H)}} \|(I - P_n)y\| \to 0.$$

Hence

$$\|T - P_nT\| \to 0.$$

Each operator $F_n := P_nT$ has range contained in the finite-dimensional space $\operatorname{span}\{e_1, \dots, e_n\}$, so F_n is finite rank. □

Corollary 5.9 (Schauder theorem for Hilbert spaces)**.** *Let H and K be Hilbert spaces and let $T \in \mathcal{B}(H, K)$. Then*

$$T \textit{ is compact} \qquad \Longleftrightarrow \qquad T^* \textit{ is compact.}$$

Proof. Assume first that T is compact. By theorem 5.8, there exist finite-rank operators F_n with $\|T - F_n\| \to 0$. Taking adjoints gives

$$\|T^* - F_n^*\| = \|(T - F_n)^*\| = \|T - F_n\| \to 0.$$

Since each F_n^* is finite rank, hence compact, and compact operators form a norm-closed subspace of $\mathcal{B}(K, H)$, we conclude that T^* is compact. The converse follows by applying the same argument to T^* and using $T^{**} = T$. □

Remark 5.10. The previous proof also shows that every compact operator on a Hilbert space can be approximated in norm by finite-rank operators built from orthogonal projections onto suitable finite-dimensional subspaces of the range.

5.2 Fredholm alternative for compact operators

The Fredholm alternative is one of the first major structural consequences of compactness. Away from the spectral point 0, compact operators behave in many respects like finite-dimensional operators: injectivity, surjectivity, and solvability of linear equations become tightly linked.

Lemma 5.11. *Let $K \in \mathcal{K}(X)$ and let $\lambda \neq 0$. If $\lambda I - K$ is injective, then there exists $c > 0$ such that*

$$\|(\lambda I - K)x\| \ge c\|x\| \qquad (x \in X).$$

In particular, $\operatorname{Ran}(\lambda I - K)$ *is closed.*

Proof. Assume no such c exists. Then there is a sequence (x_n) with $\|x_n\| = 1$ and

$$(\lambda I - K)x_n \to 0.$$

Since K is compact, passing to a subsequence if necessary, we may assume $Kx_n \to y$ in norm. Then

$$\lambda x_n = Kx_n + (\lambda I - K)x_n \to y,$$

so $x_n \to x := \lambda^{-1} y$ in norm. By continuity of $\lambda I - K$,

$$(\lambda I - K)x = \lim_{n\to\infty} (\lambda I - K)x_n = 0.$$

Because $\lambda I - K$ is injective, $x = 0$, contradicting $\|x_n\| = 1$ and $x_n \to 0$. Hence the stated inequality holds.

If $(\lambda I - K)x_n \to z$, then the inequality shows that (x_n) is Cauchy, so $x_n \to x$ for some $x \in X$. Continuity yields $(\lambda I - K)x = z$, proving that the range is closed. □

Theorem 5.12 (Fredholm alternative)**.** *Let $K \in \mathcal{K}(X)$ and let $\lambda \neq 0$. Then either λ is an eigenvalue of K, or $\lambda I - K$ is invertible. Equivalently,*

$$\sigma(K) \setminus \{0\} = \{\lambda \neq 0 : \ker(\lambda I - K) \neq \{0\}\}.$$

Proof. Scaling by λ^{-1}, it suffices to prove the statement for $\lambda = 1$. Thus let $S := K$ and assume $I - S$ is injective. We must show that $I - S$ is surjective.

By theorem 5.11, $M_1 := (I - S)(X)$ is closed. Define inductively

$$M_n := (I - S)^n(X) \qquad (n \geq 1).$$

Since S commutes with $I - S$, each M_n is S-invariant. Also, $I - S$ is injective on each M_n, so by the previous lemma applied to the restriction $S|_{M_n}$, each $M_{n+1} = (I - S)(M_n)$ is closed in M_n, hence in X.

Suppose, for contradiction, that $I - S$ is not surjective, so $M_1 \neq X$. We claim that then every inclusion

$$M_{n+1} \subsetneq M_n$$

is proper. Indeed, if $M_{n+1} = M_n$ for some $n \geq 1$, then for $y \in M_{n-1}$ we have $(I - S)y \in M_n = M_{n+1}$. Hence $(I - S)y = (I - S)z$ for some $z \in M_n$. Since $I - S$ is injective, $y = z \in M_n$. Thus $M_{n-1} \subset M_n$, and therefore $M_{n-1} = M_n$. Repeating this argument backward gives $X = M_1$, contradiction.

Hence each M_{n+1} is a proper closed subspace of M_n. By Riesz's lemma, for each n we can choose $x_n \in M_n$ with

$$\|x_n\| = 1 \qquad \text{and} \qquad \operatorname{dist}(x_n, M_{n+1}) > \frac{1}{2}.$$

If $m > n$, then $x_m \in M_m \subset M_{n+1}$. Also,

$$x_n - Sx_n = (I - S)x_n \in M_{n+1}.$$

Therefore
$$\|Sx_n - Sx_m\| \geq \operatorname{dist}(Sx_n, M_{n+1}) = \operatorname{dist}(x_n, M_{n+1}) > \frac{1}{2}.$$
Thus the sequence (Sx_n) has no Cauchy subsequence, contradicting compactness of S.

This contradiction shows that $I - S$ must be surjective. Since it is already injective and bounded, the inverse mapping theorem implies that $I - S$ is invertible. Restoring λ gives the result. $\square$

Corollary 5.13. *Let $K \in \mathcal{K}(X)$ and let $\lambda \neq 0$. Then the eigenspace*
$$E_\lambda := \ker(\lambda I - K)$$
is finite-dimensional.

Proof. If E_λ were infinite-dimensional, then the restriction of K to E_λ would equal λI_{E_λ}. Since $K|_{E_\lambda}$ is compact and $\lambda \neq 0$, the identity on E_λ would be compact. But the identity operator on an infinite-dimensional normed space is never compact. Hence E_λ must be finite-dimensional. $\square$

Remark 5.14. The full Riesz–Schauder theorem strengthens theorems 5.12 and 5.13 by showing that the nonzero spectrum of a compact operator consists of isolated eigenvalues of finite algebraic multiplicity, with 0 as the only possible accumulation point.

5.3 Spectral theorem for compact selfadjoint operators

Among infinite-dimensional spectral results, this theorem is the closest counterpart to diagonalizing a Hermitian matrix. Its proof reveals why compactness and selfadjointness together force a complete orthonormal eigenbasis, thereby providing the conceptual bridge from matrix theory to the full spectral theorem.

> *We present the spectral theorem for compact selfadjoint operators on a Hilbert space: such operators admit an orthonormal basis of eigenvectors and a convergent eigen-expansion. This is the central result of the compact-operator theory on Hilbert spaces and a model case of the general spectral theorem.*

5.3.1 Statement and proof

Theorem 5.15 (Compact selfadjoint spectral theorem)**.** *Let H be a Hilbert space over $\mathbb{C}$ (or $\mathbb{R}$), and let $K \in \mathcal{B}(H)$ be compact and selfadjoint. Then there exist an orthonormal family $(e_n)_{n\geq 1} \subset H$ and real numbers $(\lambda_n)_{n\geq 1}$ such that*
$$Ke_n = \lambda_n e_n \qquad (n \geq 1),$$
$\lambda_n \neq 0$ for all n, and $\lambda_n \to 0$ if the family is infinite. If
$$M = \overline{\operatorname{span}\{e_n : n \geq 1\}},$$

then $K|_{M^\perp} = 0$ *and, for every* $x \in H$,

$$Kx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \, e_n,$$

with convergence in norm. In particular, H *admits an orthonormal basis consisting of eigenvectors of* K*, with eigenvalue* 0 *on* $\ker K$.

Proof. We use two standard facts.

(F1) Weak compactness of the unit ball. The closed unit ball of H is compact in the weak topology.

(F2) Compact operators are weak-to-strong continuous on bounded sets. If K is compact and $x_\alpha \to x$ weakly with $\sup_\alpha \|x_\alpha\| < \infty$, then $Kx_\alpha \to Kx$ in norm.

For completeness, we justify (F2). Since $\{Kx_\alpha\}$ is relatively compact, every subnet has a further subnet converging in norm to some y. For any $z \in H$,

$$\langle y, z \rangle = \lim \langle Kx_\alpha, z \rangle = \lim \langle x_\alpha, Kz \rangle = \langle x, Kz \rangle = \langle Kx, z \rangle,$$

where we use weak convergence of x_α and selfadjointness of K in the middle identity. Hence $y = Kx$. Therefore every subnet of Kx_α has a further subnet converging to Kx, which forces $Kx_\alpha \to Kx$ in norm.

If $K = 0$, there is nothing to prove. Assume $K \neq 0$, and consider the continuous functional

$$\Phi(x) := \langle Kx, x \rangle \qquad (\|x\| = 1).$$

Let $\alpha := \sup\{\Phi(x) : \|x\| = 1\}$. Choose a net (x_β) on the unit sphere with $\Phi(x_\beta) \to \alpha$. By (F1), there is a subnet, still denoted (x_β), converging weakly to some $x \in H$ with $\|x\| \le 1$. By (F2), $Kx_\beta \to Kx$ in norm, and since $x_\beta \to x$ weakly we obtain

$$\Phi(x_\beta) = \langle Kx_\beta, x_\beta \rangle \longrightarrow \langle Kx, x \rangle = \Phi(x).$$

Thus $\Phi(x) = \alpha$. If $\alpha \neq 0$, then necessarily $x \neq 0$. Replacing x by $x/\|x\|$, we may therefore assume $\|x\| = 1$.

Claim. The maximizer x is an eigenvector of K.

Let $v \in H$ satisfy $\langle v, x \rangle = 0$, and consider

$$f(t) = \Phi\Big(\frac{x + tv}{\|x + tv\|}\Big)$$

for real t near 0. Since $t = 0$ is a maximum of f, we have $f'(0) = 0$. A direct computation, using $\langle v, x \rangle = 0$, yields

$$0 = f'(0) = 2\Re \langle Kx, v \rangle.$$

In the complex case, replacing v by iv also gives $\Im\langle Kx, v\rangle = 0$. Hence $\langle Kx, v\rangle = 0$ for every $v \perp x$, so $Kx \in \operatorname{span}\{x\}$. Therefore $Kx = \lambda x$ for some $\lambda \in \mathbb{R}$, and λ is real because K is selfadjoint. Set $e_1 := x$ and $\lambda_1 := \lambda$.

Now let $H_1 = e_1^\perp$ and consider the restriction $K_1 := K|_{H_1} : H_1 \to H_1$. Then K_1 is again compact and selfadjoint. If $K_1 \neq 0$, the same argument produces an eigenvector $e_2 \in H_1$ with eigenvalue $\lambda_2 \in \mathbb{R}$. Proceeding inductively, we obtain an orthonormal family (e_n) with $Ke_n = \lambda_n e_n$, where each $\lambda_n \neq 0$ and $|\lambda_{n+1}| \leq |\lambda_n|$.

If the construction terminates, then $K = 0$ on the remaining orthogonal complement and the proof is complete. If the construction does not terminate, compactness forces $\lambda_n \to 0$. Otherwise there would exist $\varepsilon > 0$ and infinitely many indices n such that $|\lambda_n| \geq \varepsilon$. But then the sequence $Ke_n = \lambda_n e_n$ would admit no norm-convergent subsequence, because (e_n) is orthonormal, contradicting the relative compactness of $K(B_H)$.

Let

$$M = \overline{\operatorname{span}\{e_n : n \geq 1\}}.$$

By construction, M is K-invariant and K is diagonal on $\operatorname{span}\{e_n\}$. If $x \in M^\perp$ and $Kx \neq 0$, then the restriction $K|_{M^\perp}$ would be a nonzero compact selfadjoint operator, so the preceding argument would produce an eigenvector orthogonal to every e_n, contradicting the maximality of the family. Hence $K|_{M^\perp} = 0$.

Finally, for arbitrary $x \in H$ write $x = x_M + x_\perp$ with $x_M \in M$ and $x_\perp \in M^\perp$. Then $Kx_\perp = 0$, and for the partial sums

$$S_N x := \sum_{n=1}^{N} \lambda_n \langle x, e_n\rangle e_n$$

we have

$$\|Kx - S_N x\| = \|\sum_{n>N} \lambda_n \langle x, e_n\rangle e_n\| \leq \sup_{n>N} |\lambda_n| \Big(\sum_{n>N} |\langle x, e_n\rangle|^2\Big)^{1/2} \xrightarrow[N\to\infty]{} 0,$$

because $\sup_{n>N}|\lambda_n| \to 0$ and $(\langle x, e_n\rangle) \in \ell^2$ by Bessel's inequality. This proves the claimed expansion and its convergence in norm. $\square$

Supplementary compact spectral notes

Compact selfadjoint operators

We recall several standard facts that will be used repeatedly. For a compact operator on an infinite-dimensional Hilbert space, every nonzero spectral value is an eigenvalue, and 0 is the only possible accumulation point of the spectrum. In this sense compact spectral theory is essentially an eigenvalue theory.

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. For $x, y \in H$, the map

$$x \longmapsto \langle Tx, y\rangle$$

is a bounded linear functional on H, so by the Riesz representation theorem there exists a unique vector $z = T^*y \in H$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

Thus T^* is linear and bounded, and in fact

$$\|T^*\| = \|T\|.$$

For $x, y \in H$ and $\lambda \in \mathbb{C}$ we have

$$\langle (T - \lambda I)x, y\rangle = \langle x, (T^* - \overline{\lambda} I)y\rangle.$$

Equivalently,

$$(T - \lambda I)^* = T^* - \overline{\lambda} I.$$

Therefore $T - \lambda I$ is invertible if and only if $T^* - \overline{\lambda} I$ is invertible, since for every invertible bounded operator A one has $(A^{-1})^* = (A^*)^{-1}$. Consequently,

$$\lambda \in \sigma(T) \iff \overline{\lambda} \in \sigma(T^*),$$

and hence

$$\sigma(T^*) = \overline{\sigma(T)}.$$

If $T = T^*$, then T is determined by its quadratic form, and

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Moreover, if $T = T^*$, then $\langle Tx, x\rangle$ is real for every $x \in H$, and therefore every eigenvalue of T is real.

Recall also that $T \in \mathcal{B}(H)$ is normal ($T^*T = TT^*$) if and only if

$$\|Tx\| = \|T^*x\|, \qquad \forall x \in H.$$

This identity implies that the spectral radius satisfies $r(T) = \|T\|$.

Lemma. *Let $T \in \mathcal{B}(H)$ be normal. Then for each $n \in \mathbb{N}$,*

$$\|T^n\| = \|T\|^n.$$

Proof.

$$\|T^n x\|^2 = \langle T^n x, T^n x\rangle = \langle T^* T^n x, T^{n-1} x\rangle \le \|T^*(T^n x)\|\,\|T^{n-1}x\| = \|T^{n+1}x\|\,\|T^{n-1}x\|$$

(since $\|T^* y\| = \|Ty\|$). Therefore,

$$\|T^n\|^2 \le \|T^{n+1}\|\,\|T^{n-1}\|.$$

Suppose the assertion is true for n. Then

$$\|T\|^{2n} = (\|T\|^n)^2 = \|T^n\|^2 \le \|T^{n+1}\|\,\|T\|^{n-1}.$$

Hence

$$\|T\|^{n+1} \le \|T^{n+1}\| \le \|T\|^{n+1}.$$

Since

$$r(T) = \lim \|T^n\|^{1/n},$$

we obtain $r(T) = \|T\|$. □

This suggests asking whether $\|T\| \in \sigma(T)$ when T is selfadjoint. We first need the following result.

Theorem. *If $T \in \mathcal{B}(H)$ is a selfadjoint operator, then*

$$\sigma(T) = \sigma_{\text{app}}(T)$$

(the approximate point spectrum). That is,

$$\lambda \in \sigma(T) \iff \inf_{\|x\|=1} \|(\lambda I - T)x\| = 0.$$

Proof. Suppose $\lambda \in \rho(T)$. Then $(\lambda I - T)^{-1}$ is invertible, and

$$\|x\| = \|(\lambda I - T)^{-1}(\lambda I - T)x\| \le \|(\lambda I - T)^{-1}\|\,\|(\lambda I - T)x\|.$$

Hence

$$\inf_{\|x\|=1} \|(\lambda I - T)x\| \ge \|(\lambda I - T)^{-1}\|^{-1}.$$

Conversely, if

$$\inf_{\|x\|=1} \|(\lambda I - T)x\| = \alpha > 0,$$

then

$$\|(\lambda I - T)x\| \geq \alpha \|x\|, \qquad \forall x \in H.$$

Hence $(\lambda I - T)$ is one-to-one and $R(\lambda I - T)$ is closed. If $R(\lambda I - T) \neq H$, then there exists $y_0 \in H$, $y_0 \neq 0$, such that

$$\langle (\lambda I - T)x, y_0 \rangle = \langle x, (\overline{\lambda} I - T)y_0 \rangle = 0$$

for all $x \in H$. Hence $Ty_0 = \overline{\lambda} y_0$. That is, y_0 is an eigenvector of T. Since $T = T^*$, $\overline{\lambda} = \lambda$. Hence

$$Ty_0 = \lambda y_0 \implies (\lambda I - T)y_0 = 0,$$

for $y_0 \neq 0$, which contradicts injectivity. $\square$

Theorem. *Let $T \in \mathcal{B}(H)$ be selfadjoint. Then $\sigma(T) \subset \mathbb{R}$. If we write*

$$\mu = \inf_{\|x\|=1} \langle Tx, x \rangle \qquad \text{and} \qquad \nu = \sup_{\|x\|=1} \langle Tx, x \rangle,$$

then

$$\sigma(T) \subset [\mu, \nu] \qquad \text{and} \qquad \mu, \nu \in \sigma(T).$$

Proof. Suppose H is a complex Hilbert space, and $\lambda = \alpha + i\beta \in \sigma(T)$. Then

$$\begin{aligned}\langle \lambda x - Tx, x \rangle - \langle x, \lambda x - Tx \rangle &= \lambda \|x\|^2 - \langle Tx, x \rangle - \overline{\lambda} \|x\|^2 + \langle x, Tx \rangle \\ &= (\lambda - \overline{\lambda}) \|x\|^2 = 2i\beta \|x\|^2.\end{aligned}$$

For

$$x \in S_H = \{x \in H : \|x\| = 1\},$$

we have

$$2|\beta| \leq |\langle \lambda x - Tx, x \rangle| + |\langle x, \lambda x - Tx \rangle| \leq 2\|\lambda x - Tx\|.$$

Thus

$$\inf_{\|x\|=1} \|(\lambda I - T)x\| \geq |\beta| \neq 0$$

(by previous Thm), and hence $\sigma(T) \subset \mathbb{R}$.

Note that if $S = T + \gamma I$, then $\sigma(S) = \sigma(T) + \gamma$ ($\lambda \in \rho(S)$ if and only if $(\lambda I - S)^{-1} \in \mathcal{B}(H)$ if and only if $((\lambda - \gamma)I - T)^{-1} \in \mathcal{B}(H)$; therefore $\lambda \in \sigma(S)$ iff $\lambda - \gamma \in \sigma(T)$). Hence, without loss of generality, we can assume that $0 \leq \mu \leq \nu$.

Since $\nu = \|T\|$ and $\sigma(T) \subset \mathbb{R}$, we obtain

$$\sigma(T) \subset [-\nu, \nu].$$

We claim that $\lambda = \mu - q \notin \sigma(T)$ for $q > 0$. Indeed,

$$\langle (T - \lambda I)x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle \geq \mu - \lambda = q,$$

if $\|x\| = 1$. Therefore,

$$0 < q \leq \|(T - \lambda I)x\| \, \|x\|, \qquad \forall \|x\| = 1.$$

Hence

$$\inf_{\|x\|=1} \|(T - \lambda I)x\| \geq q > 0,$$

so the previous theorem implies that $\lambda \notin \sigma(T)$. Thus,

$$\sigma(T) \subset [\mu, \nu].$$

Choose $x_n \in S_H$ such that

$$\langle Tx_n, x_n\rangle \to \nu.$$

Therefore,

$$\begin{aligned} \|(\nu I - T)x_n\|^2 &= \nu^2\|x_n\|^2 + \|Tx_n\|^2 - 2\nu\langle Tx_n, x_n\rangle \\ &\leq 2\nu^2 - 2\nu\langle Tx_n, x_n\rangle \to 0. \end{aligned}$$

Hence

$$\nu \in \sigma(T).$$

Further,

$$\sigma(T) \subset [\mu, \nu] \implies \sigma(T) - \nu I \subset [\mu - \nu, 0] \implies \sigma(\nu I - T) \subset [0, \nu - \mu].$$

By invoking $(*)$, we obtain

$$\nu - \mu \in \sigma(\nu I - T) \implies \mu \in \sigma(T).$$

$\square$

Corollary. *If there exist $x_0, y_0 \in S_H$ such that*

$$\mu = \langle Tx_0, x_0\rangle \qquad \text{and} \qquad \nu = \langle Ty_0, y_0\rangle,$$

then x_0 and y_0 are eigenvectors corresponding to the eigenvalues μ and ν, respectively.

Proof. For $\alpha \in \mathbb{C}$ and $y \in H$, by definition of μ,

$$\langle T(x_0 + \alpha y), x_0 + \alpha y \rangle \geq \mu \langle x_0 + \alpha y, x_0 + \alpha y \rangle.$$

Hence

$$\langle Tx_0, x_0 \rangle + 2\alpha \, \Re \langle Tx_0, y \rangle + |\alpha|^2 \langle Ty, y \rangle \geq \mu \langle x_0, x_0 \rangle + \mu |\alpha|^2 \langle y, y \rangle + 2\mu \, \Re \alpha \langle x_0, y \rangle.$$

Since $\mu = \langle Tx_0, x_0 \rangle$, letting

$$\alpha = \delta \, \langle (T - \mu I)x_0, y \rangle, \qquad \delta \in \mathbb{R},$$

we infer that

$$\langle y, (T - \mu I)x_0 \rangle = 0, \qquad \forall y \in H.$$

By replacing $T \mapsto -T$, we obtain the other part. □

Lemma. *Let $T \in \mathcal{B}(H)$ be selfadjoint, and let $M \subset H$ be a closed T-invariant subspace. Then $N = M^\perp$ is T-invariant. If $T_1 = T|_M$ and $T_2 = T|_N$, then T_1 and T_2 are selfadjoint,*

$$T(H) = T_1(M) \oplus T_2(N),$$

and

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2).$$

Proof. Let $y \in N$. Since $T(M) \subseteq M$, for each $x \in M$,

$$0 = \langle Tx, y \rangle = \langle x, Ty \rangle \implies T(N) \subseteq N.$$

It is immediate that T_1 and T_2 are selfadjoint on M and N, respectively, and

$$T(H) = T(M \oplus N) = T(M) \oplus T(N).$$

Now let $\lambda \in \sigma(T_1)$. Then there exists $x_n \in S_M$ such that

$$\|\lambda x_n - T_1 x_n\| = \|\lambda x_n - T x_n\| \to 0.$$

Hence $\lambda \in \sigma(T)$, so $\sigma(T_1) \subseteq \sigma(T)$. Similarly, $\sigma(T_2) \subseteq \sigma(T)$.

If $\lambda \notin \sigma(T_1) \cup \sigma(T_2)$, then there exists $k > 0$ such that for $x \in M$ and $y \in N$,

$$\|\lambda x - Tx\| \geq k\|x\|, \qquad \|\lambda y - Ty\| \geq k\|y\|.$$

Let $z \in H$. Then $z = x + y$, with $x \in M$ and $y \in N$. Note that

$$\lambda x - Tx \in M, \qquad \lambda y - Ty \in N.$$

Hence

$$\begin{aligned}\|\lambda z - Tz\|^2 &= \|\lambda x - Tx + (\lambda y - Ty)\|^2 \\ &= \|\lambda x - Tx\|^2 + \|\lambda y - Ty\|^2 \\ &\geq k\|x\|^2 + k\|y\|^2 = k\|z\|^2.\end{aligned}$$

Thus $\lambda \notin \sigma(T)$, and therefore

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2).$$

□

Eigenvectors corresponding to distinct eigenvalues of a selfadjoint operator are orthogonal, and their eigenspaces are invariant under the operator.

If T is a compact selfadjoint operator on H, then $\sigma_p(T) \neq \varnothing$. If $T = 0$, then 0 is an eigenvalue, and if $T \neq 0$, then

$$\|T\| \in \sigma(T) = \sigma_p(T) \cup \{0\}$$

(because T is compact). Hence $\|T\| \in \sigma_p(T)$.

Theorem. *Let T be a compact selfadjoint operator on an infinite-dimensional Hilbert space H. Then the set of eigenvectors of T forms an orthonormal basis for H. Moreover, for $x \in H$,*

$$Tx = \sum \lambda_n \langle x, e_n \rangle e_n,$$

where $Te_n = \lambda_n e_n$ and $|\lambda_n| \to 0$.

Proof. Let $T \neq 0$. Since $\dim H = \infty$, $0 \in \sigma(T)$. Also,

$$\|T\| \in \sigma(T) = \sigma_p(T) \cup \{0\} \implies 0 \neq \|T\| \in \sigma_p(T).$$

Hence T has a nonzero eigenvalue.

Since T is compact and selfadjoint, $\sigma_p(T)$ is countable. Let

$$\lambda_n \in \sigma_p(T) = \{\lambda_n : n = 1, 2, \dots\}$$

and write

$$N_n = \operatorname{Ker}(\lambda_n I - T).$$

Recall that $\dim N_n < \infty$ because T is compact and $\lambda_n \neq 0$.

Choose an orthonormal basis B_n for each N_n, and set

$$B = \bigcup_{n=1}^{\infty} B_n.$$

Then B is an orthonormal set because eigenvectors corresponding to distinct eigenvalues of T are orthogonal. We claim that

$$\overline{\operatorname{span}(B)} = H.$$

If $\overline{\operatorname{span}(B)} \neq H$, then set

$$G = \overline{\operatorname{span}(B)}^{\perp}.$$

Since each eigenspace N_n is T-invariant, so is $\overline{\operatorname{span}(B)}$, and hence G is also T-invariant by the previous result. Therefore,

$$\sigma(T) = \sigma(T|_{\overline{\operatorname{span}(B)}}) \cup \sigma(T|_G).$$

Since $T|_G$ is compact and selfadjoint, $T|_G$ has a nonzero eigenvalue. Thus $T|_G$ has an eigenvector $y \in G$. But then y is also an eigenvector for T. Hence,

$$y \in G \cap \overline{\operatorname{span}(B)},$$

which is a contradiction. Thus,

$$\overline{\operatorname{span}(B)} = H.$$

Now let $\{e_n\}$ be an orthonormal basis such that $Te_n = \lambda_n e_n$. Notice that the series

$$\sum \lambda_n \langle x, e_n \rangle e_n$$

converges in norm. Indeed,

$$\left\| \sum_{n=k}^{\ell} \lambda_n \langle x, e_n \rangle e_n \right\|^2 \leq \|T\|^2 \sum_{n=k}^{\ell} |\langle x, e_n \rangle|^2 \to 0.$$

Consider

$$Sx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n.$$

Then

$$\|Sx\|^2 \leq \|T\|^2 \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|T\|^2 \|x\|^2.$$

Hence

$$\|Sx\| \leq \|T\| \, \|x\|, \qquad \forall x \in H.$$

Since

$$Se_n = \lambda_n e_n = Te_n \implies S = T.$$

Finally, if $\lambda_n \not\to 0$, then there exists $\varepsilon > 0$ such that

$$|\lambda_n| \geq \varepsilon$$

for infinitely many n. Then

$$\|Te_n - Te_m\|^2 = |\lambda_n|^2 + |\lambda_m|^2 \geq 2\varepsilon^2.$$

Hence (Te_n) has no convergent subsequence. Thus, $\lambda_n \to 0$. $\square$

5.4 Compact normal operators

The compact selfadjoint case already contains the essential diagonal mechanism, but normality is the correct class for a genuinely complex-valued spectral theory. This section shows that compact normal operators remain diagonalizable with respect to an orthonormal basis, now with possibly complex eigenvalues.

Theorem 5.16 (Spectral theorem for compact normal operators)**.** *Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$ be compact and normal. Then there exist an orthonormal basis (u_n) of H and complex numbers (λ_n) such that*

$$Tu_n = \lambda_n u_n \qquad (n \geq 1),$$

and

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n, \qquad x \in H,$$

with convergence in norm. Every nonzero eigenvalue has finite multiplicity, and if infinitely many λ_n are nonzero, then $\lambda_n \to 0$.

Proof. Set

$$U := T^*T = TT^*.$$

Then U is compact, selfadjoint, and positive. By the compact selfadjoint spectral theorem (theorem 5.15), there exist mutually orthogonal finite-dimensional eigenspaces

$$E_\mu := \operatorname{Ker}(U - \mu I), \qquad \mu \in \sigma_p(U) \setminus \{0\},$$

whose orthogonal direct sum, together with $\operatorname{Ker}(U)$, spans H. Also, the nonzero eigenvalues μ of U form at most a countable set with only possible accumulation point 0.

Because T commutes with U, each eigenspace E_μ is T-invariant. Indeed, if $x \in E_\mu$, then

$$U(Tx) = T(Ux) = T(\mu x) = \mu Tx,$$

so $Tx \in E_\mu$. Since E_μ is finite-dimensional and $T|_{E_\mu}$ is normal, the finite-dimensional spectral theorem yields an orthonormal basis of E_μ consisting of eigenvectors of T. Doing this for each $\mu \neq 0$, we conclude an orthonormal family of eigenvectors of T spanning

$$M := \overline{\sum_{\mu \neq 0} E_\mu}.$$

Next,

$$\operatorname{Ker}(U) = \operatorname{Ker}(T).$$

Indeed, if $Tx = 0$, then $Ux = T^*Tx = 0$. Conversely, if $Ux = 0$, then

$$0 = \langle Ux, x\rangle = \langle T^*Tx, x\rangle = \|Tx\|^2,$$

so $Tx = 0$. Thus every vector in $\operatorname{Ker}(U)$ is an eigenvector of T with eigenvalue 0. Choose an orthonormal basis of $\operatorname{Ker}(T) = \operatorname{Ker}(U)$ and adjoin it to the orthonormal bases obtained on the nonzero eigenspaces E_μ. This gives an orthonormal basis (u_n) of H consisting entirely of eigenvectors of T.

For $x \in H$, Parseval's identity gives the norm-convergent expansion

$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n,$$

and applying T termwise yields

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n\rangle u_n.$$

The multiplicity statement follows from theorem 5.13, since every nonzero eigenvalue of T is also a nonzero spectral value of the compact operator T. Finally, if infinitely many nonzero eigenvalues occurred away from 0, this would contradict the compactness of T exactly as in the compact selfadjoint case; equivalently, the nonzero spectrum of a compact operator can accumulate only at 0. □

Remark 5.17. For selfadjoint compact operators, the eigenvalues are real and theorem 5.16 reduces to theorem 5.15. Thus the normal theorem is the natural complex extension of the selfadjoint theory developed earlier.

5.5 Polar decomposition and singular values

The polar decomposition separates an operator into a geometric part and a positive part. In the compact setting this leads to singular values, which measure the size of an operator independently of sign or phase and organize several later quantitative estimates.

Definition 5.18 (Partial isometry)**.** An operator $W \in \mathcal{B}(H, K)$ is a *partial isometry* if W restricts to an isometry on $\operatorname{Ker}(W)^{\perp}$. Equivalently, W^*W is the orthogonal projection onto $\operatorname{Ker}(W)^{\perp}$.

Theorem 5.19 (Polar decomposition)**.** *For every $T \in \mathcal{B}(H, K)$ there exists a unique partial isometry $W \in \mathcal{B}(H, K)$ such that*

$$T = W|T|, \qquad \operatorname{Ker}(W) = \operatorname{Ker}(T).$$

Moreover,

$$\overline{\operatorname{Ran}(|T|)} = \operatorname{Ker}(T)^{\perp}, \qquad \overline{\operatorname{Ran}(T)} = \operatorname{Ran}(W),$$

and if T is injective with dense range, then W is unitary.

Proof. We first define a map on $\operatorname{Ran}(|T|)$ by

$$W_0(|T|x) := Tx \qquad (x \in H).$$

This is well defined. Indeed, if $|T|x = |T|y$, then $|T|(x - y) = 0$, and therefore

$$0 = \langle |T|^2(x - y), x - y\rangle = \langle T^*T(x - y), x - y\rangle = \|T(x - y)\|^2.$$

Hence $Tx = Ty$. Moreover,

$$\|W_0(|T|x)\|^2 = \|Tx\|^2 = \langle T^*Tx, x\rangle = \langle |T|^2x, x\rangle = \||T|x\|^2,$$

so W_0 is an isometry on $\operatorname{Ran}(|T|)$. It therefore extends uniquely to an isometry

$$W_1 : \overline{\operatorname{Ran}(|T|)} \to \overline{\operatorname{Ran}(T)}.$$

Now define $W \in \mathcal{B}(H, K)$ by setting $W = W_1$ on $\overline{\operatorname{Ran}(|T|)}$ and $W = 0$ on $\overline{\operatorname{Ran}(|T|)}^{\perp} = \operatorname{Ker}(T)$. Then W is a partial isometry with initial space $\overline{\operatorname{Ran}(|T|)} = \operatorname{Ker}(T)^{\perp}$. Since $|T|x \in \overline{\operatorname{Ran}(|T|)}$ for every $x \in H$, we have

$$W|T|x = W_1(|T|x) = Tx,$$

so $T = W|T|$.

For uniqueness, suppose $T = V|T|$ with V another partial isometry satisfying $\operatorname{Ker}(V) = \operatorname{Ker}(T)$. Then V and W agree on $\operatorname{Ran}(|T|)$, because both send $|T|x$ to Tx, and they both vanish on $\operatorname{Ker}(T)$. Since

$$H = \overline{\operatorname{Ran}(|T|)} \oplus \operatorname{Ker}(T),$$

it follows that $V = W$.

Finally, if T is injective and has dense range, then $\operatorname{Ker}(T) = \{0\}$ and $\overline{\operatorname{Ran}(T)} = K$. Thus the partial isometry W has initial space H and final space K, so W is unitary. $\square$

Corollary 5.20 (Invertible case of the polar decomposition)**.** *Let $T \in \mathcal{B}(H)$ be invertible. Then in the factorization $T = W|T|$ from Theorem 5.19, the partial isometry W is unitary. Equivalently, every invertible bounded operator on a Hilbert space admits a unique decomposition*

$$T = U|T|$$

with U unitary and $|T| = (T^*T)^{1/2}$ positive.

Proof. If T is invertible, then $\operatorname{Ker}(T) = \{0\}$ and $\operatorname{Ran}(T) = H$. By Theorem 5.19, the initial space of W is $\operatorname{Ker}(T)^\perp = H$ and the final space is $\overline{\operatorname{Ran}(T)} = H$. Hence W is a surjective isometry, i.e. a unitary operator. Uniqueness follows from the uniqueness part of Theorem 5.19. □

Example 5.21 (Why the polar factor need not be unitary)**.** Let S be the unilateral shift on ℓ^2, defined by

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots).$$

Then $S^*S = I$, so $|S| = I$. The polar decomposition is therefore simply

$$S = S|S|.$$

Here the polar factor is the isometry S itself. However, S is not onto, since $e_1 \notin \operatorname{Ran}(S)$. Thus the polar factor in general is only a partial isometry, not a unitary operator on the whole space.

Remark 5.22. A useful heuristic is that the polar factor becomes unitary precisely when the "defect" between the kernel of T and the cokernel of T disappears. The invertible case is the cleanest instance of this principle and is the form most often used in matrix theory and finite-dimensional operator theory.

Definition 5.23 (Singular values)**.** Let $T \in \mathcal{K}(H, K)$ be compact. The nonnegative eigenvalues of the compact positive operator $|T|$ are called the *singular values* of T. Arranged in nonincreasing order and repeated according to multiplicity, we write them as

$$s_1(T) \geq s_2(T) \geq \cdots \geq 0.$$

If T has finite rank, then $s_n(T) = 0$ for all sufficiently large n.

Theorem 5.24 (Singular value decomposition)**.** *Let $T \in \mathcal{K}(H, K)$. Then there exist orthonormal systems (e_n) in H and (f_n) in K such that*

$$Te_n = s_n(T) f_n, \qquad T^* f_n = s_n(T) e_n,$$

and for every $x \in H$,

$$Tx = \sum_{n=1}^{\infty} s_n(T) \langle x, e_n \rangle f_n,$$

with convergence in norm. In particular, T is finite rank if and only if only finitely many singular values are nonzero.

Proof. Because T is compact, so is $|T| = (T^*T)^{1/2}$. By the compact selfadjoint spectral theorem, there exists an orthonormal basis (e_n) of H such that

$$|T|e_n = s_n(T)e_n,$$

where $s_n(T) \to 0$ if infinitely many singular values are nonzero. Let W be the partial isometry from the polar decomposition $T = W|T|$. For each index with $s_n(T) > 0$, define

$$f_n := We_n.$$

Then

$$\|f_n\| = \|e_n\| = 1, \qquad \langle f_n, f_m\rangle = \langle We_n, We_m\rangle = \langle e_n, e_m\rangle = \delta_{nm},$$

so (f_n) is orthonormal on the support of the singular values. Moreover,

$$Te_n = W|T|e_n = s_n(T)We_n = s_n(T)f_n.$$

Applying T^* gives

$$T^*f_n = T^*We_n = |T|e_n = s_n(T)e_n,$$

because W^*W is the identity on $\mathrm{Ker}(T)^\perp$, which contains each e_n corresponding to $s_n(T) > 0$. Finally, for $x \in H$ the expansion

$$|T|x = \sum_{n=1}^{\infty} s_n(T)\,\langle x, e_n\rangle e_n$$

converges in norm, and applying the bounded operator W yields

$$Tx = W|T|x = \sum_{n=1}^{\infty} s_n(T)\,\langle x, e_n\rangle f_n.$$

If only finitely many $s_n(T)$ are nonzero, this formula shows that T has finite rank; the converse is immediate from the spectral structure of $|T|$. □

Example 5.25 (Rank-one singular value decomposition)**.** Let $u \in H$ and $v \in K$ be unit vectors, and define $T : H \to K$ by

$$Tx = \langle x, u\rangle v.$$

Then $T^*y = \langle y, v\rangle u$, so

$$T^*Tx = \langle x, u\rangle u = P_{\mathrm{span}\{u\}}x.$$

Hence $|T| = (T^*T)^{1/2} = P_{\mathrm{span}\{u\}}$, the polar factor U sends u to v and vanishes on $u^\perp$, and the only nonzero singular value of T is 1. This is the atomic model from which finite-rank singular value decompositions are built.

5.6 Schatten class operators

Schatten classes refine compactness by measuring how rapidly the singular values decay. They interpolate between finite-rank, Hilbert–Schmidt, and trace-class behaviour and provide a flexible scale of operator ideals that parallels the classical ℓ^p spaces.

Definition 5.26 (Schatten classes)**.** Let $1 \le p < \infty$. For Hilbert spaces H and K, the *Schatten p-class* $S_p(H,K)$ consists of all compact operators $T \in \mathcal{K}(H,K)$ such that

$$\sum_{n=1}^{\infty} s_n(T)^p < \infty.$$

The Schatten norm is

$$\|T\|_{S_p} := \Big(\sum_{n=1}^{\infty} s_n(T)^p\Big)^{1/p}.$$

When $H = K$ we simply write $S_p(H)$.

Remark 5.27. The first two cases are the most important for MA641:

(i) $S_1(H)$ is the *trace class*;

(ii) $S_2(H,K)$ is the *Hilbert–Schmidt class*.

Moreover, every Schatten class operator is compact because $s_n(T) \to 0$ whenever $\sum s_n(T)^p < \infty$. If $T \in S_2(H,K)$, then for every orthonormal basis (u_n) of H one has

$$\|T\|_{S_2}^2 = \sum_{n=1}^{\infty} s_n(T)^2 = \sum_{n=1}^{\infty} \|Tu_n\|^2,$$

so the Hilbert–Schmidt class introduced earlier is exactly the Schatten 2-class.

Proposition 5.28 (First inclusions among Schatten classes)**.** *Let $1 \le p < q < \infty$. Then*

$$S_p(H,K) \subset S_q(H,K) \subset \mathcal{K}(H,K),$$

and

$$\|T\|_{S_q} \le \|T\|_{S_p} \qquad (T \in S_p(H,K)).$$

In particular,

$$S_1(H,K) \subset S_2(H,K) = HS(H,K) \subset \mathcal{K}(H,K).$$

Proof. If (a_n) is a nonincreasing sequence of nonnegative numbers with $\sum_n a_n^p < \infty$, then $a_n \to 0$. Hence $a_n \leq 1$ for all sufficiently large n, and for those indices one has $a_n^q \leq a_n^p$. It follows that

$$\sum_n a_n^q < \infty.$$

Applying this to the singular values $a_n = s_n(T)$ proves the inclusions. The norm inequality follows in the same way from the elementary inclusion $\ell^p \subset \ell^q$ for nonnegative sequences. $\square$

Remark 5.29 (Trace class and the Hilbert–Schmidt inner product)*.* The case $p = 1$ is the *trace class*. If $A \in S_1(H)$, one can define

$$\mathrm{tr}(A) := \sum_n \langle Ae_n, e_n \rangle,$$

and this value is independent of the orthonormal basis (e_n). For Hilbert–Schmidt operators $A, B \in HS(H)$, the product B^*A is trace class and

$$\langle A, B \rangle_{HS} = \mathrm{tr}(B^*A).$$

Thus the Hilbert–Schmidt inner product introduced earlier is the natural quadratic member of the Schatten scale.

Chapter 6

Spectral Theorem for Bounded Selfadjoint and Normal Operators

The diagonalization of compact selfadjoint operators extends in the general bounded setting to the language of continuous functional calculus and projection-valued measures. This chapter records the bounded spectral theorem in the formulations most useful for a first graduate course and explains how the compact and finite-dimensional cases fit into the general picture.

Learning objectives.

- Move from polynomial expressions in an operator to continuous functional calculus.
- Interpret spectral measures as the correct infinite-dimensional replacement for diagonal matrices.
- See the compact and finite-dimensional theorems as special cases of the bounded selfadjoint and normal spectral theorem.

Key ideas.

- The continuous functional calculus is the conceptual heart of the bounded spectral theorem.
- Projection-valued measures encode orthogonal decomposition at every Borel scale of the spectrum.
- Multiplication operators on L^2 are the universal normal models.

Common pitfalls.

- The general spectral theorem does not usually produce an orthonormal basis of eigenvectors.
- Projection-valued integration is an operator-valued analogue of scalar integration and must not be confused with a pointwise diagonalization formula.

6.1 Continuous functional calculus for selfadjoint operators

Once a selfadjoint operator is understood through its spectrum, the next natural step is to substitute that operator into scalar functions. The continuous functional calculus makes this idea precise and turns spectral information into a systematic tool for constructing new operators from old ones while preserving algebraic and norm relations.

Theorem 6.1 (Continuous functional calculus)**.** *Let $T \in \mathcal{B}(H)$ be selfadjoint. Then there exists a unique map*

$$\Phi_T : C(\sigma(T)) \to \mathcal{B}(H), \qquad f \mapsto f(T),$$

such that:

(a) $\Phi_T(1) = I$ *and* $\Phi_T(\mathrm{id}) = T$*, where* $\mathrm{id}(\lambda) = \lambda$*;*

(b) Φ_T *is a unital* $*$*-homomorphism:*

$$(fg)(T) = f(T)g(T), \qquad \overline{f}(T) = f(T)^*;$$

(c) $\|f(T)\| = \|f\|_{\infty,\sigma(T)}$ *for every* $f \in C(\sigma(T))$*;*

(d) *if* $p_n \to f$ *uniformly on* $\sigma(T)$ *for polynomials* p_n*, then* $p_n(T) \to f(T)$ *in operator norm.*

Remark 6.2. This theorem should be viewed as the correct replacement for polynomial calculus. In finite dimensions, if T is diagonal with diagonal entries $\lambda_1, \dots, \lambda_n$, then $f(T)$ is the diagonal matrix with entries $f(\lambda_1), \dots, f(\lambda_n)$.

6.2 Bounded Borel functional calculus and spectral projections

Continuous functions are sufficient for many purposes, but spectral theory becomes substantially more flexible once characteristic functions and more general Borel functions are admitted. The bounded Borel functional calculus supplies this extension and isolates spectral projections as the basic pieces of the spectral decomposition.

Theorem 6.3 (Bounded Borel functional calculus)**.** *Let* $T \in \mathcal{B}(H)$ *be selfadjoint. Then there exists a unique map*

$$\Psi_T : B_b(\sigma(T)) \to \mathcal{B}(H), \qquad f \mapsto f(T),$$

from the bounded Borel functions on $\sigma(T)$ *into* $\mathcal{B}(H)$ *such that:*

(a) Ψ_T *extends the continuous functional calculus;*

(b) Ψ_T *is a unital* $*$*-homomorphism;*

(c) $\|f(T)\| \le \|f\|_{\infty,\sigma(T)}$ *for every bounded Borel function* f*;*

(d) *if* (f_n) *is a bounded sequence in* $B_b(\sigma(T))$ *converging pointwise to* f*, then*

$$f_n(T)x \to f(T)x \qquad \text{for every } x \in H.$$

Moreover, if $S \in \mathcal{B}(H)$ *commutes with* T*, then* S *commutes with* $f(T)$ *for every bounded Borel function* f*.*

Remark 6.4. One standard construction proceeds from the continuous functional calculus and the Riesz representation theorem: for each $x, y \in H$, the functional $f \mapsto \langle f(T)x, y\rangle$ on $C(\sigma(T))$ is represented by a finite complex measure $\mu_{x,y}$, and one then defines

$$\langle f(T)x, y\rangle := \int_{\sigma(T)} f(\lambda)\, d\mu_{x,y}(\lambda)$$

for bounded Borel f. For MA641, the structural properties of Theorem 6.3 are the main point.

Corollary 6.5 (Spectral projections from characteristic functions)**.** *Let $T \in \mathcal{B}(H)$ be selfadjoint and let $B, C \subset \sigma(T)$ be Borel sets. Then*

$$E_T(B) := \mathbf{1}_B(T)$$

is an orthogonal projection, and

$$E_T(B)E_T(C) = E_T(B \cap C), \qquad E_T(B) + E_T(C) = E_T(B \cup C) \quad \textit{whenever } B \cap C = \emptyset.$$

In particular, spectral subspaces arise directly from indicator functions.

Proof. Since $\mathbf{1}_B^2 = \mathbf{1}_B = \overline{\mathbf{1}_B}$, the $*$-homomorphism property gives

$$E_T(B)^2 = E_T(B), \qquad E_T(B)^* = E_T(B),$$

so $E_T(B)$ is an orthogonal projection. The remaining identities follow from

$$\mathbf{1}_B \mathbf{1}_C = \mathbf{1}_{B \cap C} \qquad \text{and} \qquad \mathbf{1}_B + \mathbf{1}_C = \mathbf{1}_{B \cup C} \quad (B \cap C = \emptyset).$$

□

6.3 Projection-valued measures

The measure-theoretic form of the spectral theorem is expressed through projection-valued measures. They generalize the decomposition of a matrix into eigenspaces by replacing sums over eigenvalues with operator-valued integrals over the spectrum.

Definition 6.6 (Projection-valued measure)**.** A map E from the Borel σ-algebra of a compact set $K \subset \mathbb{R}$ (or $K \subset \mathbb{C}$ in the normal case) to the orthogonal projections on H is called a *projection-valued measure* if:

(i) $E(\emptyset) = 0$ and $E(K) = I$;

(ii) $E(B_1 \cap B_2) = E(B_1)E(B_2)$ for all Borel sets B_1, B_2;

(iii) for pairwise disjoint Borel sets (B_n) and every $x \in H$,

$$E\Big(\bigcup_n B_n\Big)x = \sum_n E(B_n)x$$

with convergence in norm.

Remark 6.7. For each $x \in H$, the scalar set function

$$\mu_x(B) := \langle E(B)x, x\rangle$$

is a finite positive measure. This allows one to interpret expressions such as $\int f\, dE$ via scalar integration against the family (μ_x).

Example 6.8 (Multiplication operators as spectral models)**.** Let (X, μ) be a measure space and let $\varphi \in L^\infty(X, \mu)$ be real-valued. The multiplication operator

$$(M_\varphi f)(x) = \varphi(x) f(x) \qquad (f \in L^2(X, \mu))$$

is bounded and selfadjoint. For each Borel set $B \subset \mathbb{R}$, define

$$E(B)f = \mathbf{1}_{\varphi^{-1}(B)} f.$$

Then E is a projection-valued measure and

$$M_\varphi = \int_{\sigma(M_\varphi)} \lambda \, dE(\lambda).$$

This example should be viewed as the canonical concrete model behind the abstract spectral theorem.

6.4 The bounded spectral theorem: selfadjoint case

The selfadjoint spectral theorem is the central structural statement of the course. It replaces finite-dimensional diagonalization by an operator-valued integral representation and thereby explains why selfadjoint operators may be analysed through scalar functions on their spectra.

Theorem 6.9 (Spectral theorem for bounded selfadjoint operators)**.** *Let $T \in \mathcal{B}(H)$ be selfadjoint. Then there exists a unique projection-valued measure E on the Borel subsets of $\sigma(T) \subset \mathbb{R}$ such that*

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda).$$

More generally, for every bounded Borel function f on $\sigma(T)$,

$$f(T) = \int_{\sigma(T)} f(\lambda) \, dE(\lambda).$$

This representation satisfies:

(a) $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$*;*

(b) *$E(B)$ is the orthogonal projection onto the spectral subspace corresponding to the Borel set B;*

(c) *a closed subspace M reduces T if and only if it is invariant under every spectral projection $E(B)$;*

(d) *T is unitarily equivalent to multiplication by the independent variable on a suitable L^2-space.*

Remark 6.10. The compact selfadjoint theorem proved earlier is the discrete model for this result: the projection-valued measure is then concentrated on a countable set of eigenvalues, and the operator integral reduces to an orthogonal series expansion.

6.5 The bounded spectral theorem: normal case

The normal spectral theorem extends the selfadjoint case to the full class of bounded normal operators. This extension is decisive because normality is exactly the condition under which the adjoint calculus and the spectral calculus fit together well enough to yield a complete spectral representation.

Theorem 6.11 (Spectral theorem for bounded normal operators)**.** *Let $T \in \mathcal{B}(H)$ be normal. Then there exists a unique projection-valued measure E on the Borel subsets of $\sigma(T) \subset \mathbb{C}$ such that*

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda).$$

For every bounded Borel function f on $\sigma(T)$ one has

$$f(T) = \int_{\sigma(T)} f(\lambda) \, dE(\lambda),$$

and the map $f \mapsto f(T)$ is a $*$-homomorphism from bounded Borel functions on $\sigma(T)$ into $\mathcal{B}(H)$.

Idea of construction. Write $T = A + iB$, where

$$A = \frac{T + T^*}{2}, \qquad B = \frac{T - T^*}{2i}.$$

Then A and B are commuting bounded selfadjoint operators. One first develops the joint functional calculus for the commuting pair (A, B) and then pushes it forward under the map $(s, t) \mapsto s + it$. This yields the projection-valued measure E on $\sigma(T) \subset \mathbb{C}$. $\square$

Corollary 6.12. *Every bounded normal operator is unitarily equivalent to a multiplication operator*

$$(M_\varphi f)(\omega) = \varphi(\omega) f(\omega)$$

on some $L^2(\Omega, \mu)$, where $\varphi \in L^\infty(\Omega, \mu)$ and

$$\sigma(T) = \operatorname{ess\,ran}(\varphi).$$

6.6 Finite-dimensional and compact cases as models

The abstract spectral theorem becomes much easier to internalize when viewed through familiar special cases. Finite-dimensional diagonalization and the compact spectral theorem serve as concrete models that illuminate the general measure-theoretic statement proved above.

Key ideas.

- Compact selfadjoint operators admit orthonormal eigenbases; general bounded selfadjoint operators admit spectral resolutions.

- Functional calculus encodes the slogan "apply a scalar function to the spectrum, and thereby to the operator".
- Projection-valued measures provide the correct notion of spectral decomposition in infinite dimensions.
- The multiplication-operator model is the universal normal form for bounded normal operators.

Chapter 7

Problem Sets

The problems in this chapter are intended to consolidate the theory developed in the main text and to encourage a more active command of the material. They range from routine verifications to conceptual exercises that expose the geometry, algebra, and spectral ideas underlying the course.

The exercises are grouped by theme so that they may be used either alongside the corresponding lecture sections or as a cumulative revision set. Within each section, the statements have been lightly edited for consistency of notation, punctuation, and phrasing while preserving their mathematical content.

For teaching purposes, each thematic block should be read in three layers: a diagnostic opening item, a core collection designed to reinforce the main definitions and theorems, and a final cluster of synthesis problems that mix several chapters at once. Instructors may therefore assign short weekly sets by taking only the first layer, while seminar-style or qualifying-exam preparation may use the final synthesis layer.

7.1 Foundations of Hilbert Spaces and Orthogonality

These problems are designed to reinforce the geometric core of the subject. They emphasize orthogonality, approximation, orthonormal systems, and weak convergence, and they are best approached as opportunities to reorganize the foundational theory in one's own words.

1. State whether each of the following statements is true or false, giving a justification in each case.
   (a) If $(X, \|\cdot\|)$ is a normed linear space such that $\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in X$, then there must exist an inner product on X that induces the norm $\|\cdot\|$ on X.
   (b) Every orthonormal set in a Hilbert space H must be closed in H.
   (c) If a proper subspace M of a Hilbert space H contains an orthonormal basis of H, then M cannot be a Banach space in the induced norm.
   (d) In any infinite-dimensional Hilbert space, there exists a convergent series which is not absolutely convergent.
   (e) If (x_n) is a sequence in a Hilbert space H such that $\sum\limits_{n=1}^{\infty} \|x_n\|^2 < \infty$, then the series $\sum\limits_{n=1}^{\infty} x_n$ must converge in H.
   (f) If (u_n) is an orthonormal sequence in a Hilbert space H and if $x \in H$, then the series $\sum\limits_{n=1}^{\infty} \langle x, u_n\rangle u_n$ must converge in H but not necessarily to x.

(g) If (x_n) is an unbounded sequence in a Hilbert space H, then there must exist $x \in H$ such that the sequence $(\langle x, x_n \rangle)$ is unbounded.

(h) If in a Hilbert space H, every weakly convergent sequence is norm convergent, then H must be separable.

(i) If (x_n) is a sequence in a Hilbert space H such that $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{w} x \in H$, then it is necessary that $\|x\| \leq 1$.

(j) Suppose X is a Banach space and $f_n \in X^*$ and $f_n \xrightarrow{w^*} 0$. Then f_n is necessarily a bounded sequence in X^*.

Core geometry and orthogonality.

2. Let X be an inner product space and let $x, y \in X$. Prove that
   (a) $\|x+y\|\|x-y\| \leq \|x\|^2 + \|y\|^2$.
   (b) If $\lambda > 0$, then $|\langle x, y \rangle| \leq \lambda \|x\|^2 + \frac{1}{4\lambda}\|y\|^2$.
   (c) If $\delta = \inf\{\|\alpha x + y\| : \alpha \in \mathbb{K}\}$, then $|\langle x, y \rangle|^2 \leq (\|y\|^2 - \delta^2)\|x\|^2$.
   (d) $\Big| \|x\| - \|y\| \Big| = \|x - y\|$ if and only if $tx = sy$ for some $t, s \geq 0$ with $(t, s) \neq (0, 0)$.
3. Let X be an inner product space and let $v, w \in X$ such that $\|v\|\|w\| < 1$. Show that for each $y \in X$, there exists a unique $x \in X$ such that $y = x + \langle x, v \rangle w$.
4. Show that it is impossible to define an inner product on X that induces the norm $\|\cdot\|$ when $(X, \|\cdot\|)$ is one of the following:
   (a) $(c_{00}, \|\cdot\|_\infty)$;
   (b) $(C[a, b], \|\cdot\|_\infty)$;
   (c) $\mathcal{B}(\ell^2)$ equipped with the usual operator norm.
5. Let X be a normed linear space such that every two dimensional subspace of X is an inner product space. Show that X is an inner product space.
6. Consider the following three conditions regarding the points x, y in an inner product space.
   (a) $\|x + \alpha y\| = \|x - \alpha y\|$ for all $\alpha \in \mathbb{K}$
   (b) $\|x + \alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{K}$
   (c) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

   Prove that each of (a) and (b) is a necessary and sufficient condition for $x \perp y$. Is there an analogous statement for (c)?
7. Let Y be a closed subspace of an inner product space X. Show that there exists an inner product on the quotient space X/Y which induces the quotient norm on X/Y.
8. Let M be a closed subspace of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$.
   If $\langle x_1 + M, x_2 + M \rangle_0 = \langle x_1 - P_M x_1, x_2 - P_M x_2 \rangle$ for all $x_1, x_2 \in H$, then show that $\langle \cdot, \cdot \rangle_0$ is an inner product on the quotient space H/M which induces the quotient norm on H/M.
9. If M and N are closed subspaces of a Hilbert space, then show that $(M \cap N)^\perp = \overline{M^\perp + N^\perp}$.
10. Let X be an inner product space and let $x \in X$. If $M = \{z \in X : \langle x, z \rangle = 0\}$, then determine $M^\perp$ and $M^{\perp\perp}$.
11. Let S be a nonempty subset of a Hilbert space H. Show that $S^{\perp\perp} = \overline{\text{span}(S)}$.
    Hence deduce that $\text{span}(S)$ is dense in H if and only if $S^\perp = \{0\}$.

12. Let M be a subspace of an inner product space X and let $x \in X$. Prove that $x \perp M$ if and only if $\|x\| \leq \|x + y\|$ for all $y \in M$.
13. Let M be a closed subspace of a Hilbert space H and let $x \in H \setminus M$. Prove that $d(x, M) = \sup\{|\langle x, y\rangle| : y \in M^\perp,\ \|y\| \leq 1\}$.
14. Let H be a Hilbert space. Let $M \subset H$ and let $T : H \to H$ be linear such that $Tx \in M$ and $x - Tx \in M^\perp$ for each $x \in H$. Show that M is a closed subspace of H.
15. Let M be a nonempty subset of a Hilbert space H and let $z \in H$. Show that there exists $u \in M^\perp$ such that $\langle x, z\rangle = \langle x, u\rangle$ for all $x \in M^\perp$.
16. If $\omega = e^{2\pi i/3}$, determine the point in the subspace

$$\operatorname{span}\{(1, \omega, \omega^2), (1, \omega^2, \omega)\} \subset \mathbb{C}^3$$

that is nearest to $(1, -1, 1)$.
17. Let C be a nonempty convex subset of an inner product space X and let $x \in X$. Prove that for $y \in C$, $d(x, C) = \|x - y\|$ if and only if $\operatorname{Re}\langle x - y, z - y\rangle \leq 0$ for all $z \in C$.
18. Let C be a nonempty convex set in an inner product space X and let (x_n) be a sequence in C such that $\lim\limits_{n\to\infty} \|x_n\| = \inf\limits_{x\in C} \|x\|$. Show that (x_n) is a Cauchy sequence in X.

Weak convergence, duality, and Fourier expansions.

19. Let M be a closed subspace of a Hilbert space H. If $x \in M$ and if (x_n) is a sequence in M, then show that $x_n \xrightarrow{w} x$ in H if and only if $x_n \xrightarrow{w} x$ in M.
20. Let (x_n) be a sequence in a Hilbert space H such that for each $x \in H$, the sequence $(\langle x_n, x\rangle)$ converges in $\mathbb{K}$. Show that there exists $y \in H$ such that $x_n \xrightarrow{w} y$ in H.
21. Using Riesz representation theorem, show that $\left\{(x_n) \in \ell^2 : \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}} x_n = 0\right\}$ is not a closed subset of the Hilbert space ℓ^2.
22. Let $\{u_1, ..., u_n\}$ be an orthonormal set in a Hilbert space H. Prove that $\|x - \sum\limits_{i=1}^{n} \alpha_i u_i\| \geq \|x - \sum\limits_{i=1}^{n} \langle x, u_i\rangle u_i\|$ for all $x \in H$ and for all $(\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$.
23. Let (T_n) be a Cauchy sequence in $\mathcal{B}(X)$, where X is an inner product space. Let $y \in X$ and let $f_n(x) = \langle T_n x, y\rangle$ for all $x \in X$. Show that (f_n) is a convergent sequence in X^*.
24. Let (u_n) be an orthonormal sequence in an inner product space X and let (α_n) be a sequence in $\mathbb{K}$. If $s_n = \sum\limits_{i=1}^{n} \alpha_i u_i$ for all $n \in \mathbb{N}$, then show that (s_n) is a Cauchy sequence in X if and only if $(\alpha_n) \in \ell^2$.
25. Let H be an infinite-dimensional Hilbert space. Show that no orthonormal basis of H can be a Hamel basis of H.
26. Let $f(x) = \int\limits_0^1 t x(t)\, dt$ for all $x \in C[0, 1]$. Show that $f \in (C[0, 1], \|\cdot\|_2)^*$ and find $\|f\|$.

27. Let $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space H and let $f \in H^*$. Prove that $y = \sum_{n=1}^{\infty} \overline{f(u_n)} u_n$ is the unique element in H such that $f(x) = \langle x, y \rangle$ for all $x \in H$ and that $\|f\|^2 = \sum_{n=1}^{\infty} |f(u_n)|^2$.
28. Let $(H, \|\cdot\|)$ be a separable Hilbert space with an orthonormal basis $\{u_n : n \in \mathbb{N}\}$. If $\|x\|_0 = \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, u_n \rangle|$ for all $x \in H$, then show that $\|\cdot\|_0$ is a norm on H which is not equivalent to $\|\cdot\|$.
29. Let $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space H and let $\{v_n : n \in \mathbb{N}\}$ be an orthonormal set in H such that $\sum_{n=1}^{\infty} \|u_n - v_n\|^2 < 1$. Show that $\{v_n : n \in \mathbb{N}\}$ is an orthonormal basis of H.

Synthesis and challenge problems.

30. Find $\min_{a,b,c \in \mathbb{R}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx$.
31. Give an example to show that the range of a one-to-one continuous linear map from a Hilbert space H to itself need not be closed in H.
32. If $\{u_n : n \in \mathbb{N}\}$ is a countably infinite orthonormal basis of a Hilbert space H, then show that there exists a discontinuous linear map $T : H \to H$ such that $Tu_n = 0$ for all $n \in \mathbb{N}$.
33. Let H be a Hilbert space and let (T_n) be a sequence in $\mathcal{B}(H)$ such that for each $x, y \in H$, $\lim_{n\to\infty} \langle T_n x, y \rangle$ exists in $\mathbb{K}$. Show that $\sup\{\|T_n\| : n \in \mathbb{N}\} < \infty$.
34. If H is a non-zero Hilbert space, then show that there cannot exist $T, S \in \mathcal{B}(H)$ such that $TS - ST = I$, where I denotes the identity operator on H.
35. Let X and Y be normed linear spaces. Suppose $T : X \to Y$ is a linear map that sends every weakly convergent sequence in X to a weakly convergent sequence in Y. Show that T is bounded.
36. Suppose X is a Banach space and $x, x_n \in X$. Prove that $x_n \xrightarrow{w} x$ if and only if $\{\|x_n\|\}$ is bounded and
$$f(x_n) \to f(x) \qquad \text{for every } f \in S,$$
where $\overline{\operatorname{span} S} = X^*$.
37. A sequence $\{f_n\}$ in X^* is weak-$*$ convergent if and only if $\{\|f_n\|\}$ is bounded and $\{f_n(x)\}$ is a Cauchy sequence for each $x \in S$, where $\overline{\operatorname{span} S} = X$.

7.2 Adjoints, Projections, and Compact Operators

The exercises in this section revisit the algebraic and geometric operator classes developed in the middle chapters. They are intended to consolidate fluency with adjoints, projections, compactness, and the interplay between operator structure and Hilbert-space geometry.

1. State whether each of the following statements is true or false, giving a justification in each case.

(a) If H is a Hilbert space and $T \in \mathcal{B}(H)$ is selfadjoint, then it is necessary that $T \geq 0$ or $T \leq 0$.

(b) If H is a Hilbert space and $T \in \mathcal{B}(H)$ is selfadjoint such that $\mathrm{Ran}(T)$ is dense in H, then T must be one-to-one.

(c) If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is selfadjoint, then there must exist $\alpha \in \mathbb{R}$ with $\alpha > 0$ such that $\alpha T \leq I$.

(d) If X is an infinite-dimensional normed linear space, $p(t)$ is a polynomial in t with coefficients in $\mathbb{K}$ and $T \in \mathcal{B}_0(X)$, then $p(T) \in \mathcal{B}_0(X)$ if and only if $p(0) = 0$.

(e) If H is a Hilbert space and $T, S \in \mathcal{B}(H)$ such that TS is compact, then at least one of T and S must be compact.

(f) Every $T \in \mathcal{B}_0(\ell^2) \setminus \mathcal{B}_{00}(\ell^2)$ is normal.

(g) Every bounded linear operator from $(c_0, \|\cdot\|_\infty)$ to $(\ell^2, \|\cdot\|_2)$ is compact.

(h) If H is a Hilbert space and $T \in \mathcal{B}_0(H)$, then $T(B_H)$ must be a compact subset of H.

(i) If $T \in \mathcal{B}_0(H)$ such that $Tx = x$ for some $x \in H$ with $\|x\| = 1$, then there must exist $y \in H$ such that $\|y\| = 1$ and $T^*y = y$.

(j) If H is a Hilbert space, $T \in \mathcal{B}(H)$ and (T_n) is a sequence in $\mathcal{B}_0(H)$ such that $T_n x \to Tx$ for each $x \in H$, then T must be compact.

(k) If H is a Hilbert space and $T, S \in \mathcal{B}(H)$ such that $TT^* + SS^* = 0$, then it is necessary that $T = S = 0$.

(l) If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is bounded below, then T cannot be compact.

(m) If H is a Hilbert space and $T \in \mathcal{B}(H)$ is normal such that T is bounded below, then T must be invertible in $\mathcal{B}(H)$.

Adjoints and operator identities.

2. Let X and Y be normed linear spaces, and let $T \in \mathcal{B}(X, Y)$. Show that $T^* : Y^* \to X^*$ is one-to-one if and only if $T(X)$ is dense in Y.

3. Let H be a Hilbert space and let $y \in H$. If $f(x) = \langle x, y\rangle$ for all $x \in H$, then determine the adjoint operator f^*.

4. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be such that $\dim(\mathrm{Ran}(T)) = 1$. Show that there exist $y, z \in H$ such that $Tx = \langle x, y\rangle z$ for all $x \in H$. Also, find T^*.

5. Consider $T \in \mathcal{B}(\ell^2)$, defined by $T((x_n)) = (0, 3x_1, x_2, 3x_3, x_4, ...)$ for all $(x_n) \in \ell^2$. Determine T^*.

6. Let (α_n) be a sequence in $\mathbb{K}$ and let $T((x_n)) = (\alpha_n x_n)$ for all $(x_n) \in \ell^2$. Prove that $T \in \mathcal{B}(\ell^2)$ if and only if $(\alpha_n) \in \ell^\infty$ and in such case determine $\|T\|$ and T^*. Also, prove that

   (a) T is selfadjoint if and only if $\alpha_n \in \mathbb{R}$ for each $n \in \mathbb{N}$.

   (b) $T \geq 0$ if and only if $\alpha_n \geq 0$ for each $n \in \mathbb{N}$.

   (c) T is unitary if and only if $|\alpha_n| = 1$ for all $n \in \mathbb{N}$.

7. Let $\{u_n : n \in \mathbb{N}\}$ be a countably infinite orthonormal basis of a Hilbert space H. Consider the bounded linear operator $T : \ell^2 \to H$ defined by

$$T((\alpha_n)) = \sum_{n=1}^{\infty} \alpha_n u_n \qquad ((\alpha_n) \in \ell^2).$$

   Determine the adjoint operator T^*.

8. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. Prove that
   (a) $\ker(T) = (\mathrm{Ran}(T^*))^\perp = \ker(T^*T)$.
   (b) $\ker(T^*) = (\mathrm{Ran}(T))^\perp = \ker(TT^*)$.
   (c) $(\ker(T))^\perp = \overline{\mathrm{Ran}(T^*)} = \overline{\mathrm{Ran}(T^*T)}$.
   (d) $(\ker(T^*))^\perp = \overline{\mathrm{Ran}(T)} = \overline{\mathrm{Ran}(TT^*)}$.
9. Let H be a Hilbert space and let $T, S \in \mathcal{B}(H)$. Show that $(\mathrm{Ran}(T) + \mathrm{Ran}(S))^\perp = \ker(T^*) \cap \ker(S^*)$.
10. Let H be a Hilbert space. Show that $T \in \mathcal{B}(H)$ is invertible in $\mathcal{B}(H)$ if and only if both T and T^* are bounded below.
11. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ with $\|T\| = 1$. If $x \in H$ such that $Tx = x$, then show that $T^*x = x$.
12. Let H be a Hilbert space and let $x \in H$, $T \in \mathcal{B}(H)$. Show that $T^*Tx = \|T\|^2 x$ if and only if $\|Tx\| = \|T\|\|x\|$.
13. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ such that $T \geq 0$. If $x \in H$ such that $\langle Tx, x\rangle = 0$, then show that $x \in \ker(T)$.
14. Let M, N be closed subspaces of a Hilbert space H and let $T \in \mathcal{B}(H)$. Show that $T(M) \subset N$ if and only if $T^*(N^\perp) \subset M^\perp$.
15. Let $\{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space H and let $T \in \mathcal{B}(H)$. Show that $\left\| \sum\limits_{n=1}^{\infty} \langle Tx, u_n\rangle T u_n \right\|^2 = \sum\limits_{n=1}^{\infty} |\langle Tx, T^* u_n\rangle|^2$.
16. Let H be a Hilbert space and let $T, S \in \mathcal{B}(H)$ such that T is selfadjoint. Show that $TS = 0$ if and only if $\mathrm{Ran}(T) \perp \mathrm{Ran}(S)$.
17. Let H be a Hilbert space. If $T \in \mathcal{B}(H)$ is selfadjoint and if $T \neq 0$, then show that $T^n \neq 0$ for each $n \in \mathbb{N}$.
18. Let (T_n) be a sequence of (bounded) selfadjoint operators on a Hilbert space H and let $T \in \mathcal{B}(H)$. If $T_n x \xrightarrow{w} Tx$ for each $x \in H$, then show that T is selfadjoint.
19. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be selfadjoint. If $\|x\| = d(x, \mathrm{Ran}(T))$ for all $x \in \ker(T)$, then show that $\|\cdot\|$ is a norm on $\ker(T)$.

   Positivity, projections, and normality.

20. If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is positive, then show that
   (a) $|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle \langle Ty, y\rangle$ for all $x, y \in H$.
   (b) $\|Tx\|^2 \leq \|T\| \langle Tx, x\rangle$ for all $x \in H$.

21. If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is positive, then show that $I + T : H \to H$ is invertible.
22. If H is a Hilbert space and if $T \in \mathcal{B}(H)$, then show that both $I + T^*T : H \to H$ and $I + TT^* : H \to H$ are invertible.
23. Let H be a Hilbert space. If $T, S \in \mathcal{B}(H)$ are selfadjoint and satisfy $T \geq S$, is it necessary that $T^2 \geq S^2$?
24. If H is a Hilbert space and if $T \in \mathcal{B}(H)$ such that $\|T\| \leq 1$, then show that $I - T^*T \geq 0$.
25. Let H be a Hilbert space. If $T \in \mathcal{B}(H)$ is positive and invertible, then show that $T^{-1} \geq 0$.
26. Let H be a Hilbert space. If $T \in \mathcal{B}(H)$ such that $0 \leq T \leq I$, then show that $T^2 \leq T$.
27. Let H be a Hilbert space and let (T_n) be a sequence in $\mathcal{B}(H)$ such that $T_n \geq 0$ for all $n \in \mathbb{N}$. If $T_n \xrightarrow{\text{WOT}} 0$, then show that $T_n \xrightarrow{\text{SOT}} 0$.
28. Let H be a Hilbert space. If $T \in \mathcal{B}(H)$ and if $\alpha, \beta \in \mathbb{K}$ such that $|\alpha| = |\beta|$, then show that $\alpha T + \beta T^*$ is normal.
29. Let H be a Hilbert space. Show that $T \in \mathcal{B}(H)$ is normal if and only if $\langle Tx, Ty \rangle = \langle T^*x, T^*y \rangle$ for all $x, y \in H$.
30. If H is a Hilbert space and $T \in \mathcal{B}(H)$ is normal, then show that $\ker(T^2) = \ker(T)$.
31. If H is a Hilbert space and if $T \in \mathcal{B}(H)$ is normal, then show that $\|T^n\| = \|T\|^n$ for each $n \in \mathbb{N}$.
32. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. If $TT^* \leq T^*T$, then show that $\|T^n\|^2 \leq \|T^{n+1}\| \|T^{n-1}\|$ for all $n \in \mathbb{N}$.
33. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. If $T^2 = T$, then show that T is an orthogonal projection.
34. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be selfadjoint such that $T^3 = T^2$. Show that T is an orthogonal projection.
35. If H is a Hilbert space, then show that every orthogonal projection $P \in \mathcal{B}(H)$ must satisfy $0 \leq P \leq I$.
36. Let H be a Hilbert space and let $P \in \mathcal{B}(H)$ such that $P \neq 0$ and $P^2 = P$. Show that P is an orthogonal projection if and only if $\|P\| = 1$.
37. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. If $T^2 = T$, $M = \operatorname{Ran}(T)$ and $N = \ker(T)$, then show that $P_M - P_N$ is invertible in $\mathcal{B}(H)$ and that $(P_M - P_N)^{-1} = T + T^* - I$.
38. If M, N are closed subspaces of a Hilbert space H, then show that $\|P_M - P_N\| \leq 1$, where P_M and P_N are the orthogonal projection operators on M and N respectively.
39. If H is a Hilbert space and if $P, Q \in \mathcal{B}(H)$ are orthogonal projections, then prove that the following statements are equivalent.
    (a) $P \leq Q$
    (b) $\|Px\| \leq \|Qx\|$ for all $x \in H$
    (c) $\ker(Q) \subset \ker(P)$
    (d) $\operatorname{Ran}(P) \subset \operatorname{Ran}(Q)$

(e) $QP = P$
(f) $PQ = P$
(g) $QPQ = P$
(h) $Q - P$ is an orthogonal projection with $\mathrm{Ran}(Q-P) = \mathrm{Ran}(Q) \cap (\mathrm{Ran}(P))^{\perp}$.

40. If H is a Hilbert space and if $P, Q \in \mathcal{B}(H)$ are orthogonal projections, then show that PQ is an orthogonal projection if and only if $PQ = QP$ and in such case $\mathrm{Ran}(PQ) = \mathrm{Ran}(P) \cap \mathrm{Ran}(Q)$.
41. If H is a Hilbert space and if $P, Q \in \mathcal{B}(H)$ are orthogonal projections such that $PQ = QP$, then show that $P+Q-PQ$ is an orthogonal projection with $\mathrm{Ran}(P+Q-PQ) = \mathrm{Ran}(P)+\mathrm{Ran}(Q)$.
42. Let H be a Hilbert space. Show that $T \in \mathcal{B}(H)$ is selfadjoint and unitary if and only if $T = 2P - I$ for some orthogonal projection $P \in \mathcal{B}(H)$.
43. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ such that $T^*(T-I) = 0$. Show that T is an orthogonal projection.
44. Let H be a Hilbert space and let $A(\neq \emptyset) \subset H$. If $T \in \mathcal{B}(H)$ is unitary, then show that $T(A^{\perp}) = T(A)^{\perp}$.
45. Let $T((x_n)) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, ...)$ for all $(x_n) \in \ell^1$. Prove that $T : (\ell^1, \|\cdot\|_1) \to (\ell^{\infty}, \|\cdot\|_{\infty})$ is linear and bounded but not compact.
46. If $Tx = x$ for all $x \in \ell^1$, then examine whether $T : (\ell^1, \|\cdot\|_1) \to (\ell^2, \|\cdot\|_2)$ is a compact linear operator.
47. Let $T((x_n)) = (x_2, x_1, \frac{1}{2}x_4, \frac{1}{2}x_3, ..., \frac{1}{n}x_{2n}, \frac{1}{n}x_{2n-1}, ...)$ for all $(x_n) \in \ell^2$. Show that $T \in \mathcal{B}_0((\ell^2, \|\cdot\|_2))$.
48. Show that every bounded linear operator from $(\ell^2, \|\cdot\|_2)$ to $(\ell^1, \|\cdot\|_1)$ is compact.
49. Let $(Tx)(t) = x(t^2)$ for all $x \in C[0,1]$ and for all $t \in [0,1]$. Show that $T : (C[0,1], \|\cdot\|_{\infty}) \to (C[0,1], \|\cdot\|_{\infty})$ is linear and bounded but not compact.
50. Let X, Y be normed linear spaces and let $y \in Y$, $f \in X^*$. If $Tx = f(x)y$ for all $x \in X$, then show that $T \in \mathcal{B}_0(X,Y)$.
51. Let X, Y be a normed linear spaces and let $x(\neq 0) \in X$, $y \in Y$. Show that there exists $T \in \mathcal{B}_0(X,Y)$ such that $Tx = y$.
52. Let X, Y be normed linear spaces and let $T \in \mathcal{B}_0(X,Y)$. If $S(x + \ker(T)) = Tx$ for all $x \in X$, then show that $S \in \mathcal{B}_0(X/\ker(T), Y)$.

Compactness, invariance, and asymptotic behaviour.

53. Let H be a Hilbert space and let $T \in \mathcal{B}_0(H)$. Show that there exists $x_0 \in H$ such that $\|x_0\| \leq 1$ and $\|Tx_0\| = \|T\|$.
54. Let X be a Banach space and let $T \in \mathcal{B}_0(X) \setminus \mathcal{B}_{00}(X)$. If $S = \{x \in X : \|x\| = 1\}$, then show that $0 \in \overline{T(S)}$.
55. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. If T^*T is compact, then show that T is compact.
56. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. If T^2 is compact, then show that T is compact.
57. Let X be a normed linear space and let $T \in \mathcal{B}_0(X)$ such that $T^2 = T$. Show that T is a finite rank operator.

58. Let X, Y be Banach spaces and let $T \in \mathcal{B}_0(X, Y)$. If Y is infinite-dimensional, then show that T is not onto.
59. Let X, Y be Banach spaces and let $T \in \mathcal{B}_0(X, Y)$. If $\mathrm{Ran}(T)$ is infinite-dimensional, then show that $\mathrm{Ran}(T)$ cannot be closed in Y.
60. Let X be a Banach space and let (T_n) be a sequence in $\mathcal{B}(X)$ such that for each $x \in X$, $T_n x \to 0$. If $S \in \mathcal{B}_0(X)$, then show that $\|T_n S\| \to 0$.
61. Let X and Y be Banach spaces. Show that the class of all completely continuous linear maps from X to Y is closed in $\mathcal{B}(X, Y)$.
62. Let $X \neq \{0\}$, Y be normed linear spaces such that $\mathcal{B}_0(X, Y)$ is a Banach spaces. Prove that Y is a Banach space.
63. Let H be a Hilbert space and let (T_n), (S_n) be sequences in $\mathcal{B}(H)$ such that $T_n \xrightarrow{\text{SOT}} T$ and $S_n \xrightarrow{\text{SOT}} S$, where $T, S \in \mathcal{B}(H)$. Show that $T_n S_n \xrightarrow{\text{SOT}} TS$.
64. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. If $\lambda \in \mathbb{K}$, then show that $\ker(T - \lambda I)$ is a reducing subspace in H for T.
65. Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be normal. Show that $\{x \in H : \|Tx\| = \|T\|\|x\|\}$ is a reducing subspace in H for T.
66. Let H be an infinite-dimensional Hilbert space and let $T, S \in \mathcal{B}(H)$ such that $S \neq 0, I$ and $STS = TS$. Show that there exists a nonzero proper invariant subspace in H under T.
67. Let H be a Hilbert space and $T \in \mathcal{B}_0(H)$. If M is a closed subspace of H which is invariant under T and if $S(x + M) = Tx + M$ for all $x \in H$, then show that $S \in \mathcal{B}_0(H/M)$.
68. If H is a non-separable Hilbert space and if $T \in \mathcal{B}(H)$, then show that H contains a nonzero proper invariant subspace for T.
69. Show that there is no nonzero proper reducing subspace of the right shift operator on ℓ^2.
70. Let H be a Hilbert space. Let (T_n) and (S_n) be sequences in $\mathcal{B}(H)$ and let $T, S \in \mathcal{B}(H)$. If $T_n \to T$ (in norm) and $S_n \xrightarrow{\text{WOT}} S$, then show that $T_n S_n \xrightarrow{\text{WOT}} TS$.
71. Let H be a Hilbert space. Let (T_n) be a sequence in $\mathcal{B}(H)$ and let $T \in \mathcal{B}(H)$. If for each $x \in H$, $\|T_n x\| \to \|Tx\|$ and $\langle T_n x, x\rangle \to \langle Tx, x\rangle$ as $n \to \infty$, then show that $T_n \xrightarrow{\text{SOT}} T$.
72. Let H be a Hilbert space. Let $T_n \in \mathcal{B}(H)$ be normal for each $n \in \mathbb{N}$ and let $T \in \mathcal{B}(H)$ be normal. If $T_n \xrightarrow{\text{SOT}} T$, then show that $T_n^* \xrightarrow{\text{SOT}} T^*$.

7.3 Spectrum, Resolvent, and Normal Operators

These problems focus on the spectral viewpoint that culminates in the final chapters. They invite the reader to test the abstract theory against model operators and to sharpen intuition about resolvents, spectral decomposition, and normality.

1. State whether each of the following statements is true or false, giving a justification in each case.
   (a) If H is a complex Hilbert space, then for each $T \in \mathcal{B}(H)$, there exist $A, B \in \mathcal{B}(H)$ such that A and B are invertible in $\mathcal{B}(H)$ and $T = A + B$.

(b) If H is a complex Hilbert space and $T \in \mathcal{B}(H)$ such that $3T^3 + 4I = 2T^2$, then $\sigma(T)$ must be a finite set.

(c) If H is a complex Hilbert space and $T \in \mathcal{B}(H)$ such that $r_\sigma(T) = \|T\|$, then it is necessary that $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$.

(d) If H is a complex Hilbert space and if $T \in \mathcal{B}(H)$ is selfadjoint, then $I + iT$ must be invertible in $\mathcal{B}(H)$.

(e) If H is an infinite-dimensional complex Hilbert space and $T \in \mathcal{B}(H)$ such that T^2 is compact, then it is necessary that $0 \in \sigma(T)$.

(f) If X is a nonzero Banach space over $\mathbb{C}$ and $T \in \mathcal{B}_0(X)$, then it is necessary that $\sigma(T) = \sigma_{ap}(T)$.

(g) There does not exist any nonzero proper reducing subspace in the Hilbert space $\mathbb{C}^2$ for $T \in \mathcal{B}(\mathbb{C}^2)$, where $T(x, y) = (y, 0)$ for all $(x, y) \in \mathbb{C}^2$.

(h) If H is an infinite-dimensional separable Hilbert space over $\mathbb{C}$ and $T \in \mathcal{B}(H)$ is normal, then $\sigma_p(T)$ must be countable.

(i) If H is a complex Hilbert space and $T \in \mathcal{B}_0(H)$ such that $I + T$ is one-to-one, then $I + T$ must be invertible in $\mathcal{B}(H)$.

Model computations and first spectral tests.

2. Let X be a Banach space and let $T \in \mathcal{B}(X)$ such that $\|T^m\| < 1$ for some $m \in \mathbb{N}$. Show that $I - T$ is invertible in $\mathcal{B}(X)$ and that $(I - T)^{-1} = \sum\limits_{n=0}^{\infty} T^n$.

3. Let X be a Banach space and $T \in \mathcal{B}(X)$. Prove that $\exp(T) = \sum_{n=0}^{\infty} \frac{T^n}{n!}$ is invertible and $\sigma(\exp T) = \exp(\sigma(T))$.

4. Let X be a nonzero Banach space over $\mathbb{C}$ and let $T \in \mathcal{B}(X)$. If E is the set of all eigenvectors of T and if $E^0 \neq \emptyset$, then show that there exists $\lambda \in \mathbb{C}$ such that $T = \lambda I$.

5. Let S be the left shift operator on ℓ^2. Show that there does not exist any $T \in \mathcal{B}(\ell^2)$ such that $T^2 = S$.

6. Let R be the right shift operator on ℓ^2. Prove that
   (a) resolvent set $\rho(R) = \{\lambda \in \mathbb{C} : \ |\lambda| > 1\}$,
   (b) point spectrum $\sigma_p(R) = \emptyset$,
   (c) continuous spectrum $\sigma_c(R) = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}$,
   (d) residual spectrum $\sigma_r(R) = \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}$.

7. Let $T : (C[0, 1], \|.\|_\infty) \to (C[0, 1], \|.\|_\infty)$ be a linear map defined by $Tf(t) = f\left(\frac{t}{3}\right)$. Find the spectral radius of T and show that $0 \in \sigma(T)$. Is T compact?

8. Let $T : L^2[0, 1] \to L^2[0, 1]$ be a linear map defined by $T(f)(x) = \int\limits_0^x f(t)dt$. Show that spectral radius $r(T) = 0$ and $0 \in \sigma_c(T)$, continuous spectrum.

9. Let $g \in C[0, 1]$ and $T : L^2[0, 1] \to L^2[0, 1]$ be a linear map defined by $T(f)(t) = g(t)f(t)$. Find the spectrum $\sigma(T)$ and deduce that T is not compact.

10. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$. For $x = (x_k)_{-\infty}^{\infty} \in \ell^2(\mathbb{Z})$, define $T(x) = (x_{k-1})_{-\infty}^{\infty}$ (right shift operator). Show that

(a) the point spectrum $\sigma_p(T) = \emptyset$,
(b) $\mathrm{Im}(\lambda I - T) = \ell^2(\mathbb{Z})$ if $|\lambda| \neq 1$,
(c) spectrum $\sigma(T) = \{\lambda : \ |\lambda| = 1\}$.

11. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$. For $x \in \ell^2(\mathbb{N})$, define $T(x) = (x_2, x_3, \ldots)$. Prove that
    (a) $\rho(T) = \{\lambda \in \mathbb{C} : \ |\lambda| > 1\}$.
    (b) $\sigma_c(T) = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}$.
    (c) $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}$.
    (d) $\sigma_r(T) = \emptyset$.
12. Let g be a continuous and bounded function on $\mathbb{R}$. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be linear map defined by $T(f)(t) = g(t)f(t)$. Show that T is bounded and spectrum $\sigma(T) = \overline{\{g(x) : x \in \mathbb{R}\}}$.
13. Let P be an orthogonal projection on a Hilbert space H. Show that $\sigma(P) = \sigma_p(P) = \{0, 1\}$. Further, derive that the resolvent function for P is given by $R_P(\lambda) = (\lambda I - P)^{-1} = \frac{I}{\lambda} + \frac{1}{\lambda(1-\lambda)} P$.
14. A bounded linear operator T on a separable Hilbert space H is called Hilbert-Schmidt operator if there exists an orthonormal basis $\{e_n : \ n \in \mathbb{N}\}$ such that $\sum \|Te_n\|^2 < \infty$. Write $\|T\|_{\text{H.S.}} = \left(\sum \|Te_n\|^2\right)^{1/2}$. Show that
    (a) T is a compact operator.
    (b) $\|T^*\|_{\text{H.S.}} = \|T\|_{\text{H.S.}}$.
    (c) Hilbert-Schmidt norm is independent of choice of orthonormal basis.
15. Let $y \in C([0, 1], \mathbb{R})$ and let $T : (C([0, 1], \mathbb{R}), \| \cdot \|_\infty) \to (C([0, 1], \mathbb{R}), \| \cdot \|_\infty)$ be defined by $Tx(t) = x(t)y(t)$ for all $t \in [0, 1]$. Determine $\sigma(T)$.

    Resolvent identities and abstract spectral structure.
16. If X is a complex Banach space and if $T \in \mathcal{B}(X)$ such that $T^n = 0$ for some $n \in \mathbb{N}$, then show that $\sigma(T) = \{0\}$.
17. If X is a complex Banach space and if $T \in \mathcal{B}(X)$ is invertible in $\mathcal{B}(X)$, then show that $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.
18. If X is a complex Banach space and if $T, S \in \mathcal{B}(X)$, then show that $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$, although it is not necessary that $\sigma(TS) = \sigma(ST)$.
19. Let X be a nonzero Banach space over $\mathbb{C}$ and let $T, S \in \mathcal{B}(X)$. If T is invertible in $\mathcal{B}(X)$, then show that $\sigma(TS) = \sigma(ST)$.
20. Let $T : (\ell^\infty, \| \cdot \|_\infty) \to (\ell^\infty, \| \cdot \|_\infty)$ be defined by $T((x_n)) = (x_2, x_3, ...)$ for all $(x_n) \in \ell^\infty$. Determine $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.
21. Let $T : \ell^2 \to \ell^2$ be defined by $T((x_n)) = \left(\frac{n}{n+1} x_n\right)$ for all $(x_n) \in \ell^2$. Determine $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$, $\sigma_{ap}(T)$ and $\sigma_{cp}(T)$.
22. Let $T((x_n)) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...)$ and $S((x_n)) = (x_2, \frac{x_3}{2}, \frac{x_4}{3}, ...)$ for all $(x_n) \in \ell^2$. Prove the following:
    (a) $T, S \in \mathcal{B}_0(\ell^2)$.
    (b) $\sigma(T) = \{0\}$, $\sigma_p(T) = \emptyset$ and $\sigma(S) = \sigma_p(S) = \{0\}$.
    (c) $\overline{\mathrm{Ran}(T)} \neq \ell^2$ and $\overline{\mathrm{Ran}(S)} = \ell^2$.

23. Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$ such that $|\langle Tx, x\rangle| \geq \langle x, x\rangle$ for all $x \in H$. Show that $0 \notin \sigma(T)$.
24. Let H be a nonzero Hilbert space over $\mathbb{C}$ and let $T \in \mathcal{B}(H)$. Show that $\sigma(T^*) = \sigma_{ap}(T^*) \cup \{\overline{\lambda} : \lambda \in \sigma_{ap}(T)\}$.
25. Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$. Show that $\{\lambda^n : \lambda \in \sigma_{ap}(T)\} \subset \sigma_{ap}(T^n)$ for all $n \in \mathbb{N}$.
26. If X is a complex Banach space and if $T \in \mathcal{B}(X)$, then show that $\sigma_{ap}(T)$ is a compact subset of $\mathbb{C}$.
27. Let X be a nonzero Banach space over $\mathbb{C}$. If (T_n) is a sequence in $\mathcal{G}(X)$ such that $T_n \xrightarrow{\|\cdot\|} T \in \mathcal{B}(X) \setminus \mathcal{G}(X)$, then show that $0 \in \sigma_{ap}(T)$.
28. If X is a complex Banach space and if $T \in \mathcal{B}(X)$, then show that $\partial\sigma(T) \subset \sigma_{ap}(T)$.

Hilbert-space spectral geometry and normal operators.

29. Let H be a complex Hilbert space and $y, z \in H$. If $Tx = \langle x, y\rangle z$ for all $x \in H$, then show that $\sigma(T) = \{0, \langle z, y\rangle\}$.
30. If H is a complex Hilbert space and if $T \in \mathcal{B}(H)$ is normal, then show that $\sigma_r(T) = \emptyset$, $\sigma_{ap}(T) = \sigma(T)$ and $r_\sigma(T) = \|T\|$.
31. If H is a complex Hilbert space and $T \in \mathcal{B}(H)$, then show that $\sigma_r(T) \subset \{\lambda \in \mathbb{C} : |\lambda| < \|T\|\}$.
32. Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$. If there is no nonzero proper invariant subspace in H under T, then show that $\sigma(T) = \sigma_c(T)$.
33. Let X be a complex Banach space and let $T(\neq 0, I) \in \mathcal{B}(X)$ such that $T^2 = T$. Show that $\sigma(T) = \sigma_p(T) = \{0, 1\}$.
34. Let H be a complex Hilbert space and let $T(\neq -I, I) \in \mathcal{B}(H)$ such that $T^2 = I$. Show that $\sigma(T) = \{-1, 1\}$.
35. Let X be a complex Banach space and let $T, T_n \in \mathcal{B}(X)$ for all $n \in \mathbb{N}$ such that $T_n \to T$. If $\lambda_n \in \sigma(T_n)$ for all $n \in \mathbb{N}$ and if $\lambda_n \to \lambda \in \mathbb{C}$, then show that $\lambda \in \sigma(T)$.
36. If H is a complex Hilbert space and if $T \in \mathcal{B}(H)$, then show that $\sigma_c(T^*) = \{\overline{\lambda} : \lambda \in \sigma_c(T)\}$.
37. If H is a non-separable Hilbert space and if $T \in \mathcal{B}_0(H)$, then show that $0 \in \sigma_p(T)$.
38. Let K be a nonempty compact subset of $\mathbb{C}$. Show that there exists $T \in \mathcal{B}((\ell^2, \|\cdot\|_2))$ such that $\sigma(T) = K$.
39. Let H be a complex Hilbert space. If $T \in \mathcal{B}(H)$ and $\lambda \in W(T)$ such that $|\lambda| = \|T\|$, then show that $\lambda \in \sigma_p(T)$.
40. Let H be an infinite-dimensional Hilbert space over $\mathbb{C}$ and let $T \in \mathcal{B}(H)$. If $T^n \xrightarrow{\text{WOT}} 0$, then show that $\sigma_p(T) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.
41. Let H be a complex Hilbert space. If $T \in \mathcal{B}(H)$ is normal and $\sigma(T) = \{0, 1\}$, then show that $T^2 = T$.

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