DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA550: Measure Theory — MidSem
Instructor: Rajesh Srivastava — March 1, 2024
Time duration: Two hours — Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) Whether an infinite σ-algebra on $\mathbb{Z}$ is countable? **[1]**

   (b) What is cardinality of the class of all G_δ sets in $\mathbb{R}$? **[1]**

   (c) Does there exist a decreasing sequence (E_n) of Lebesgue measurable sets in $\mathbb{R}$ such that $\lim m(E_n) = m(\cap E_n)$? **[1]**

   (d) Let E be subset of $\mathbb{R}$ and let f be function on $\mathbb{R}$ such that $f = 1$ on E and 0 elsewhere. If $\{x \in \mathbb{R} : f(x) > a\} \in M(\mathbb{R})$ for all $a \in \mathbb{R}$, does it imply that $E \in M(\mathbb{R})$? **[1]**

2. Let $A \in M(\mathbb{R})$ and $m^*(A \triangle B) = 0$. Show that $B \in M(\mathbb{R})$ and $m(A) = m(B)$. **[2]**

3. Let E be subset of $\mathbb{R}$. Show that $m^*(\sin(E)) \leq m^*(E)$. **[3]**

4. Let Let $E \in M(\mathbb{R})$ and $m(E) < \infty$. Show that for each $\epsilon > 0$ there exists a compact set K such that $m(E \triangle K) < \epsilon$. **[3]**

5. Let E and F be two disjoint compact subsets of $\mathbb{R}$. Show that $m^*(E \cup F) = m^*(E) + m^*(F)$. (Without using measurability of compact sets.) **[4]**

6. Let $E \in M(\mathbb{R})$ and $m(E) = 2$. Show that there exists a compact subset K of E such that $m(K) = 1$. **[4]**

7. Let $E \in M(\mathbb{R})$ be such that $m(E \cap (0, 2)) > \frac{3}{2}$. Show that for each $x \in (-1, 1)$, $(E \cap (0, 2)) \cap ((E \cap (0, 2)) + x) \neq \emptyset$. Further, deduce that $(-1, 1) \subset E - E = \{x - y : x, y \in E\}$. **[4+2]**

8. Let S be a σ-algebra on a non-empty set X, and $\mu : S \to [0, \infty]$ be finitely additive with $\mu(\emptyset) = 0$. If for every decreasing sequence (E_n) of S having $\cap E_n = \emptyset$ implies $\lim \mu(E_n) = 0$, show that μ is a measure. **[4]**

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