Measure Theory Lecture Notes

MA550: Measure Theory

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Contents

Index of Notation

$\mathcal{P}(X)$	Power set of a set X.
$\sigma(\mathcal{A})$	σ-algebra generated by a family $\mathcal{A} \subseteq \mathcal{P}(X)$.
$\mathcal{B}(\mathbb{R})$	Borel σ-algebra on $\mathbb{R}$.
χ_E **or** $\mathbf{1}_E$	Indicator function of a set E.
m^*	Outer measure (typically Lebesgue outer measure on $\mathbb{R}$).
m	Lebesgue measure (restriction of m^* to Lebesgue measurable sets).
(X, Σ, μ)	A measure space: Σ a σ-algebra on X, μ a measure on Σ.
$E \triangle F$	Symmetric difference: $(E \setminus F) \cup (F \setminus E)$.
$f^{-1}(A)$	Preimage $\{x \in X : f(x) \in A\}$.
$\|f\|_\infty$	Essential supremum norm: $\|f\|_\infty = \inf\{M : \|f\| \leq M \text{ a.e.}\}$.
$L^p(X, \mu)$	Lebesgue space of measurable functions with $\int \|f\|^p \, d\mu < \infty$ (modulo equality a.e.).
$\|f\|_p$	L^p norm: $\|f\|_p = (\int \|f\|^p \, d\mu)^{1/p}$ for $1 \leq p < \infty$.
a.e.	Almost everywhere (outside a μ-null set).
$\operatorname{ess\,sup}$	Essential supremum.

Introduction

These lecture notes present a self-contained development of measure and integration theory, with Lebesgue measure as the guiding example and general measure spaces as the natural setting. Along the way we emphasize the structural principles that make the theory powerful: countable *additivity, approximation by simple objects, and stability under limits. The final chapters treat three indispensable extensions of the basic theory: signed measures and the Radon–Nikodym theorem, product measures and the Tonelli–Fubini machinery, and regular Borel measures with the Riesz representation theorem.*

Motivation: from Riemann to Lebesgue. The idea of measuring physical quantities (such as length, area, and volume) for geometric objects is classically approached via Riemann integration. However, there are sets and functions whose "size" or accumulated value cannot be captured by the Riemann method—most notably, when the relevant objects are highly discontinuous or unbounded. This limitation calls for a strengthened framework.

Lebesgue's viewpoint (building on contributions of Borel, Baire, Vitali, and others) is to measure sets by decomposing them into *countably many* elementary pieces, with the intuition that one may "recollect" these pieces without quantitative alteration. The elementary pieces are chosen to be nearly open (or nearly closed) sets, and the sets that can be assembled in this way form the class of *measurable sets*. Lebesgue's program also reveals the existence of sets whose irregularity is too severe to admit any consistent notion of size; such sets are called *non-measurable*.

A parallel generalization occurs for integration. For regions bounded by sufficiently regular curves, classical methods reduce area computations to Riemann integrals. When the boundary becomes too irregular to be parametrized by finitely many "almost continuous" functions, one must enlarge the class of admissible parametrizations. This leads naturally to *measurable functions*, which may be viewed as nearly continuous maps (continuous outside sets of arbitrarily small measure). The corresponding method of computing physical quantities is the *Lebesgue integral*, built by approximation with simple functions and distinguished by its excellent limit theorems.

Course scope and organization. These notes develop the foundations of Lebesgue measure and integration, starting from outer measure and Carathéodory measurability and culminating in differentiation of integrals, Radon–Nikodym theory, product measures, and representation theorems. The exposition is written with the syllabus in mind, but it is also intended to serve as a coherent reference: each chapter begins with a short abstract, and proofs are organized to make the logical dependencies transparent.

Roadmap.

Unit I. **Lebesgue outer measure, measurable sets, and Lebesgue measure.** Construction of Lebesgue outer measure; Carathéodory measurability; the induced complete measure.

Unit II. **Algebras and σ-algebras; Borel sets; Carathéodory construction.** Set systems, generated σ-algebras, Borel σ-algebra; outer measures and measures; extension of premeasures and completion.

Unit III. **Measurable functions; Lusin and Egoroff theorems.** Equivalent criteria for measurability; simple functions; approximation and almost-everywhere notions; quantitative approximation by continuity and uniform convergence.

Unit IV. **Lebesgue integration and convergence theorems.** Integral of nonnegative measurable functions; extension to integrable functions; monotone convergence, Fatou's lemma, dominated convergence, and consequences.

Unit V. **L^p-spaces.** The spaces $L^p(X, \Sigma, \mu)$; Hölder and Minkowski inequalities; completeness; basic functional-analytic features.

Unit VI. **Signed measures and the Radon–Nikodym theorem.** Hahn and Jordan decompositions; total variation; absolute continuity and singularity; Radon–Nikodym derivatives; Lebesgue decomposition; duality of L^p.

Unit VII. **Product measures; Tonelli and Fubini.** Product σ-algebra and product measure; iterated integration; measurability issues.

Unit VIII. **Absolute continuity and the Lebesgue fundamental theorem of calculus.** Absolute continuity; differentiation of indefinite integrals; recovery from derivatives almost everywhere.

Unit IX. **Regular Borel measures and the Riesz representation theorem.** Regularity on compact metric spaces; Urysohn functions; representation of positive linear functionals by measures.

Chapter 1

Real analysis preliminaries

Measure theory repeatedly appeals to a small collection of principles from real analysis: order completeness, approximation by countable constructions, and the elementary topology of $\mathbb{R}$. This chapter records the conventions and tools that will be used throughout the notes, so that later arguments (for example, those involving countable coverings, limiting procedures, and regularity) can be written efficiently and without ambiguity.

1.1 Completeness of $\mathbb{R}$ and bounds

Recall that the set of rational numbers is:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0 \right\}.$$

The real line $\mathbb{R}$ strictly contains $\mathbb{Q}$. For instance, $\sqrt{2} \notin \mathbb{Q}$.

Proposition 1.1. *$\sqrt{2}$ is irrational; that is, $\sqrt{2} \notin \mathbb{Q}$.*

Proof. Assume $\sqrt{2} = p/q$ with integers p, q that are coprime and $q \neq 0$. Then $p^2 = 2q^2$, hence p^2 is even and therefore p is even. Write $p = 2m$. Substituting gives $4m^2 = 2q^2$, so $q^2 = 2m^2$ and q is even as well, contradicting $\gcd(p, q) = 1$.

□

Definition 1.2 (Upper and lower bounds)**.** Let $A \subset \mathbb{R}$. A number $M \in \mathbb{R}$ is an *upper bound* of A if $a \leq M$ for every $a \in A$. A number $m \in \mathbb{R}$ is a *lower bound* of A if $m \leq a$ for every $a \in A$.

Definition 1.3 (Supremum and infimum)**.** Assume $A \subset \mathbb{R}$ is nonempty.

(i) The *supremum* of A, denoted $\sup A$, is the least upper bound of A; that is, $\sup A$ is an upper bound and $\sup A \leq M$ for every upper bound M of A.

(ii) The *infimum* of A, denoted $\inf A$, is the greatest lower bound of A; that is, $\inf A$ is a lower bound and $m \leq \inf A$ for every lower bound m of A.

Example 1.4. If $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$, then $\inf A = 0$ and $\sup A = 1$. Note that $\sup A \notin A$.

Theorem 1.5 (Completeness axiom of $\mathbb{R}$)**.** *Every nonempty subset $A \subset \mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$. Equivalently, every nonempty subset that is bounded below has an infimum in $\mathbb{R}$.*

We adopt the standard conventions:

$$\sup \varnothing = -\infty, \qquad \inf \varnothing = \infty,$$

and we write $\sup A = \infty$ if A is not bounded above, and $\inf A = -\infty$ if A is not bounded below.

Proposition 1.6 (Monotonicity)**.** *If $A \subset B \subset \mathbb{R}$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$.*

1.2 Archimedean property and density

Theorem 1.7 (Archimedean property)**.** *If $x > 0$ and $y \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof. Assume, towards a contradiction, that $nx \leq y$ for every $n \in \mathbb{N}$. Then y is an upper bound of $S = \{nx : n \in \mathbb{N}\}$. Let $\alpha = \sup S$. Since $x > 0$, we have $\alpha - x < \alpha$, hence $\alpha - x$ cannot be an upper bound of S. Therefore there exists $n \in \mathbb{N}$ with $nx > \alpha - x$, which implies $(n+1)x > \alpha$, contradicting that α is an upper bound of S.

□

Corollary 1.8 (Density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$)**.** *Between any two distinct real numbers there exists both a rational and an irrational number.*

Proof. Let $x < y$. By the Archimedean property, choose $n \in \mathbb{N}$ such that $n(y - x) > 1$. Then there exists $m \in \mathbb{Z}$ with $nx < m < ny$. Dividing by n yields $x < \frac{m}{n} < y$, and $\frac{m}{n} \in \mathbb{Q}$. For the existence of an irrational in (x, y), apply the rational-density statement to $\left(x/\sqrt{2},\ y/\sqrt{2}\right)$ and multiply the resulting rational by $\sqrt{2}$.

□

Example 1.9. Let $A = \{r \in \mathbb{Q} : r > 0,\ r^2 < 2\}$. Then $\sup A = \sqrt{2} \notin \mathbb{Q}$, illustrating that $\mathbb{Q}$ is not complete.

Example 1.10. Set $A = \left\{\frac{m}{mn+1} : m, n \in \mathbb{N}\right\}$. Then $\inf A = 0$ and $\sup A = 1$. Indeed, $0 < \frac{m}{mn+1} \leq \frac{m}{m+1} < 1$, so 0 is a lower bound and 1 is an upper bound. Moreover, $\frac{1}{n+1} \in A$ (take $m = 1$) and $\frac{m}{m+1} \in A$ (take $n = 1$), so elements of A can be made arbitrarily close to 0 and to 1, respectively.

Proposition 1.11 (Approximation of inf and sup)**.** *Let $A \subset \mathbb{R}$ be nonempty and bounded below, and set $\alpha = \inf A$. Then for every $\varepsilon > 0$ there exists $a \in A$ such that $a < \alpha + \varepsilon$. Similarly, if A is bounded above and $\beta = \sup A$, then for every $\varepsilon > 0$ there exists $b \in A$ such that $b > \beta - \varepsilon$.*

Proof. If $a \geq \alpha + \varepsilon$ for every $a \in A$, then $\alpha + \varepsilon$ would be a lower bound of A, contradicting the definition of α. The supremum statement is analogous.

□

1.3 Sequences, limits, and subsequences

A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$), written $a_n = a(n)$. We say $a_n \to \ell$ as $n \to \infty$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - \ell| < \varepsilon$.

Example 1.12. The sequence $a_n = \frac{1}{n}$ converges to 0. Indeed, given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon}$; then $n \geq N$ implies $|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

Definition 1.13 (Subsequence)**.** If $n_1 < n_2 < \cdots$ is a strictly increasing sequence of integers, then $(a_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of $(a_n)_{n\in\mathbb{N}}$.

1.4 Monotone sequences, nested intervals, and compactness

Proposition 1.14 (Monotone convergence)**.** *If (x_n) is increasing and bounded above, then (x_n) converges and $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$. Similarly, if (x_n) is decreasing and bounded below, then (x_n) converges and its limit equals the infimum of its range.*

Proof. Let $\alpha = \sup\{x_n : n \in \mathbb{N}\}$. Given $\varepsilon > 0$, by the approximation property of the supremum there exists n_0 such that $x_{n_0} > \alpha - \varepsilon$. Since the sequence is increasing, $n \geq n_0$ implies $x_n \geq x_{n_0} > \alpha - \varepsilon$, while always $x_n \leq \alpha$. Hence $|x_n - \alpha| < \varepsilon$ for all $n \geq n_0$.

□

Theorem 1.15 (Nested interval theorem)**.** *Let $I_n = [a_n, b_n]$ be closed intervals with $I_{n+1} \subset I_n$ for all n, and assume $\lim_{n\to\infty}(b_n - a_n) = 0$. Then $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one point.*

Proof. The sequence (a_n) is increasing and bounded above by b_1, hence $a_n \to a$. Similarly (b_n) is decreasing and bounded below by a_1, hence $b_n \to b$. Passing to the limit gives $b - a = \lim(b_n - a_n) = 0$, so $a = b$. Since $a_n \leq a \leq b_n$ for all n, we have $a \in \bigcap_n I_n$. If $x \in \bigcap_n I_n$, then $a_n \leq x \leq b_n$ for all n, hence $x = a$ after letting $n \to \infty$.

□

Theorem 1.16 (Bolzano–Weierstrass)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let $(x_n) \subset [a, b]$. Bisect $I_1 = [a, b]$ into two closed subintervals and choose $I_2 \subset I_1$ that contains infinitely many terms of (x_n). Continuing inductively, obtain nested closed intervals I_k with lengths $b_k - a_k = (b - a)/2^{k-1} \to 0$, each containing infinitely many x_n. Choose n_k strictly increasing such that $x_{n_k} \in I_k$. By the nested interval theorem,

$\bigcap_k I_k = \{x\}$ for some $x \in \mathbb{R}$. Given $\varepsilon > 0$, choose k such that $I_k \subset (x-\varepsilon, x+\varepsilon)$. Then $j \geq k$ implies $x_{n_j} \in I_j \subset I_k \subset (x-\varepsilon, x+\varepsilon)$, so $x_{n_j} \to x$.

□

1.5 Limit superior and limit inferior

For a real sequence (x_n), define:

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty} \sup\{x_n : n \geq k\}, \qquad \liminf_{n\to\infty} x_n = \lim_{k\to\infty} \inf\{x_n : n \geq k\}.$$

One always has $\liminf x_n \leq \limsup x_n$. Moreover, (x_n) converges if and only if $\liminf x_n = \limsup x_n$, in which case both equal $\lim x_n$.

Example 1.17. For $x_n = (-1)^n$, we have $\liminf x_n = -1$ and $\limsup x_n = 1$, so the sequence does not converge.

Example 1.18. If $X_n = (x_n, y_n) \in \mathbb{R}^2$ is bounded, then (x_n) and (y_n) are bounded real sequences. By Bolzano–Weierstrass, there exist subsequences $x_{n_k} \to x$ and $y_{n_{k_\ell}} \to y$, yielding a subsequence $X_{n_{k_\ell}} \to (x, y)$ in $\mathbb{R}^2$.

1.6 Elementary topology of $\mathbb{R}$

A set $U \subset \mathbb{R}$ is *open* if for every $x \in U$ there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subset U$. A set $F \subset \mathbb{R}$ is *closed* if $\mathbb{R} \setminus F$ is open.

Definition 1.19 (Closure)**.** For $A \subset \mathbb{R}$, the *closure* $\overline{A}$ is the set of points $x \in \mathbb{R}$ such that every open interval containing x intersects A. Equivalently, $x \in \overline{A}$ if and only if there exists a sequence $(a_n) \subset A$ with $a_n \to x$.

Proposition 1.20. *The set $\overline{A}$ is closed, contains A, and is the smallest closed subset of $\mathbb{R}$ that contains A.*

Open Sets in $\mathbb{R}$:

A countable union of open intervals is an open set. On the other hand, any open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals.

Theorem 1.21 (Decomposition of open sets in $\mathbb{R}$)**.** *Let $O \subset \mathbb{R}$ be open. Then there exists a countable family of pairwise disjoint open intervals $\{I_n\}_{n\geq 1}$ such that:*

$$O = \dot{\bigcup_{n=1}^{\infty}} I_n.$$

Moreover, this representation is unique up to a permutation of the intervals.

Proof. Fix $x \in O$. Since O is open, there exists an open interval (a, b) with $x \in (a, b) \subset O$. Define the *maximal* interval contained in O and containing x by:

$$a_x := \inf\{a \in \mathbb{R} : (a, x] \subset O\}, \qquad b_x := \sup\{b \in \mathbb{R} : [x, b) \subset O\},$$

and set $I_x := (a_x, b_x)$.

Step 1: $I_x \subset O$. Let $z \in (a_x, b_x)$. Choose $\eta > 0$ so small that $a_x + \eta < z < b_x - \eta$. By definition of the infimum, there exists $a < a_x + \eta$ with $(a, x] \subset O$, hence $(a_x + \eta, x] \subset O$. Similarly, by definition of the supremum there exists $b > b_x - \eta$ with $[x, b) \subset O$, hence $[x, b_x - \eta) \subset O$. Therefore $(a_x + \eta, b_x - \eta) \subset O$, and in particular $z \in O$.

Step 2: maximality and disjointness. By construction, I_x is an open interval with $x \in I_x \subset O$, and it is maximal with respect to these properties. If $x, y \in O$ and $I_x \cap I_y \neq \varnothing$, then $I_x \cup I_y$ is an open interval contained in O that contains x. Maximality forces $I_y \subset I_x$, and by symmetry $I_x \subset I_y$, hence $I_x = I_y$. Consequently, the family $\{I_x : x \in O\}$ consists of pairwise disjoint open intervals, and:

$$O = \bigcup_{x \in O} I_x.$$

Step 3: countability. Each nonempty open interval contains a rational number. Choose $q_x \in I_x \cap \mathbb{Q}$. If $I_x \neq I_y$, then $I_x \cap I_y = \varnothing$, so $q_x \neq q_y$. Thus the map $I_x \mapsto q_x$ is injective into $\mathbb{Q}$, and therefore the collection $\{I_x : x \in O\}$ is countable. Renaming these intervals as $\{I_n\}_{n \geq 1}$ gives the desired representation.

Step 4: uniqueness. Suppose:

$$O = \dot{\bigcup_{n=1}^{\infty}} I_n = \dot{\bigcup_{m=1}^{\infty}} J_m,$$

where $\{I_n\}$ and $\{J_m\}$ are pairwise disjoint families of open intervals. Fix n. Since $I_n \subset O$, we have:

$$I_n = I_n \cap O = \bigcup_{m=1}^{\infty} (I_n \cap J_m).$$

The sets $I_n \cap J_m$ are pairwise disjoint, and each is (possibly empty) open in $\mathbb{R}$. Since I_n is a connected open set (an interval), only one of these intersections can be nonempty; hence $I_n \subset J_{m_0}$ for some m_0. By the same argument with the roles reversed, $J_{m_0} \subset I_n$, and therefore $I_n = J_{m_0}$. This proves uniqueness up to a permutation.

□

Chapter 2

Measures and measurable sets

The inadequacy of the Riemann integral for highly oscillatory or discontinuous functions motivates the measure-theoretic approach. This chapter constructs Lebesgue outer measure and uses Carathéodory's criterion to single out the class of measurable sets, thereby producing Lebesgue measure as a complete, countably additive, translation-invariant measure on $\mathbb{R}$. *We develop the basic calculus of outer measure and measurable sets (monotonicity, countable subadditivity, continuity from above/below), and we work through canonical examples that illuminate the theory, including the Cantor set and the phenomenon of non-measurable sets. The emphasis is on the structural viewpoint: measurability is characterized by how sets interact with all other sets, and measure is built to be compatible with countable limiting operations.*

2.1 Limitations of Riemann integration

The Riemann integral is well suited to bounded functions on compact intervals whose oscillation is sufficiently controlled. However, several natural examples fall outside that framework or are handled only indirectly as improper integrals. This is one of the main motivations for the Lebesgue theory.

Example 2.1 (A highly discontinuous bounded function)**.** Define the *Dirichlet function* on $[0,1]$ by

$$f(x) = \begin{cases} 1, & x \in [0,1] \setminus \mathbb{Q}, \\ 0, & x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

Every subinterval of $[0,1]$ contains both rational and irrational points. Hence for every partition P of $[0,1]$ we have

$$L(P,f) = 0, \qquad U(P,f) = 1.$$

Therefore f is not Riemann integrable.

Example 2.2 (An unbounded function on a finite interval)**.** The function $h(t) = t^{-1/2}$ is not bounded on $(0, 1]$, so it is not Riemann integrable on $[0, 1]$ in the classical sense. Nevertheless, its improper integral exists and

$$\int_0^1 \frac{dt}{\sqrt{t}} := \lim_{n\to\infty} \int_{1/n}^1 \frac{dt}{\sqrt{t}} = \lim_{n\to\infty} 2\Big(1 - \frac{1}{\sqrt{n}}\Big) = 2.$$

This suggests that one should have a theory that can integrate nonnegative functions by approximation from below, without first requiring boundedness.

Example 2.3 (An integral over an unbounded interval)**.** Similarly,

$$\int_0^\infty \frac{dt}{1+t^2} := \lim_{n\to\infty} \int_0^n \frac{dt}{1+t^2} = \lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2}.$$

Again, the natural domain is not compact, but the limiting value exists.

Remark 2.4*.* Lebesgue's idea is to reverse the order of approximation used in the Riemann theory. Instead of partitioning the *domain* into small intervals and controlling oscillation on each piece, one first measures the size of the level sets of the function. This requires a satisfactory notion of length for very general subsets of $\mathbb{R}$, which leads to outer measure.

2.2 Lebesgue Outer Measure

For any bounded interval I with endpoints $a < b$, we write $\ell(I) = b - a$, regardless of whether the interval is open, closed, or half-open. If I is unbounded, we set $\ell(I) = \infty$.

Definition 2.5 (Lebesgue outer measure)**.** For a set $A \subset \mathbb{R}$, the *Lebesgue outer measure* of A is defined by

$$m^*(A) := \inf \left\{ \sum_{n=1}^\infty \ell(I_n) : A \subset \bigcup_{n=1}^\infty I_n,\ I_n \text{ open intervals} \right\}.$$

If $A = \emptyset$, the empty covering shows that $m^*(\emptyset) = 0$.

Remark 2.6*.* The restriction to open intervals is not essential. One obtains the same quantity if one allows arbitrary intervals, or even only bounded open intervals when covering bounded sets.

Lemma 2.7 (Finite interval-covering lemma)**.** *If a compact interval* $[a, b]$ *is covered by finitely many open intervals* $J_1, \dots, J_N$*, then*

$$b - a \le \sum_{k=1}^N \ell(J_k).$$

Proof. We argue by induction on N.

For $N = 1$, if $[a, b] \subset J_1$, then $\ell(J_1) \ge b - a$.

Assume the statement true for $N - 1$ intervals and suppose $[a, b] \subset \bigcup_{k=1}^N J_k$. Reorder the intervals so that $a \in J_1$. Since J_1 is open and contains a, there exists $c > a$ such that $[a, c] \subset J_1$. If $b \le c$, then $[a, b] \subset J_1$ and we are done. Otherwise $[c, b]$ is covered by the remaining intervals together with $J_1 \cap (c, b)$; equivalently, there exists $d \in (c, b]$ such that $[c, d]$ is covered by some subfamily of $J_2, \dots, J_N$ and d is maximal with this property. By induction applied to $[c, b]$, we obtain

$$b - c \leq \sum_{k=2}^{N} \ell(J_k).$$

Since also $c - a \leq \ell(J_1)$, adding gives

$$b - a = (c - a) + (b - c) \leq \sum_{k=1}^{N} \ell(J_k).$$

□

2.3 Properties of the outer measure

Proposition 2.8 (Basic properties of m^*)**.** *For arbitrary sets $A, B, A_1, A_2, \dots \subset \mathbb{R}$ the following hold:*

(i) If $A \subset B$, then $m^*(A) \leq m^*(B)$.

(ii) m^* is countably subadditive:

$$m^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

(iii) m^* is translation invariant: for every $x \in \mathbb{R}$,

$$m^*(A + x) = m^*(A), \qquad A + x := \{a + x : a \in A\}.$$

Proof. (i) Every cover of B is also a cover of A, so taking infima gives $m^*(A) \leq m^*(B)$.

(ii) Fix $\varepsilon > 0$. For each n, choose open intervals $\{I_{n,k}\}_{k \geq 1}$ covering A_n such that

$$\sum_{k=1}^{\infty} \ell(I_{n,k}) < m^*(A_n) + \frac{\varepsilon}{2^n}.$$

Then $\{I_{n,k} : n, k \in \mathbb{N}\}$ covers $\bigcup_n A_n$, and therefore

$$m^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \ell(I_{n,k}) < \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ yields the claim.

(iii) If $A \subset \bigcup_n I_n$, then $A + x \subset \bigcup_n (I_n + x)$ and $\ell(I_n + x) = \ell(I_n)$ for every n. Hence $m^*(A + x) \leq m^*(A)$. Replacing A by $A + x$ and x by $-x$ gives the reverse inequality. □

Example 2.9 (Countable sets have outer measure zero)**.** If $A = \{a_1, a_2, \dots\}$ is countable, then for every $\varepsilon > 0$ the intervals

$$I_n := \Big(a_n - \frac{\varepsilon}{2^{n+1}},\ a_n + \frac{\varepsilon}{2^{n+1}}\Big)$$

cover A and satisfy $\sum_n \ell(I_n) = \varepsilon$. Hence $m^*(A) = 0$. In particular,

$$m^*(\mathbb{Q}) = 0.$$

Example 2.10 (The irrationals have infinite outer measure)**.** We have $m^*(\mathbb{R} \setminus \mathbb{Q}) = \infty$. Indeed, if $m^*(\mathbb{R} \setminus \mathbb{Q}) < \infty$, then by countable subadditivity,

$$m^*(\mathbb{R}) \leq m^*(\mathbb{Q}) + m^*(\mathbb{R} \setminus \mathbb{Q}) < \infty,$$

which is impossible because $[-N, N] \subset \mathbb{R}$ and later we shall prove $m^*([-N, N]) = 2N$ for every $N \in \mathbb{N}$.

Proposition 2.11 (Approximation from outside by open sets)**.** *For every $A \subset \mathbb{R}$,*

$$m^*(A) = \inf\{m^*(O) : A \subset O,\ O \text{ open}\}.$$

In particular, if $m^*(A) < \infty$, then for every $\varepsilon > 0$ there exists an open set $O \supset A$ such that

$$m^*(O) < m^*(A) + \varepsilon.$$

Proof. If $m^*(A) = \infty$, then the statement is immediate from monotonicity. Assume $m^*(A) < \infty$ and fix $\varepsilon > 0$. Choose open intervals I_n covering A with

$$\sum_{n=1}^{\infty} \ell(I_n) < m^*(A) + \varepsilon.$$

Let $O := \bigcup_n I_n$. Then O is open, $A \subset O$, and by countable subadditivity,

$$m^*(O) \leq \sum_{n=1}^{\infty} \ell(I_n) < m^*(A) + \varepsilon.$$

Since always $m^*(A) \leq m^*(O)$ when $A \subset O$, taking the infimum over open supersets O gives the result. □

Corollary 2.12 (G_δ hull)**.** *For every set $A \subset \mathbb{R}$ there exists a G_δ set $G \supset A$ such that*

$$m^*(G) = m^*(A).$$

Proof. If $m^*(A) = \infty$, take $G = \mathbb{R}$. Otherwise, for each $n \in \mathbb{N}$ choose an open set $O_n \supset A$ with

$$m^*(O_n) < m^*(A) + \frac{1}{n}.$$

Set $G := \bigcap_{n=1}^{\infty} O_n$. Then G is a G_δ set containing A. By monotonicity,

$$m^*(A) \leq m^*(G) \leq m^*(O_n) < m^*(A) + \frac{1}{n} \qquad (n \in \mathbb{N}),$$

so $m^*(G) = m^*(A)$. □

2.4 Outer measure of intervals

Theorem 2.13 (Outer measure of an interval)**.** *Let $I \subset \mathbb{R}$ be an interval. Then $m^*(I) = \ell(I)$.*

Proof. We first prove the statement for a compact interval $[a,b]$.

The inequality $m^*([a,b]) \leq b-a$ is immediate, since $[a,b] \subset (a-\varepsilon, b+\varepsilon)$ for every $\varepsilon > 0$, and therefore

$$m^*([a,b]) \leq b-a+2\varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives $m^*([a,b]) \leq b-a$.

For the reverse inequality, let $[a,b] \subset \bigcup_{n=1}^{\infty} I_n$ with each I_n open. By compactness, a finite subfamily already covers $[a,b]$. Hence, by Lemma 2.7,

$$b-a \leq \sum_{k=1}^{N} \ell(I_{n_k}) \leq \sum_{n=1}^{\infty} \ell(I_n).$$

Taking the infimum over all covers yields $b-a \leq m^*([a,b])$. Thus $m^*([a,b]) = b-a$.

Now let $I = (a,b)$. For every $\varepsilon > 0$,

$$[a+\varepsilon, b-\varepsilon] \subset (a,b) \subset (a-\varepsilon, b+\varepsilon),$$

so by monotonicity and the already proved compact-interval case,

$$b-a-2\varepsilon \leq m^*((a,b)) \leq b-a+2\varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives $m^*((a,b)) = b-a$.

The same squeezing argument gives

$$m^*([a,b)) = m^*((a,b]) = b-a.$$

Finally, if I is unbounded, then for every $N > 0$ it contains a bounded subinterval of length at least N; by monotonicity, $m^*(I) \geq N$ for all N, hence $m^*(I) = \infty = \ell(I)$. □

Proposition 2.14 (Open sets and sums of interval lengths)**.** *Let $O \subset \mathbb{R}$ be open, and write it as a pairwise disjoint union of open intervals:*

$$O = \dot{\bigcup_{n=1}^{\infty}} I_n.$$

Then

$$m^*(O) = \sum_{n=1}^{\infty} \ell(I_n).$$

Proof. By countable subadditivity and the theorem above,

$$m^*(O) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \ell(I_n).$$

For the reverse inequality, fix $N \in \mathbb{N}$ and $\varepsilon > 0$. Write $I_n = (a_n, b_n)$ for $1 \le n \le N$, and define

$$K_{N,\varepsilon} := \bigcup_{n=1}^{N} \left[a_n + \frac{\varepsilon}{2^{n+1}},\ b_n - \frac{\varepsilon}{2^{n+1}}\right].$$

For ε sufficiently small, each interval in the union is nonempty, $K_{N,\varepsilon}$ is compact, and $K_{N,\varepsilon} \subset O$. Therefore every open cover of O is also an open cover of $K_{N,\varepsilon}$. Since $K_{N,\varepsilon}$ is a finite union of pairwise disjoint compact intervals, repeated use of Lemma 2.7 yields

$$m^*(O) \ge m^*(K_{N,\varepsilon}) = \sum_{n=1}^{N} \left(\ell(I_n) - \frac{\varepsilon}{2^n}\right).$$

Letting $\varepsilon \downarrow 0$ gives

$$m^*(O) \ge \sum_{n=1}^{N} \ell(I_n).$$

Now let $N \to \infty$ to obtain

$$m^*(O) \ge \sum_{n=1}^{\infty} \ell(I_n).$$

Combining the two inequalities proves the result. □

Corollary 2.15 (Compact approximation of bounded open sets)**.** *If $G \subset \mathbb{R}$ is open and bounded, then for every $\varepsilon > 0$ there exists a compact set $K \subset G$ such that*

$$m^*(G \setminus K) < \varepsilon.$$

Equivalently,

$$m^*(K) > m^*(G) - \varepsilon.$$

Proof. Write $G = \bigcup_{n=1}^{\infty}(a_n, b_n)$, so that

$$m^*(G) = \sum_{n=1}^{\infty}(b_n - a_n) < \infty.$$

Choose N such that

$$\sum_{n>N}(b_n - a_n) < \frac{\varepsilon}{2}.$$

Now choose $\delta_n > 0$ for $1 \le n \le N$ so that

$$\sum_{n=1}^{N} 2\delta_n < \frac{\varepsilon}{2},$$

and define

$$K := \bigcup_{n=1}^{N} [a_n + \delta_n, b_n - \delta_n].$$

Then K is compact and $K \subset G$. Moreover,

$$G \setminus K \subset \bigcup_{n=1}^{N} ((a_n, a_n + \delta_n) \cup (b_n - \delta_n, b_n)) \cup \bigcup_{n>N} (a_n, b_n).$$

Hence,

$$m^*(G \setminus K) \leq \sum_{n=1}^{N} 2\delta_n + \sum_{n>N} (b_n - a_n) < \varepsilon.$$

□

Remark 2.16. We have now seen that intervals and open sets behave additively with respect to outer measure. The next question is to characterize those sets $E \subset \mathbb{R}$ for which the decomposition

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

holds for every set $A \subset \mathbb{R}$. This is exactly the content of Carathéodory's criterion for measurability.

2.5 Carathéodoryś criterion for measurability

Let $E \in \mathcal{M}$ and E is bounded. Then $E \subseteq (a, b)$ for some $a < b$.

Notice that, in this case, we have:

$$m^*((a, b)) = m^*(E) + m^*((a, b) \setminus E)$$

Let $I = (a, b)$. Then:

$$(\star) \quad m^*(I) = m^*(I \cap E) + m^*(I \setminus E)$$

We will soon see that I can be replaced by any subset $A \subseteq \mathbb{R}$. Thus, it is an interesting question to see that if $(\star)$ holds for a given set $E \subseteq \mathbb{R}$, does it imply E is Lebesgue measurable?
See Figure 2.1.

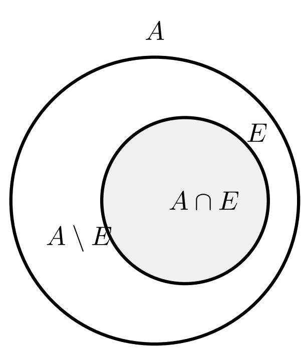


Figure 2.1: Carathéodory's criterion partitions A into $A \cap E$ and $A \setminus E$ and requires additivity of m^* across this partition.

Theorem 2.17. *$E \in \mathcal{M}$ if and only if for all $A \subseteq \mathbb{R}$,*

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E). \tag{2.1}$$

(Carathéodory's Criterion)

Proof. Since $A = (A \cap E) \cup (A \setminus E)$, for proving (2.8), it is enough to prove:

$$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E).$$

Now, suppose $E \in \mathcal{M}$. If $m^*(A) = \infty$, then (2.8) is true. Let $m^*(A) < \infty$. Then there exists a set $G \supseteq E$ such that $m^*(A) = m^*(G) < \infty$.

Therefore,

$$m^*(A \cap E) + m^*(A \setminus E) \leq m^*((A \cap E) \cup (A \setminus E)) + m^*(G \setminus E)$$

Since $G = (A \cap E) \cup (G \setminus E)$ and $G \setminus E$, $A \setminus E \in \mathcal{M}$,

It follows that:

$$m^*(A \cap E) + m^*(A \setminus E) \leq m^*(G_1) = m^*(A)$$

Conversely, suppose that (2.8) holds for each $A \subset \mathbb{R}$. Claim: $E \in \mathcal{M}$.

Firstly, let $m^*(E) < \infty$. Then there exists G_1-set $G \supseteq E$ such that $m^*(G_1) = m^*(E)$. Since (2.8) is true for all $A \subset \mathbb{R}$, take $A = G_1$, then:

$$m^*(G_1) = m^*(G_1 \cap E) + m^*(G_1 \setminus E) = m^*(E) + m^*(G_1 \setminus E)$$

Thus $m^*(G_1 \setminus E) = 0$ (since $m^*(G_1) = m^*(E) < \infty$). Hence $G_1 \setminus E \in \mathcal{M}$. Thus, $E = G_1 \setminus (G_1 \setminus E) \in \mathcal{M}$.

If $m^*(E) = \infty$, then we can decompose:

$$E = \bigcup_{n \in \mathbb{Z}} (E \cap (n, n+1]) = \bigcup_{n \in \mathbb{Z}} E_n$$

To prove $E \in \mathcal{M}$, we need to prove that if E_1 and E_2 satisfy Carathéodory's criterion (2.8), then $E_1 \cap E_2$ will also satisfy (2.8).

From the bounded case $(n, n+1 \in \mathcal{M}$ if and only if $(n, n+1]$ satisfies (2.8). (Notice this.))

Hence, each $E_n = E \cap (n, n+1]$ satisfies (1). Thus, by the bounded case, $E_n \in \mathcal{M}$. Therefore, $E = \bigcup E_n \in \mathcal{M}$ ($\because \mathcal{M}$ is closed under countable unions).

Lemma: If E_1 and E_2 satisfy (1), then $E_1 \cap E_2$, $(E_1 \cup E_2)$, E_1^c, etc., will satisfy (1).

Proof:

$$m^*(A) = m^*(A \cap E_1) + m^*(A \setminus E_1) \tag{2.2}$$

$$m^*(A) = m^*(A \cap E_2) + m^*(A \setminus E_2) \tag{2.3}$$

By replacing $A \mapsto A \cap E_2$, $A \mapsto A \setminus E_2$ and using them in (2.2), implies:

$$m^*(A) \geq m^*(A \cap (E_1 \cap E_2)) + m^*(A \cap E_1 \setminus E_2) + m^*(A \setminus E_1 \cap E_2) + m^*(A \setminus (E_1 \cup E_2))$$

$$\geq m^*(A \cap (E_1 \cap E_2)) + m^* \left\{A \cap [(E_1 \triangle E_2) \cup (E_1 \cup E_2)^c]\right\}$$

Since $E_1 \cup E_2 = (E_1 \triangle E_2) \cup (E_1 \cap E_2)$,

$$m^*(A) \geq m^*(A \cap (E_1 \cap E_2)) + m^*(A \cap (E_1 \cap E_2)^c)$$

Hence, $E_1 \cap E_2$ satisfies (1). As E satisfies (1), so does E^c.

$$(E_1 \cup E_2)^c = E_1^c \cap E_2^c$$

etc. will give the other conclusions.

Corollary: If $\{E_i\}$ satisfies (1), then so is $\bigcup E_i$.

(Aim: Let $E = \bigcup E_i$, then $E \in \mathcal{M}$ implies E satisfies (1).)

In fact, (1) is equivalent to saying that:

$$m^*(I) = m^*(I \cap E) + m^*(I \setminus E) \tag{2.4}$$

holds for each bounded open interval $I \subset \mathbb{R}$.

First, let $m^*(A) = 0$. Then (2.4) is true for A instead of I. Now, consider $A = 0$ and $m^*(0) < \infty$. Then:

$$0 = \sum_n I_n \text{ and} b_n - a_n < \alpha_1 + \frac{\alpha_1}{2}$$

Hence, $0 \in \mathcal{M}$ and satisfies (2.4) in place of I.

Finally, let $m^*(A) < \infty$. Then, there exists a G_δ-set $G \supset A$ such that $m^*(G_1) = m^*(A)$, where:

$$G = \bigcap_n O_n, \quad O_n \text{ open in } \mathbb{R}$$

Since (2.4) is true for each O_n implies $G = \bigcap O_n \in \mathcal{M}$.

implies (2.4) holds for G. Thus,

$$m^*(A) = m^*(G_1) = m^*(G \cap E) + m^*(G \setminus E) \geq m^*(A \cap E) + m^*(A \setminus E)$$

Notice that once again we come to a conclusion that measurability of E is equivalent to test-measurability of each section of E by bounded intervals.

This will give way to generalize m^* to an abstract set rather than real line.

Observation:

Let function $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$, such that.

1.$m^*(\varnothing) = 0$ (definiteness).

2.$m^*(A) \leq m^*(B)$, if $A \subseteq B \subseteq \mathbb{R}$ (monotone).

3.$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$ (countable subadditivity).

Notice that (i) can be replaced by any set $E \in \mathcal{M}$ with $m(E) < \infty$. To see this, we have:

$$E = E \cup \varnothing \text{ implies } m(E) = m(E) + m(\varnothing) \text{ implies } m(\varnothing) = 0.$$

(ii) is followed by (iii). Let $B = (B \setminus A) \cup A$,

$$m^*(B) \leq m^*((B \setminus A) + m^*(A)), \text{ if } A, B \in \mathcal{M}$$

i.e., if $m : \mathcal{M} \to [0, \infty]$ $(: m = m^*|_{\mathcal{M}})$, then (ii) can be merged with countable additivity.

Outer Measures!

Let X be a non-empty set. A set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is said to be an outer measure on X if.

(i)$\mu^*(\varnothing) = 0$ (definiteness property) .

1.

(ii)$\mu^*(A) \leq \mu^*(B)$, if $A \subseteq B \subseteq X$. (monotone property).

(iii)$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$, for any sequence of sets A_i in $P(X)$. (countable subadditivity).

□

Example 2.18.

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset \end{cases}$$

is an outer measure on X.

Example 2.19.

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases}$$

is an outer measure on X.

Example 2.20. Let X be an infinite set. Then,

$$\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is a finite set} \\ 1 & \text{if } A \text{ is not a finite set} \end{cases}$$

does not define an outer measure on X as μ^* fails to be countably subadditive. If $\{A_i : i \in I\}$ is a disjoint family of finite non-empty subsets in X, then:

$$\mu^*\left(\dot{\bigcup_{i\in I}} A_i\right) = 1 \quad \text{but} \quad 0 = \sum_{i\in I} \mu^*(A_i).$$

This leads to separating out those sets which satisfy:

$$\mu^*\left(\dot{\bigcup_{i\in I}} A_i\right) = \sum_{i\in I} \mu^*(A_i).$$

Definition 2.21. A set $E \subseteq X$ is said to be μ^*-measurable (or *Carathéodory measurable*) if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c), \qquad \text{for all } A \subseteq X.$$

By countable subadditivity of μ^*, the inequality

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

always holds. Thus it is enough to verify

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c), \qquad \text{for all } A \subseteq X. \tag{2.5}$$

Since (2.5) is symmetric in E and E^c, it follows that if E is measurable, then so is E^c.

Let $\mathcal{M}_{\mu^*}$ denote the class of all μ^*-measurable subsets of X. Clearly $\emptyset, X \in \mathcal{M}_{\mu^*}$.

If $\mu^*(E) = 0$, then for every $A \subseteq X$ we have $\mu^*(A \cap E) = 0$, and hence

$$\mu^*(A) \geq \mu^*(A \cap E^c) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Therefore $E \in \mathcal{M}_{\mu^*}$. In particular, every subset of a μ^*-null measurable set is again measurable, so the restricted measure

$$\mu := \mu^*|_{\mathcal{M}_{\mu^*}}$$

will be complete.

It remains to verify that $\mathcal{M}_{\mu^*}$ is closed under countable unions and that μ is countably additive on $\mathcal{M}_{\mu^*}$. For finite unions, let $E_1, E_2 \in \mathcal{M}_{\mu^*}$ and $A \subseteq X$. Applying the measurability of E_1 to A, and then the measurability of E_2 to $A \cap E_1^c$, we obtain

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)\\ &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c).\end{aligned}$$

Since

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_1^c \cap E_2)$$

is a disjoint union, countable subadditivity gives

$$\mu^*(A \cap (E_1 \cup E_2)) \leq \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2).$$

Hence

$$\mu^*(A) \geq \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c),$$

so $E_1 \cup E_2 \in \mathcal{M}_{\mu^*}$. By induction, every finite union of measurable sets is measurable.

Lemma 2.22. *If $\{E_i\}_{i=1}^n$ is a disjoint family of sets in $\mathcal{M}_{\mu^*}$, then for any $A \subseteq X$,*

$$\mu^*\left(A \cap \left(\bigsqcup_{i=1}^n E_i\right)\right) = \sum_{i=1}^n \mu^*(A \cap E_i).$$

Proof. Since $E_i \in \mathcal{M}_{\mu^*}$,

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c), \quad \text{for all } A \subseteq X.$$

Replace A by $A \cap (E_1 \cup E_2)$ in above. Then:

$$\begin{aligned}\mu^*[A \cap (E_1 \cup E_2)] &= \mu^*[A \cap (E_1 \cup E_2) \cap E_1] + \mu^*[A \cap (E_1 \cup E_2) \cap E_1^c] \\ &= \mu^*(A \cap E_1) + \mu^*(A \cap E_2).\end{aligned}$$

(since $E_1 \cap E_2 = \emptyset$).

Hence, by induction, the above lemma follows.

Moreover, if $\{E_i\}_{i=1}^n$ is a disjoint family in $\mathcal{M}_{\mu^*}$, then by monotone property of μ^*, we get:

$$\mu^*\left(A \cap \left(\dot{\bigcup_{i=1}^m} E_i\right)\right) \geq \mu^*\left(A \cap \bigcup_{i=1}^m E_i\right) = \sum_{i=1}^m \mu^*(A \cap E_i) \quad \text{(by previous lemma)}$$

Letting $m \to \infty$, we have:

$$\mu^*\left(A \cap \dot{\bigcup_{i=1}^\infty} E_i\right) \geq \sum_{i=1}^\infty \mu^*(A \cap E_i).$$

By countable subadditivity of μ^*, it follows that:

$$\mu^*\left(A \cap \dot{\bigcup_{i=1}^\infty} E_i\right) = \sum_{i=1}^\infty \mu^*(A \cap E_i), \quad \text{for all } A \subseteq X.$$

For $A = X$, we get:

$$\mu\left(\dot{\bigcup_{i=1}^\infty} E_i\right) = \sum_{i=1}^\infty \mu(E_i).$$

Thus μ is countably additive on $\mathcal{M}_{\mu^*}$.

□

Corollary 2.23. *If $\{E_i\}_{i=1}^\infty$ is a family in $\mathcal{M}_{\mu^*}$, then $\bigcup_{i=1}^\infty E_i \in \mathcal{M}_{\mu^*}$.*

Proof. Let $E = \bigcup_{i=1}^\infty E_i$. Define $E_i' = E_i \setminus \bigcup_{k \neq i} E_k$. Then $E_i' \in \mathcal{M}_{\mu^*}$ (by finite case), $E = \dot{\bigcup}_i E_i'$, $E_i' \cap E_j' = \emptyset$ if $i \neq j$.

Hence, without loss of generality, we can assume that $\{E_i\}_{i=1}^{\infty}$ is a disjoint family in M_{ax} and $E = \bigcup_{i=1}^{\infty} E_i$. Let $E_m = \bigcup_{j=1}^{m} E_j$.

Then:

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap E_m) + \mu^*(A \cap E_m^c) \\ &\geq \sum_{i=1}^{m} \mu^*(A \cap E_i) + \mu^*(A \cap E^c) \quad \text{for any } m \qquad \text{[by previous lemma]}.\end{aligned}$$

Letting $m \to \infty$, we get:

$$\begin{aligned}\mu^*(A) &\geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap E^c) \\ &= \mu^*(A \cap E) + \mu^*(A \cap E^c).\end{aligned}$$

Hence, $E = \bigcup_{i=1}^{\infty} E_i \in M_{ax}$ for any family $\{E_i\}_{i=1}^{\infty} \subset M_{ax}$. Thus, M_{ax} is a σ-algebra.

Notice that:

$$\mu : M_{ax} \to [0, \infty] \text{ such that:}$$

1.$\mu(\emptyset) = 0$,

2.$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$,

Hence, monotonicity of μ will be followed by (ii).

$$\mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A)$$

for $A, B \in M_{ax}$ and $A \subset B$.

Thus, if $\mathcal{S}$ is any σ-algebra of sets in X, then a set function:

$$\mu : \mathcal{S} \to [0, \infty]$$

satisfies.

(i)$\mu(\emptyset) = 0$,

(ii)$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$,

is called a measure on $(X, \mathcal{S})$.

Now, we will elaborate the idea of σ-algebra.

□

Definition 2.24. Let $X \neq \emptyset$. Let $\mathcal{S} \subset \mathcal{P}(X)$ be such that.

(i) $\emptyset \in \mathcal{S}$

(ii) $A \in \mathcal{S}$ implies $A^c \in \mathcal{S}$

(iii) $\{A_i\}_{i=1}^{\infty} \subset \mathcal{S}$ implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$

Then $\mathcal{S}$ is called a σ-algebra on X and $(X, \mathcal{S})$ is called a measurable space with each member of $\mathcal{S}$ as measurable set.

Examples:

- $\mathcal{S}_1 = \{\emptyset, X\}$ and $\mathcal{S}_2 = \mathcal{P}(X)$ are the smallest and the largest σ-algebra on X respectively.
- For $X \neq \emptyset$, let $\mathcal{S} = \{A \subset X : A \text{ or } A^c \text{ is countable}\}$. Then $\mathcal{S}$ is a σ-algebra.
  Hence, $\emptyset \in \mathcal{S}$ and if $A \in \mathcal{S}$, then $A^c \in \mathcal{S}$.

  Let $\{E_i\}_{i=1}^{\infty} \subset \mathcal{S}$, and write:

$$I_1 = \{i \in \mathbb{N} : A_i \text{ countable}\}$$
$$I_2 = \{i \in \mathbb{N} : A_i^c \text{ countable}\}$$

  Then:

$$\left(\bigcup_{i \in \mathbb{N}} A_i\right)^c = \left(\bigcup_{i \in I_1} A_i \cup \bigcup_{i \in I_2} A_i\right)^c = \left(\bigcup_{i \in I_1} A_i\right)^c \cap \left(\bigcup_{i \in I_2} A_i\right)^c$$

  is countable.

Example 2.25. Let X be an infinite set. Then:

$$\mathcal{S} = \{A \subset X : A \text{ or } A^c \text{ is finite}\}$$

is an algebra (i.e., closed under complement and finite union, $\emptyset \in \mathcal{S}$), but $\mathcal{S}$ is not a σ-algebra.

(Hint: First do for $X = \mathbb{N},\ A_m = \{2m\}$.
Write $X = X_1 \cup X_2$, X_1: countable and X_2: infinite.)

Example 2.26. Let $\mathcal{A}$ be an algebra of sets in X. Then show that $\mathcal{A}$ is a σ-algebra if $\mathcal{A}$ is closed under countable increasing unions.

(Give $A_i \in \mathcal{A},\ A_i \subset A_{i+1}$ implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$).
(Hint: $A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{j=1}^{\infty} B_j$, $B_i = A_i \setminus \bigcup_{k=1}^{i-1} A_k$.)

σ-algebra generated by a family of sets

Let $X \neq \emptyset$, let $A \subset X$, and let $\mathcal{E} = \{A\}$.
Then $\{\emptyset, X, A, A^c\}$ is a σ-algebra, so in this case

$$\sigma(\mathcal{E}) = \{\emptyset, X, A, A^c\}.$$

For a general family $\mathcal{E} \subset \mathcal{P}(X)$, it is usually neither practical nor enlightening to list all elements of $\sigma(\mathcal{E})$ explicitly. The correct structural description is

$$\sigma(\mathcal{E}) = \bigcap \{\mathcal{S} : \mathcal{E} \subset \mathcal{S},\ \mathcal{S} \text{ is a } \sigma\text{-algebra}\}.$$

Thus $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$. In particular, if $\mathcal{B}(\mathbb{R})$ denotes the σ-algebra generated by the open subsets of $\mathbb{R}$, then $\mathcal{B}(\mathbb{R})$ may be generated by any of the following collections:

(*i*) $F_1 = \{\text{all closed sets in } \mathbb{R}\}$

(*ii*) $F_2 = \{(-\infty, b] : b \in \mathbb{R}\}$

(*iii*) $F_3 = \{(a, b] : a < b,\ a, b \in \mathbb{R}\}$

Let $B_i = \sigma(F_i)$, $i = 1, 2, 3$. We prove that $\mathcal{B}(\mathbb{R}) \supset B_1 \supset B_2 \supset B_3 \supset \mathcal{B}(\mathbb{R})$.

Since $\mathcal{B}(\mathbb{R})$ contains all open sets and is closed under complement, $\mathcal{B}(\mathbb{R}) \supset F_1$. Given $\mathcal{B}(\mathbb{R})$ is a σ-algebra, $\mathcal{B}(\mathbb{R}) \supset \sigma(F_1)$.

As $(-\infty, b]$ is closed, it follows that $F_1 \supset F_2$ implies $\sigma(F_1) \supset \sigma(F_2)$:

$$\text{implies } B_1 \supset B_2$$

Observe that $(a, b] = (-\infty, b] \cap (-\infty, a]^c$, so $\sigma(F_2) \supset F_3$ implies $B_2 \supset B_3$.

Notice that $(a, b) = \bigcup_{n=1}^{\infty}(a, b - \frac{1}{n}) \subset B_3 = \sigma(F_3)$.

Hence, each bounded open set of the form $O = \bigcup_{j=1}^{\infty}(a_j, b_j) \in \sigma(F_3)$.

Since $(-\infty, b)' = (-\infty, a) \cup (a, b) \in \sigma(\mathcal{F}_3)$ and similarly, $(a, \infty) \in \sigma(\mathcal{F}_3)$, it follows that $\mathcal{B}_3$ contains each open subset of $\mathbb{R}$. Thus,

$$\mathcal{B}_3 \supseteq \{O \subset \mathbb{R} : O \text{ open}\} = \mathcal{B}(\mathbb{R}).$$

If μ is a measure on the σ-algebra $\mathcal{S}$ of subsets of X, the triple $(X, \mathcal{S}, \mu)$ is called a **measure space**.

- $(X, \mathcal{S}, \mu)$ is called a **finite measure space** if $\mu(X) < \infty$.
- $(X, \mathcal{S}, \mu)$ is called a **σ-finite measure space** if X can be expressed as a countable union of sets of finite measure, i.e.,

$$X = \bigcup_{i=1}^{\infty} E_i,\ \mu(E_i) < \infty, \text{ for all } i.$$

Example: $(\mathbb{R}, \mathcal{M}, m)$ is a σ-finite measure space but not a finite measure space.

Example: Let $Y \subset X$ and $\mathcal{S}$ be a σ-algebra on X. Then:

$$\mathcal{S} \cap Y = \{A \cap Y : A \in \mathcal{S}\}$$

is a σ-algebra, which can be thought of as the relative σ-algebra of Y.

Example: $([0, 1], \mathcal{M}|_{[0,1]}, m|_{[0,1]})$ is a finite measure space.

Example: $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu)$, where:

$$\mu(A) = \begin{cases} \#(A), & A \text{ finite} \\ \infty, & \text{otherwise} \end{cases}$$

then μ is neither finite nor σ-finite.

Proposition 2.27. *Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space. Then.*

(i) There exists an increasing sequence (F_n) in $\mathcal{S}$ that satisfies the σ-finite condition.

(ii) There exists a disjoint sequence (G_n) in $\mathcal{S}$ that satisfies the σ-finite condition.

Proof. Given that $X = \bigcup_{i=1}^{\infty} E_i$, with $\mu(E_i) < \infty$.

(i) Let $F_n = \bigcup_{i=1}^{n} E_i$. Then $F_n \uparrow$, $\mu(F_n) < \infty$, and $X = \bigcup_{n=1}^{\infty} F_n$.

(ii) Let $G_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$. Then $X = \bigcup_{n=1}^{\infty} G_n$, $\mu(G_n) < \infty$, and $G_n \cap G_m = \varnothing$ if $n \neq m$.

Example: Let $(X, \mathcal{S}, \mu)$ be a measure space.

(i) If $F, E \in \mathcal{S}$, then for $\mu(F) < \infty$ and $F \subset E$, we have

$$\mu(E \setminus F) = \mu(E) - \mu(F).$$

(ii) For any $E, F \in \mathcal{S}$,

$$\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \cup F).$$

Proof. (ii) is formal as $\mu(E \setminus F) + \mu(E \cap F) = \mu(E)$ and similarly for F. Now,

$$E = (E \setminus F) \cup (E \cap F), \quad F = (F \setminus E) \cup (E \cap F)$$

so:

$$\mu(E) + \mu(F) = \mu(E \setminus F) + \mu(F \setminus E) + \mu(E \cap F) + \mu(E \cap F) = \mu(E \cup F) + \mu(E \cap F).$$

□

Proposition 2.28. *Let $(X, \mathcal{S}, \mu)$ be a measure space.*

(i) If $E_n \uparrow E$ is an increasing sequence in $\mathcal{S}$, then:

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

(ii) If $E_n \downarrow E$ is a decreasing sequence in $\mathcal{S}$ with $\mu(E_1) < \infty$, then:

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

(Hint: Proof is similar to that of Lebesgue measure).

Premeasures:

Let $\mathcal{A}$ be an algebra of sets (in X). A set function $\mu_0 : \mathcal{A} \to [0, \infty]$ satisfies.

(i) $\mu_0(\varnothing) = 0$

(ii) If $\{A_n\}_{n=1}^{\infty}$ *is a disjoint sequence in* $\mathcal{A}$ *with* $\bigcup A_n \in \mathcal{A}$*, then:*

$$\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu_0(A_n)$$

is called a ***premeasure*** *on* $\mathcal{A}$*. Obviously,* μ_0 *is finitely additive.*

Now, for $A \subset X$*, define:*

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \mu_0(E_i) : A \subset \bigcup_{i=1}^{\infty} E_i, E_i \in \mathcal{A}\right\}$$

Then μ^* *is an outer measure on* X*.*

Proofs:

(i) $\mu^*(\varnothing) = 0$

(ii) $\mu^*(A) \le \mu^*(B)$ *whenever* $A \subset B$*.*

are obvious.

For countable subadditivity, let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{P}(X)$*. Then for each* $\varepsilon > 0$*, there is a cover* $\{E_{n,j}\}$ *of* A_n *such that:*

$$\sum_{j=1}^{\infty} \mu_0(E_{n,j}) < \mu^*(A_n) + \frac{\varepsilon}{2^n}$$

Hence, $\{E_{n,j} : n \in \mathbb{N}, j \in \mathbb{N}\}$ *is a cover of* $\bigcup A_n$*. Thus,*

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \mu_0(E_{n,j}) \le \sum_{n=1}^{\infty}[\mu^*(A_n) + \varepsilon/2^n] = \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

Hence,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n)$$

Lemma 2.29. *Any set* $E \in \mathcal{A}$ *is a* μ^**-measurable set and* $\mu^*(E) = \mu_0(E)$*.*

Proof. Let $A \subset X$ and $\varepsilon > 0$. Then there is a cover $\{E_i\}_{i=1}^{\infty}$ of A such that:

$$\mu^*(A) > \sum_{i=1}^{\infty} \mu_0(E_i) - \varepsilon \tag{2.6}$$

Now, $E_i = (E_i \cap E) \cup (E_i \cap E^c)$.

From (2.6), then:

$$\mu^*(A) \ge \sum_{i=1}^{\infty} \mu_0(E_i \cap E) + \mu_0(E_i \cap E^c)$$

(since μ_0 is finitely additive).

$$\varepsilon + \mu^*(A) > \mu^*\left(\left(\bigcup_{i=1}^{\infty} E_i\right) \cap E\right) + \mu^*\left(\left(\bigcup_{i=1}^{\infty} E_i\right) \cap E^c\right)$$
$$\geq \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

$$\text{for all } \varepsilon > 0, \text{ hence}$$

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

That is, $E \in \mathcal{M}_{\mu^*}$ (class of μ^*-measurable sets).

Next, $\mu^*(E) \leq \mu_0(E)$, since E covers itself. On the other hand, let $E \subseteq \bigcup_{j=1}^{\infty} E_j$, $E_j \in \mathcal{A}$.

Write $E_j' = (E_j \cup E_1) \cap E$. Then $E_j' \cap E_k' = \varnothing$ for $j \neq k$, and $E = \bigcup_{j=1}^{\infty} E_j'$, $E_j' \in \mathcal{A}$.

Now,

$$\mu_0(E) = \mu_0\left(\bigcup_{j=1}^{\infty} E_j'\right) = \sum_{j=1}^{\infty} \mu_0(E_j') \leq \sum_{j=1}^{\infty} \mu_0(E_j).$$

$$\text{implies } \mu_0(E) \leq \mu^*(E). \text{ Hence } \mu_0(E) = \mu^*(E).$$

Moreover, let:

$$\mu = \mu^*\big|_{\mathcal{M}_{\mu^*}}.$$

Then, as usual, μ is a measure on $\mathcal{M}_{\mu^*}$. Notice that μ extends μ_0 to $\mathcal{M}_{\mu^*}$.

□

Theorem 2.30. *If μ_0 is a σ-finite premeasure on $\mathcal{A}$, then there is one and only one measure μ on $\mathcal{M}_{\mu^*}$ such that:*

$$\mu\big|_{\mathcal{A}} = \mu_0 \quad \textit{(i.e., there is a unique } \mu \textit{ on } \mathcal{M}_{\mu^*} \textit{ that extends } \mu_0\textit{)}.$$

Proof. Let ν be another extension of μ_0. That is, $\nu\big|_{\mathcal{A}} = \mu_0$. Then for $E \in \mathcal{M}_{\mu^*}$ and a cover $\{E_i\}_{i=1}^{\infty}$ of E, we have:

$$\nu(E) \leq \sum_{j=1}^{\infty} \nu(E_j) \leq \sum_{j=1}^{\infty} \mu_0(E_j) \quad [\text{as } E_j \in \mathcal{A}]$$

$$\text{implies } \nu(E) \leq \mu^*(E) = \mu(E). \tag{2.7}$$

If $\mu(E) < \infty$, then for $\varepsilon > 0$, there is a set $F = \bigcup_{i=1}^{\infty} E_i \supseteq E$ such that:

$$\mu(F) \leq \sum \mu_0(E_i) < \mu(E) + \varepsilon.$$

$$\text{implies } \mu(F \setminus E) < \varepsilon \quad (\text{since } \mu(E) < \infty)$$

Now,

$$\nu(F) = \lim_{n\to\infty} \nu\left(\bigcup_{i=1}^{n} E_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right) = \mu(F).$$

Since $E \subseteq F = \bigcup_{i=1}^{\infty} E_i$, we set:

$$\begin{aligned}\mu(E) \leq \mu(F) &= \nu(F)\\ &= \nu(E) + \nu(F \setminus E)\\ &\leq \nu(E) + \mu(F \setminus E) \quad \text{(by (2.7) applied to } F \setminus E \in \mathcal{M}_{\mu^*})\\ &< \nu(E) + \varepsilon, \quad \text{for all } \varepsilon > 0, \text{ all } E \in \mathcal{M}_{\mu^*}\\ &\text{implies } \mu(E) \leq \nu(E) \leq \mu(E)\\ &\text{implies } \mu(E) = \nu(E), \quad \text{for all } E \in \mathcal{M}_{\mu^*} \text{ with } \mu(E) < \infty.\end{aligned}$$

If $\mu(E) = \infty$, then by the fact that μ_0 is σ-finite, we have:

$$E = \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{A},$$

$$\begin{aligned}\therefore \mu(E) = \mu\left(\bigcup_{i=1}^{\infty} (E \cap E_i)\right) &= \sum \mu(E \cap E_i)\\ &= \sum \nu(E \cap E_i)\\ &= \nu(E) \qquad \text{(by finite case).}\end{aligned}$$

□

2.6 Inner regularity of Lebesgue measurable sets

We shall show that every Lebesgue measurable set can be approximated by compact sets in the set itself. Before proving this result, we need to work out the following lemma.

Lemma 2.31. *Let $E \in \mathcal{M}$ and $m^*(E) < \infty$. Then for each $\epsilon > 0$, there is a compact set $K \subseteq E$ such that $m^*(E \setminus K) < \epsilon$,*
(i.e. $m^*(E) < m^*(K) + \epsilon$.)

Notice that $m^*(K) < m^*(E) < m^*(K) + \epsilon$.

:

$$\textit{implies}\ m^*(E) = \sup\{m^*(K) : K \subseteq E,\ K \ \textit{compact}\}$$

Proof. Let:

$$E = \bigcup_{n=1}^{\infty} (E \cap (-n, n)) = \bigcup_{n=1}^{\infty} E_n.$$

Then $E_n = E \cap (-n, n)$ is an increasing sequence and hence $m^*(E) = \lim m^*(E_n)$. Since $m^*(E) < \infty$, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that:

$$m^*(E) - m^*(E_N) < \frac{\epsilon}{2} \tag{2.8}$$

Moreover, $E_N \in \mathcal{M}$ and E_N is bounded. Hence, for each $\epsilon > 0$, there is a closed set $K_N \subseteq E_N$ such that $m^*(E_N \setminus K_N) < \frac{\epsilon}{2}$
so:

$$m^*(E_N) - m^*(K_N) < \frac{\epsilon}{2} \tag{2.9}$$

(from above).

Thus $m^*(E) - m^*(K_N) < \epsilon$ when K_N is bounded and hence compact.

□

Theorem 2.32. *Let $E \in \mathcal{M}$. Then:*

$$m^*(E) = \sup \{m^*(K) : K \subseteq E,\ K \ \textit{compact}\}.$$

Proof. Consider, if $m^*(E) = \infty$. In this case,

$$E = \bigcup (E \cap (-n, n)) = \bigcup E_n,\ E_n \uparrow$$

and:

$$\lim_{n\to\infty} m^*(E_n) = m^*(E) = \infty \quad (*)$$

Since $m^*(E_n) < \infty$, by the previous lemma, for each $\epsilon > 0$, there is a compact set $K_n \subseteq E_n$ such that:

$$m^*(E_n) < m^*(K_n) + \epsilon$$

Take $\epsilon = 1$. Then:

$$m^*(E_n) - 1 < m^*(K_n) \leq \sup m^*(K)$$

$$\text{implies } \infty = m^*(E) = \lim_{n\to\infty} m^*(E_n) = \sup m^*(K)$$

Conversely, if $m^*(E) < \infty$, then for each $\epsilon > 0$, there is a compact set $K \subseteq E$ such that $m^*(E \setminus K) < \epsilon$.
:

$$\text{implies } m^*(E) \leq m^*(K) + \epsilon \tag{2.10}$$

Notice that $m^*(K) \leq m^*(E)$, for all $K \subseteq E$, K compact

:

$$\sup\{m^*(K) : K \subseteq E, K \text{ compact}\} \leq m^*(E) \tag{2.11}$$

From (2.10) & (2.11), we get:

$$m^*(E) = \sup\{m^*(K) : K \subseteq E,\ K \text{ compact}\}$$

□

Notice that if $E \in \mathcal{M}$, then we have shown that:

$$m^*(E) = \inf\{m^*(O) : O \supseteq E,\ O \text{ open}\} = \sup\{m^*(K) : K \subseteq E,\ K \text{ compact}\}$$

Thus, if E is Lebesgue measurable, we can say that $m^*(E)$ is a true length of the set $E \subseteq \mathbb{R}$.

Now onward, we write:

$$m^*(E) = m(E),\ \text{ if } E \in \mathcal{M}$$

The set function $m : \mathcal{M} \to [0, \infty]$, which is countably additive, is known as Lebesgue measure. The set function m is satisfying continuity-like condition in the sense that for any $\epsilon > 0$, there is an open set $O \supseteq E$ such that $m^*(O \setminus E) < \epsilon$.

Exercise 2.33. For $x \in \mathbb{R}$, write $f(x) := m(E \cap (-\infty, x])$. Where $E \in \mathcal{M}$ and $m(E) < \infty$. Then show that f is uniformly continuous on $\mathbb{R}$.

2.7 The Cantor set

The *Cantor set* is a closed, uncountable subset of $[0,1]$ of Lebesgue measure zero. It provides one of the most instructive examples in real analysis and topology, and we record some of its basic properties below.

Let $C_0 = [0, 1]$.

See Figure 2.2.

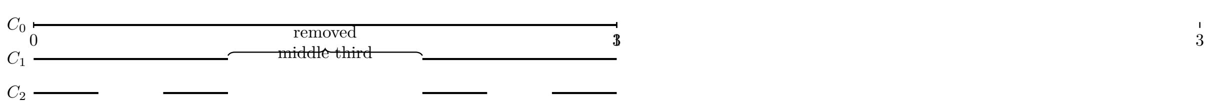


Figure 2.2: First steps in the construction of the Cantor set: $C_0 = [0, 1]$, C_1 obtained by removing the open middle third, and C_2 obtained by repeating the procedure on each remaining interval.

$$C_0 = [0, 1], \quad 1 \text{ closed interval, length } = 1$$

$$C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1], \quad 2 \text{ disjoint closed intervals, each of length } = \tfrac{1}{3}$$

$$C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$

Consists of 4 disjoint closed intervals, each of length 3^{-2}.

By induction, C_n consists of 2^n intervals, each having length 3^{-n}.

(i) $\{C_n\}_{n\geq 0}$ decreasing sequence of closed sets, hence $C_n \in \mathcal{M}$.

(ii) Let $C = \bigcap_{n=0}^{\infty} C_n$, then C contains all the end-points of the deleted open intervals.

(iii) $C = [0,1] \setminus \Big((\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \cdots \Big)$

(iv) Since $C \subseteq C_n$, for all $n \geq 0$,

$$m^*(C) \leq m^*(C_n) = 2^n \cdot \tfrac{1}{3^n} \to 0.$$

(v) The Cantor ternary set C (later, we just say *Cantor set*) is nowhere dense.

Indeed, C is closed as an intersection of closed sets. Since $m(C) = 0$, the set C cannot contain any nontrivial interval. Therefore C has empty interior, and because it is closed, it is nowhere dense.

(vi) C is **totally disconnected** (i.e., the only connected subsets of C are singletons).

(vii) Every point of C is a limit point of C itself; equivalently, C is a **perfect** set.

To prove (vii), fix $x \in C$. For each $n \geq 0$, let $I_n = [a_n, b_n]$ be the unique closed interval of C_n containing x. Then $|I_n| = 3^{-n}$ and $a_n, b_n \in C$. Choose an endpoint $u_n \in \{a_n, b_n\}$ with $u_n \neq x$ whenever possible. Then

$$|u_n - x| \leq |I_n| = 3^{-n} \to 0,$$

so $u_n \to x$ with $u_n \in C \setminus \{x\}$ for infinitely many n. Thus every point of C is a limit point of C.

If E denotes the set of all endpoints of the stage intervals, then E is countable and $E \subset C$. The argument above shows that every $x \in C$ is the limit of a sequence from E, so E is dense in C. Hence C is separable. The next section gives the ternary description of C, from which it follows that C is uncountable.

2.8 Representation of the Cantor set

Consider the endpoint $\frac{1}{3} \in C$. We can rewrite:

$$\frac{1}{3} = \frac{0}{3} + \frac{2}{3} + \frac{2}{3^2} + \cdots = (0.022\ldots)_3$$

Similarly,

$$\frac{2}{3} = (0.2)_3.$$

In a similar way, it can be shown that all endpoints can be expressed as $x = \frac{q_1}{3} + \frac{q_2}{3^2} + \cdots$, $q_i \in \{0, 2\}$.

Now, consider the set:

$$F = \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{q_i}{3^i},\, q_i \in \{0,1,2\} \right\}$$

$-$ { endpoints } $= [0,1] \setminus E$.

For $x \in F$, we have $x = \frac{q_1}{3} + \frac{q_2}{3^2} + \cdots$.

Notice that $q_1 = 1$ if and only if $x \in (\frac{1}{3}, \frac{2}{3})$, if and only if $x \notin C_1$.

Moreover, $q_1 \neq 1$, $q_2 = 1$ if and only if $x \in (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$ if and only if $x \notin C_2$.

Thus, $q_{i_0} = 1$ if and only if $x \in C_{i_0}$ for some i_0.

Since $C = \bigcap_{n=0}^{\infty} C_n$, we can show that:

$$C = \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{q_i}{3^i},\, q_i \in \{0,2\} \right\}$$

Let $x \in C = \bigcap_n G_n$, and $x = \sum_{i=1}^{\infty} \dfrac{q_i}{3^i}$. Suppose some of $q_{i_0} = 1$. Then $x \notin G_{i_0}$ implies $x \notin C$. (4!)

First, all the $q_i < 0$ or 2.

We $C \subset \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{q_i}{3^i},\ q_i \in \{0,2\} \right\}$. $(*)$.

On the other hand, let $x \notin C$. Then $x \notin G_{i_0}$ for some i_0, that implies $q_{i_0} = 1$.

Therefore $x \notin$ Right Hand Side of $(*)$.

Representation is unique.

For every $x \in C$, there exists ! sequence $q_i \in \{0,2\}$ such that:

$$x = \sum_{i=1}^{\infty} \frac{q_i}{3^i} \tag{2.12}$$

$$x = \sum_{i=1}^{\infty} \frac{b_i}{3^i}. \tag{2.13}$$

Then claim $q_i = b_i$.

If not, let $q_{i_0} \neq b_{i_0}$ for some i_0. Let i_0 be the smallest integer such that $q_{i_0} \neq b_{i_0}$. Then $q_{i_0}, b_{i_0} \in \{0,2\}$, so $q_{i_0} = 0$, $b_{i_0} = 2$ (otherwise $q_{i_0} \neq b_{i_0}$ implies $q_{i_0} = 0, b_{i_0} = 2$ or vice versa).

Then from (1), $x \in [0, 1/3]$ and from (2) $x \in [2/3, 1]$ which is a contradiction.

Exercise 2.34. Conclude without assuming $i_0 = 1$.

Cantor set is uncountable.

Define $f : C \longrightarrow [0,1] = \left\{ x = \sum_{i=1}^{\infty} \frac{q_i}{2^i},\ q_i \in \{0,1\} \right\}$.

by $f(x) = f\left(\sum_{i=1}^{\infty} \frac{b_i}{3^i} \right) = \sum_{i=1}^{\infty} \frac{(b_i/2)}{2^i}$.

Then $b_i/2 \in \{0,1\}$, & $f(x) \in [0,1]$. Since each $x \in C$ has unique representation, the map f is well defined.

f is not one-one!

$$f\left(\frac{1}{3}\right) = f(0.022\ldots_3) = (0.011\ldots)_2 = (0.1)_2 = \frac{1}{2}$$

and:

$$f\left(\frac{2}{3}\right) = f((0.2\ldots)_3) = (0.1)_2 = \frac{1}{2}$$

implies $f(1/3) = f(2/3)$

(binary repr. not unique)

Exercise 2.35. Show that $f(x) = f(y)$ if x, y are end points of one of the deleted open intervals.

f is an onto map!

Let $f : C \to [0,1] \to y$ such that $f(x) = y = \sum_{i=1}^{\infty} b_i 2^{-i}$, let:

$$x = \sum \frac{2b_i}{3^i}$$

then $f(x) = y$ holds.

Hence C is an uncountable set.

f is monotone increasing.

Let $x, y \in C$ and $x < y$. Then there exists the least positive integer $n \in \mathbb{N}$ such that $a_i = b_i,\ i = 1, 2, \ldots, n-1$, and $a_n < b_n$. Hence $q_i = b_i$ for $i = 1, 2, \ldots, n-1$. Thus, while comparing $f(x), f(y)$, we can ignore the first $n-1$ terms. Thus, without loss of generality we can assume $n = 1$. That is, $a_1 < b_1$ implies $a_1 = 0,\ b_1 = 2$.

$\therefore f(x) \leq \frac{0}{2} + \frac{y}{2^2} + \frac{y}{2^3} + \cdots = \frac{1}{2}$ and $f(y) = \frac{1}{2} + \frac{b_2}{2^2} + \cdots > \frac{1}{2}$.

implies $f(x) < f(y)$.

Notice that $f(1/3) = f(2/3) = 1/2$. Hence, by keeping f constant on each deleted interval, we can extend f out to $[0,1]$. Thus,

$y' : [0,1] \to [0,1]$ such that $y'|_C = f$ is a monotone function which is onto.

Hence, f is continuous.

Exercise 2.36. Let $f : [a,b] \to [a,b]$ be onto and monotone. Show that f is continuous. (See Carothers's book.)

Hence $f : C \to [0,1]$ is a continuous onto map.

We have shown that the class $\mathcal{M}$, the collection of all Lebesgue measurable subsets of $\mathbb{R}$, is closed under countable union and complement and containing empty set. Such a collection of sets is called a σ-algebra of sets.

Let $\mathcal{B}(\mathbb{R})$ denote the collection of sets in $\mathbb{R}$ which are made of countable union of open sets and their complement.

i.e., $O_i \in \mathcal{J}_u = \{$ all open sets in $\mathbb{R}\}$, then $\bigcup_i O_i \in \mathcal{B}(\mathbb{R})$, $O_i^c \in \mathcal{B}(\mathbb{R})$, $\mathcal{J}_u \subset \mathcal{B}(\mathbb{R})$.

Since each open set is made of countable intervals,

$$\#(\mathcal{B}(\mathbb{R})) = 2^{\aleph_0} = \mathfrak{c}$$

However, $C \subset \mathcal{M}(\mathbb{R})$, is uncountable and $m^*(C) = 0$ implies $P(C) \subset \mathcal{A}(\mathbb{R})$ implies $\#(\mathcal{M}(\mathbb{R})) \geq 2^{\mathfrak{c}}$.

Thus,

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{M}(\mathbb{R})$$

We will give a concrete example of Lebesgue measurable set which is not Borel measurable (e.g., the "Sierpiński set") later, while discussing measurable functions.

However, any Lebesgue set dif and only ifers with a Borel measurable set on a null set (set of outer measure zero).

Theorem 2.37. *Let E be a subset of $\mathbb{R}$. Then the following are equivalent:*

1. *E is Lebesgue measurable.*
2. *For each $\epsilon > 0$, there exists an open set $O \supset E$ such that $m^*(O \setminus E) < \epsilon$.*
3. *For each $\epsilon > 0$, there exists a closed set $F \subset E$ such that $m^*(E \setminus F) < \epsilon$.*
4. *There exists a G_δ-set $G \supset E$ such that $m^*(G \setminus E) = 0$. (Like $E = G \setminus N$, $N = G \setminus E$, $m^*(N) = 0$).*
5. *There exists an F_σ-set $F \subset E$ such that $m^*(E \setminus F) = 0$. (Like $E = F \cup N$, $N = E \setminus F$).*

Proof. We prove (1) implies (2) implies (4) implies (1) and (1) implies (3) implies (5) implies (1).

(1) implies (2): Since $E \in \mathcal{M}$, if $\epsilon > 0$, there exists an open set O and closed set F such that $F \subset E \subset O$ and $m^*(O \setminus F) < \epsilon$.

implies $m^*(O \setminus E) \leq m^*(O \setminus F) < \epsilon$.

(2) implies (4): For $\epsilon = \frac{1}{n} > 0$, there exists an open set $O_n \supset E$ such that $m^*(O_n \setminus E) < \frac{1}{n}$.

Let $G = \bigcap_n O_n$. Then $m^*(G \setminus E) \leq m^*(O_n \setminus E) < \frac{1}{n}$ for all $n \in \mathbb{N}$.

implies $m^*(G \setminus E) = 0$. implies $G \setminus E \in \mathcal{M}$.

(4) implies (1): $E = G \setminus (G \setminus E) \in \mathcal{M}$.

(1) implies (3): Obvious.

For (3) implies (5), let $\epsilon = \frac{1}{n} > 0$, then there exists a closed set $F_n \subset E$ such that $m^*(E \setminus F_n) < \frac{1}{n}$.

Write $F = \bigcup F_n$. Then.

$m^*(E \setminus F) \leq m^*(E \setminus F_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$.

implies $m^*(E \setminus F) = 0$ implies $E \setminus F \in \mathcal{M}$.

(5) implies (1): $E = (E \setminus F) \cup F \in \mathcal{M}$.

<u>Non-measurable set</u>

Since $m^*(\mathbb{Q}) = 0$, while searching for non-Lebesgue measurable set, we need to ignore $\mathbb{Q}$.

For $x, y \in \mathbb{R}$, define $x \sim y$ if and only if $x - y \in \mathbb{Q}$.

Then $\sim$ is an equivalence relation on $\mathbb{R}$.

Hence it partitions $\mathbb{R}$ into disjoint equivalence classes.

Let $x + \mathbb{Q} = \{x + q : q \in \mathbb{Q}\}$. Then $x + \mathbb{Q}$ is an equivalence class under $\sim$.

(i)$(x + \mathbb{Q}) \cap [0,1] \neq \varnothing$, for all $x \in \mathbb{R}$ (easy).

(ii)Let E be a subset of $[0,1]$ that contains exactly one member from each $x + \mathbb{Q}$, where $x \in [0,1]$.

1.Let $\mathbb{Q} \cap [0,1] = \{q_1, q_2, \ldots\}$ and write $E_i = E + q_i$; $i = 1, 2, \ldots$.

2.Then.

(iii)$E_i \cap E_j = \varnothing$, if $i \neq j$. To see this, let $x \in E_i \cap E_j$, then $x = x_1 + q_i = y + q_j$, where $x, y \in E$. Hence $x - y = q_j - q_i \in \mathbb{Q}$ implies $x \sim y$, is a contradiction to the definition of E, as E contains exactly one member from each $x + \mathbb{Q}$.

(iv)$[0,1] \subset \bigcup_i E_i \subset [-1,2]$.

3.Let $x \in [0,1]$. Then by definition of E (see (iii)), $x + \mathbb{Q}$ must contain a point of E, that is, there exists $y \in (x + \mathbb{Q}) \cap E$. That is, $y - x \in \mathbb{Q} \cap [0,1]$. Thus $y - x = q_{i_0}$. implies $x = y - q_{i_0} \in E_{i_0}$.

(vi) The set E is **not** Lebesgue measurable. On the contrary, if $E \in \mathcal{M}$, then each of $E_i \in \mathcal{M}$ and hence from (iv),

$$1 \leq m^*\left(\bigcup_i E_i\right) \leq 3.$$

$$\text{implies } 1 \leq \sum_{n=1}^{\infty} m^*(E) \leq 3 \quad (\because m^*(E_i) = m^*(E))$$

which is **not possible**. Note that $m^*(E) > 0$. If not, then for $m^*(E) = 0$ implies $m^*(E_i) = 0$ as $[0,1] \subset \bigcup_i E_i$:

$$\text{implies } 1 \leq \sum m^*(E_i) = 0.$$

Remark (i): The Lebesgue outer measure m^* is **not** countably additive. To see this, let $A = \bigcup_i E_i$. Then $1 \le m^*(A) \le 3$.

But $\sum_i m^*(E_i) = \sum_i m^*(E) = \infty$.

So $m^*(E) = m^* \left(\bigcup_i E_i\right) \le 3 < \infty = \sum_i m^*(E_i)$.

(ii) Whether m^* is finitely additive?

Suppose $m^* \left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n m^*(A_i)$,

where $A_i \subset \mathcal{P}(\mathbb{R})$ = power set of $\mathbb{R}$. (Like in other words, let m^* be finitely additive.)

Firstly, let $E \subset [0,1]$ and assume $m^*(E) > 0$. Let $N \subset [0,1]$ be a non-measurable set and enumerate $\mathbb{Q}$ as $\{q_i\}$. Set $N_i := N + q_i$. Then $N_i \cap N_j = \varnothing$ for $i \neq j$ (easy) and $\mathbb{R} = \bigcup_i N_i$.

Let $x \in \mathbb{R}$, then $x \in [0,1] + q_k$ for some $q_k \in \mathbb{Q}$.
implies $x - q \in [0,1] \subset \bigcup_i N_i \subset [-1,2]$,
where $N_i = N + q_i,\ q_i \in \mathbb{Q} \cap [-1,1]$
implies $x - q \in N_{i_0}$ for some i_0. But $q \in \mathbb{Q}$
implies $x \in N_{i_0} + q = N_{j_0}$ for some j_0..

Now $E = \bigcup_i (E \cap N_i)$. If $E \cap N_i$ were Lebesgue measurable for every i, then $m^*(E \cap N_i) = 0$ for all i, hence $m^*(E) = 0$, a contradiction.

Hence, there exists i_0 such that $E \cap N_{i_0} \subset E$, $E \cap N_{i_0} \notin \mathcal{M}$..

Finally, let $E \subset \mathbb{R}$ and $m^*(E) > 0$ implies $E \notin \mathcal{M}(\mathbb{R})$..

Then:

$$m^*(E) = \sum_{n \in \mathbb{Z}} m^*(E \cap (n_0, n_0 + 1)) > 0.$$

Hence, there exists n_0 such that $m^*(E \cap (n_0, n_0+1)) > 0$.
Let $F = E \cap (n_0, n_0+1)$. Then
implies $F - n_0 \subset (0,1)$ and $m^*(F - n_0) > 0$.
Hence, there exists $H \subset F - n_0$, $H \notin \mathcal{M}$.
implies $H + n_0 \subset F \subset E$ with $H + n_0 \notin \mathcal{M}$..

□

Proposition 2.38. *Let $(E_n) \subset \mathcal{M}$ be an increasing sequence of sets. Then:*

$$\lim_{n \to \infty} m^*(E_n) = m^* \left(\bigcup_{n=1}^{\infty} E_n \right).$$

Proof. Let $E = \bigcup_{n=1}^{\infty} E_n$. Then $m(E_n) \uparrow$ and
$\lim m(E_n) = \sup m(E_n) \le m^*(E)$.

Now,

$$\bigcup_{n=1}^{\infty} E_n = E_1 \cup \dot{\bigcup}_{n=1}^{\infty}(E_{n+1} \setminus E_n)$$

$$\text{implies } m^*(E) = m^*(E_1) + \sum_{n=1}^{\infty} m^*(E_{n+1} \setminus E_n)$$

$$= m^*(E_1) + \lim_{K\to\infty} \sum_{n=1}^{K} m^*(E_{n+1} \setminus E_n).$$

$$= \lim_{K\to\infty} \left\{ m^*(E_1) + \sum_{n=1}^{K} m^*(E_{n+1} \setminus E_n) \right\}$$

$$= \lim_{K\to\infty} m^* \left(E_1 \cup \dot{\bigcup}_{n=1}^{K}(E_{n+1} \setminus E_n) \right)$$

$$\text{implies } m(E) = \lim_{k\to\infty} m^*(E_{k+1}).$$

□

Proposition 2.39. *Let $(E_n) \subset \mathcal{M}$ be a decreasing sequence such that $m^*(E_1) < \infty$. Then:*

$$\lim m(E_n) = m\left(\bigcap_{n=1}^{\infty} E_n\right).$$

$$\lim m^*(E_n) = m^*\left(\bigcap_{n=1}^{\infty} E_n\right).$$

Proof. Since $m^*(E_n) \geq m^*(E_{n+1}) \geq m^*(\bigcap_{n=1}^{\infty} E_n)$, we have $\lim m^*(E_n) \geq \inf m^*(E_n) \geq m^*(\bigcap_{n=1}^{\infty} E_n)$.

$$E_1 \setminus \bigcap_{n=1}^{\infty} E_n = \dot{\bigcup}_{n=1}^{\infty}(E_n \setminus E_{n+1}) \quad \text{(disjoint union)}$$

$$\text{implies } m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m^*(E_n \setminus E_{n+1})$$

$$= \lim_{K\to\infty} \sum_{n=1}^{K} m^*(E_n \setminus E_{n+1})$$

$$= \lim_{K\to\infty} (m^*(E_1) - m^*(E_{K+1}))$$

Since $m^*(E_1) < \infty$,

$$m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim m^*(E_n).$$

Alternative: If (E_n) is decreasing, apply the previous result to the complements.

□

Example 2.40. Let $E_n = \mathbb{R} \setminus (-n, n)$ for $n \in \mathbb{N}$. Then (E_n) is decreasing and

$$\bigcap_{n=1}^{\infty} E_n = \varnothing, \qquad \text{hence} \qquad m^*\Big(\bigcap_{n=1}^{\infty} E_n\Big) = 0.$$

But $m^*(E_n) = \infty$.

Chapter 3

Measurable functions

Once a measure is in place, the next step is to identify which functions are compatible with the underlying σ-algebra. This chapter introduces measurable spaces and measurable maps, framing the usual Borel/Lebesgue measurability on $\mathbb{R}$ as a special case of a general concept. The point of view is deliberately abstract: measurability is defined through inverse images and is stable under algebraic operations and limiting procedures. We develop practical criteria for checking measurability, study the behavior of measurable functions under composition and pointwise limits, and prepare the approximation machinery (in particular, simple functions) that will be used to construct the Lebesgue integral and to analyze convergence in L^p spaces.

3.1 Measurable spaces and measurable functions

Let $\mathcal{J}_u$ be the collection of all open subsets of $\mathbb{R}$ with respect to the usual metric u on $\mathbb{R}$,

$$\mathcal{J}_u = \left\{ O \subset \mathbb{R} : O = \bigcup_{n=1}^{\infty} I_n,\ I_n = (a_n, b_n) \right\}.$$

And $\mathcal{M}$ is the class of all Lebesgue measurable subsets of $\mathbb{R}$.

Let $\mathcal{J}_{d_0}$ be the collection of all open subsets of $\mathbb{R}$ with respect to d_0, the discrete metric on $\mathbb{R}$ so that $\mathcal{J}_{d_0} = \mathcal{P}(\mathbb{R})$.

Thus,

$$\mathcal{J}_u \subset \mathcal{M} \subset \mathcal{J}_{d_0} = \mathcal{P}(\mathbb{R}).$$

Since $\mathcal{J}_u$ is not closed under countable intersections (and complements) of open sets,

$$\mathcal{J}_u \subset \mathcal{M}, \quad \mathcal{M} \subset \mathcal{J}_{d_0},$$

because every subset of $\mathbb{R}$ need not be Lebesgue measurable.

Consider $f : (\mathbb{R}, \mathcal{J}_u) \to (\mathbb{R}, \mathcal{J}_u)$ continuous. Then:

$$f^{-1}(O) \in \mathcal{J}_u, \quad \text{for all } O \in \mathcal{J}_u \qquad \text{(from the definition of continuity).}$$

Now, if $f : (\mathbb{R}, \mathcal{M}) \to (\mathbb{R}, \mathcal{J}_u)$, what can we say about $f^{-1}(O)$?

If $f : (\mathbb{R}, \mathcal{J}_u) \to (\mathbb{R}, \mathcal{J}_u)$ is continuous, then $f^{-1}(O) \in \mathcal{M}$ for every open set O, because $f^{-1}(O)$ is open.

In addition, consider $f(x) = \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$. Then f cannot be made continuous at 0, but $f(x) = \infty$ if and only if $x = 0$ *(important!)*.
If we want to take $f(x) = \frac{1}{x}$ into consideration, we have to extend the range $(-\infty, \infty)$ to $[-\infty, \infty]$.

Let $\overline{\mathbb{R}} = (-\infty, \infty) \cup \{-\infty, \infty\} = [-\infty, \infty]$.

Therefore, the sets $[-\infty, a)$ and $(b, \infty]$ for $a, b \in \mathbb{R}$ should be added to $\mathcal{J}_u$.

That is,

$$\mathcal{J}_{\overline{u}} = \mathcal{J}_u \cup \{[-\infty, a), (b, \infty] : \ a, b \in \mathbb{R}\}$$

Notice that $[-\infty, a) \cup (b, \infty]$ is the complement of $[a, b]$ in $\mathbb{R}$ union with $\{\pm\infty\}$. That is, $(\overline{\mathbb{R}}, \mathcal{J}_{\overline{u}})$ is the two-point compactification of $(\mathbb{R}, \mathcal{J}_u)$. But $f(x) = 1/x$ is still not continuous, because for $a < 0$,

$$f^{-1}((a, \infty]) = (-\infty, 0] \notin \mathcal{J}_u.$$

Definition 3.1. $f : (\mathbb{R}, \mathcal{M}) \to (\overline{\mathbb{R}}, \mathcal{J}_{\overline{u}})$ (or $\overline{\mathbb{R}}$ itself) is said to be Lebesgue measurable if $f^{-1}(O) \in \mathcal{M}$ for all $O \in \mathcal{J}_{\overline{u}}$.

Similarly, if $f : (X, S) \to \overline{\mathbb{R}}$, then f is said to be S-measurable if $f^{-1}(O)$ belongs to S for all $O \in \mathcal{J}_{\overline{u}}$.

Lemma 3.2. *If $f : (X, S) \to \overline{\mathbb{R}}$, then the following are equivalent (for all $\alpha \in \mathbb{R}$):*

1. $f^{-1}((\alpha, \infty]) \in S$,
2. $f^{-1}([\alpha, \infty]) \in S$,
3. $f^{-1}((-\infty, \alpha)) \in S$,
4. $f^{-1}((-\infty, \alpha]) \in S$,
5. $f^{-1}((a, b)) \in S$ *for all* $a, b \in \mathbb{R}$ *and* f *takes values in* S.

Proof. (i) implies (ii):

$$[\alpha, \infty] = \bigcap_{n=1}^{\infty} (\alpha - 1/n, \infty]$$

Let x not in RHS, then there exists n_0 such that $x \le \alpha - \frac{1}{n_0} < \alpha$. Hence x not in LHS. Since S closed under complement, (vi) implies (iii). Now, (iii) implies (iv) because $(-\infty, \alpha] = \bigcap_n (-\infty, \alpha + 1/n)$.

Now, (iv) implies (i). Thus, (i) to (iv) are equivalent.

Hence $f^{-1}(\{\infty\}) = \bigcap f^{-1}([n,\infty]) \in S$ by (i),

and $f^{-1}(-\infty) = \bigcap f^{-1}((-\infty,-n)) \in S$ by (iii).

Also, $(a,b) = (a,\infty) \cap (-\infty, b)$, we get (v).

Finally, (v) implies (i) as follows:

$$(a,\infty] \;=\; (a,\infty) \cup \{\infty\} \;=\; \bigcup_{n\in\mathbb{N}} (a,n) \;\cup\; \{\infty\}.$$

□

Example 3.3. Let $E \in S$, and define:

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

$$\chi_E^{-1}([\alpha,\infty]) = \{x \in X : \chi_E(x) > \alpha\} = \begin{cases} X, & \alpha < 0 \\ E, & 0 \le \alpha < 1 \\ \varnothing, & \alpha \ge 1 \end{cases}$$

See Figure 3.1.

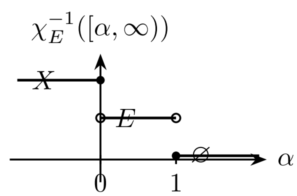


Figure 3.1: Upper level sets of the characteristic function: the preimage $\chi_E^{-1}([\alpha,\infty))$ equals X for $\alpha < 0$, equals E for $0 \le \alpha < 1$, and is empty for $\alpha \ge 1$.

Hence, the characteristic function χ_E is measurable if $E \in S$.

Example 3.4. Let (X, S) be a measurable space. Then *constant function* is S-measurable.

Let $f(x) = c$, for all $x \in X$, c is a finite constant.

$$\{x \in X : f(x) > \alpha\} = \begin{cases} \varnothing & \alpha \ge c \\ X & \alpha < c \end{cases}$$

If $f(x) = \alpha$, for all $x \in X$, then:

$$\{x \in X : f(x) > \alpha\} = \varnothing, \quad X$$

Notice that for $\alpha \in \mathbb{R}$, there exists $\gamma_n \in \mathbb{Q}$ such that $\gamma_n \uparrow \alpha$. Thus, $f(x) > \alpha$ implies $f(x) > \gamma_n$, for all n.

i.e.

$$\{x : f(x) > \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \gamma_n\}$$

Thus, f is S-measurable if and only if $\{f > q\} \in S$ for all $q \in \mathbb{Q}$.

Example 3.5. Let $D \subset \mathbb{R}$ be dense, and let $f : X \to \overline{\mathbb{R}}$ be a function such that

$$\{x \in X : f(x) > \gamma\} \in S \qquad \text{for every } \gamma \in D.$$

Then f is S-measurable.

Indeed, fix $\alpha \in \mathbb{R}$. Since D is dense, there exists a sequence $(\gamma_n) \subset D$ such that $\gamma_n \uparrow \alpha$. Therefore

$$\{x \in X : f(x) > \alpha\} = \bigcap_{n=1}^{\infty} \{x \in X : f(x) > \gamma_n\} \in S.$$

By the usual characterization of measurability in terms of upper level sets, f is measurable.

Example 3.6. Suppose $f, g : (X, S) \to \overline{\mathbb{R}}$ are measurable and $f(x) + g(x) \neq +\infty, -\infty$ for any $x \in X$, then $f + g$ is measurable.

To see this, we need to show that:

$$A = \{x \in X : f(x) + g(x) = +\infty\} \in S$$

and:

$$B = \{x \in X : \infty > f(x) + g(x) > \alpha\} \in S, \quad \text{for all } \alpha \in \mathbb{R}$$

Now,

$$A = \{x : f(x) = +\infty \text{ if } g(x) \text{ is finite (or otherwise)}\}$$

Thus, $A \in S$.

For $x \in B$,

$$\alpha < f(x) + g(x) < \infty$$

Then, there exists $\gamma \in \mathbb{Q}$ such that $f(x) > \gamma > \alpha - g(x)$.

implies $x \in \bigcup_{\gamma \in \mathbb{Q}} \{x : f(x) > \gamma\} \cap \{x : g(x) > \alpha - \gamma\}$.

Hence,

$$B = \bigcup_{\gamma \in \mathbb{Q}} \{x : f(x) > \gamma\} \cap \{x : g(x) > \alpha - \gamma\} \in S$$

Example 3.7. If $f : (X, S) \to \overline{\mathbb{R}}$ is measurable, then:

$$\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}$$

$f < 0$

Hence f^2 is measurable.

Example 3.8. $fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$, implies that if f, g are measurable, then fg is measurable.

Definition 3.9. A property P is said to hold "almost everywhere" if the places (sets) where it fails has measure zero.

Let (X, S, μ) be a measure space. Then:

$$\mu^*\{x \in X : P \text{ is false}\} = 0$$

Example 3.10. Let $f : (X, S, \mu) \to \overline{\mathbb{R}}$ be such that $f = 0$ almost everywhere (a.e.), then f is measurable, if (X, S, μ) is *complete*.

Let $E = \{x \in X : f(x) \neq 0\}$. Then $\mu^*(E) = 0$.

$$\{x \in X : f(x) > \alpha\} = \begin{cases} E^c \cup A, & A \subseteq E,\ \alpha < 0 \\ B, & B \subseteq E,\ \alpha > 0 \end{cases}$$

Since (X, S, μ) is complete, and $A, B \subseteq E$, implies $A, B \in S$. Thus, f is measurable.

Notice that for $A \subset X$, we have:

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\alpha} \mu(E_i) : E_i \in S,\ A \subseteq \bigcup_{i=1}^{\alpha} E_i\right\} = \inf\{\mu(E) : E \in S,\ A \subseteq E\}$$

Example 3.11. If $f : (X, S) \to \overline{\mathbb{R}}$ is measurable, then $|f|$ is also measurable.

$$\{x : |f(x)| > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : f(x) < -\alpha\},\ \alpha > 0$$

But converse need not be true.

Example 3.12. Let $N \subset \mathbb{R}$ be a non-measurable set. Then:

$$f(x) = \begin{cases} 1 & x \in N \\ -1 & x \notin N \end{cases}$$

is not Lebesgue-measurable, but $|f| = 1$ is measurable.

Let $L(X, S)$ and $L(X, S, \mu)$ denote the space of all measurable functions on X.

Define $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$.

Then $f^+ = \frac{f+|f|}{2}$ and $f^- = \frac{|f|-f}{2}$.

Hence, if $f \in L(X, S)$, then $f^+, f^- \in L(X, S)$.

Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Example 3.13. $f : [0, 2\pi] \to \mathbb{R}$,

$$f(x) = \sin nx, \quad f^+(x) = \begin{cases} \sin(nx) & \text{if } 0 < x < \pi \\ 0 & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

and:

$$f^-(x) = \begin{cases} 0 & 0 \leq x \leq \pi \\ -\sin nx & \pi < x \leq 2\pi \end{cases}$$

Example 3.14. Let $f_n \in L(X, S)$. Then $\inf f_n$, $\sup f_n$, $\liminf f_n$, and $\limsup f_n$ (if exists) are in $L(X, S)$.

$$\{x : \liminf f_n(x) < \alpha\} = \bigcup_{n \in \mathbb{N}} \{x : f_n(x) < \alpha\}$$

If $x \in L(X, S)$, then there exists n_0 such that $f_{n_0}(x) < \alpha$.
Hence, $x \in$ RHS and vice-versa.

Lemma 3.15. *Let $f : (X, S) \to \mathbb{R}$. The following are equivalent.*

(i) f is measurable, i.e. $f^{-1}(U) \in S$ for every open set $U \subset \mathbb{R}$.

(ii) $f^{-1}(F) \in S$ for every closed set $F \subset \mathbb{R}$.

(iii) $f^{-1}(B) \in S$ for every Borel set $B \subset \mathbb{R}$.

Proof. (i)⇒(ii): if F is closed, then F^c is open and

$$f^{-1}(F) = X \setminus f^{-1}(F^c) \in S.$$

(ii)⇒(iii): let

$$\mathcal{A} := \{B \subset \mathbb{R} : f^{-1}(B) \in S\}.$$

Then $\mathcal{A}$ is a σ-algebra on $\mathbb{R}$. Since it contains every closed set, it contains the Borel σ-algebra generated by the closed sets. Hence every Borel preimage is measurable.

(iii)⇒(i): every open set is Borel.

□

In particular, every Borel measurable function is Lebesgue measurable, because $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}$.

Monotone functions:

Proposition 3.16. *Let $f : (a, b) \to \mathbb{R}$ be monotone. Then for every $c \in (a, b)$ the one-sided limits*

$$f(c-) := \lim_{x \uparrow c} f(x), \qquad f(c+) := \lim_{x \downarrow c} f(x)$$

exist. Moreover, the set of discontinuities of f is at most countable.

Proof. Suppose first that f is increasing. Set

$$L := \sup\{f(x) : a < x < c\}, \qquad M := \inf\{f(x) : c < x < b\}.$$

Then $L \le f(c) \le M$.

To prove $f(x) \to L$ as $x \uparrow c$, fix $\varepsilon > 0$. By definition of supremum, choose $x_0 < c$ with $f(x_0) > L - \varepsilon$. For every $x \in (x_0, c)$,

$$L - \varepsilon < f(x_0) \le f(x) \le L,$$

so $|f(x) - L| < \varepsilon$. Hence $f(c-) = L$. The proof for $f(c+) = M$ is analogous.

The function f is discontinuous at c exactly when $f(c-) < f(c+)$, that is, when there is a nontrivial jump interval $(f(c-), f(c+))$.

If $c < d$ are two discontinuity points, then monotonicity gives $f(c+) \le f(d-)$. Therefore the jump intervals $(f(c-), f(c+))$ and $(f(d-), f(d+))$ are disjoint.

Choose a rational number in each jump interval. This gives an injective map from the set of discontinuities into $\mathbb{Q}$, so the discontinuity set is at most countable.

The decreasing case follows by applying the same argument to $-f$.

□

Example 3.17. If $f : [a, b] \to [c, d]$ is monotone and onto, then f is continuous.

Let f be $\uparrow$. Then $f(a) = c$ and $f(b) = d$.

See Figure 3.2.

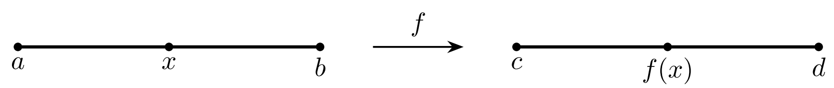


Figure 3.2: A monotone map $f : [a, b] \to [c, d]$ sends endpoints to endpoints when it is onto; gaps in the image correspond to discontinuities.

If $f(a) > c$, then for $y \in [c, f(a))$, there does not exist any $x \in [a, b]$ such that $f(x) = y$. If so, then $f(x) = y < f(a)$ implies $x < a$ (since f is increasing).

Moreover, if possible, let $f(c-) < f(c)$.

See Figure 3.3.

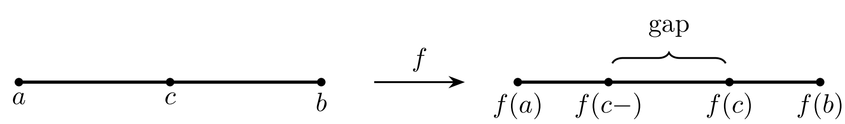


Figure 3.3: If f is increasing and $f(c-) < f(c)$, then values in $(f(c-), f(c))$ have no preimage: a jump discontinuity creates a gap in the image.

Then $y \in (f(c-), f(c))$ has no pre-image.

If there exists $x_0 \in (a, c)$ such that $f(x_0) = y$, then:

$$L := \sup\{f(x) : a < x < c\} = f(c-) \geq f(x_0) > y$$

for $a < x < c$, which contradicts the fact that L is the supremum on (a, c).

Thus, $f(c-) = f(c) = f(c+)$.

Another example: If $f : [a, b] \to [c, d]$ is monotone, then f is continuous.

Proof is similar for the above case.

Observe that if f is monotone onto, then f need not be one-one. However, if f is strictly monotone and onto, then $f^{-1} : (c, d) \to (a, b)$ is continuous, because in this case, f^{-1} is also strictly monotone. To see this, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. If not, then $f(x_1) > f(x_2)$ implies $x_1 > x_2$, but $f(x_1) < f(x_1) = y_1$.

Note that $f : [c, d]$ onto (e, f) need not be continuous, if f is monotone, else $f([a, b])$ would be compact.

Finally, if $f : \mathbb{R} \to \mathbb{R}$ is one-one onto cont(inuous), then f and f^{-1} both are continuous.

Proposition 3.18 (Monotone functions are Borel measurable)**.** *Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be monotone. Then f is Borel measurable.*

Proof. It suffices to treat the case in which f is increasing, since the decreasing case follows by applying the argument to $-f$. Fix $\alpha \in \mathbb{R}$ and consider

$$E_\alpha := \{x \in I : f(x) > \alpha\}.$$

If $E_\alpha = \varnothing$ or $E_\alpha = I$, then E_α is Borel. Assume therefore that E_α is nonempty and proper. If $x \in E_\alpha$ and $y \in I$ with $y > x$, then monotonicity gives $f(y) \geq f(x) > \alpha$, hence $y \in E_\alpha$. Thus E_α is an upper tail of the interval I.

Set $x_0 := \inf E_\alpha$, where the infimum is taken in $\mathbb{R}$. Then for every $y \in I$ with $y < x_0$ we have $y \notin E_\alpha$, so $f(y) \leq \alpha$. On the other hand, if $y \in I$ and $y > x_0$, then by the definition of the infimum there exists $x \in E_\alpha$ with $x < y$, and therefore $f(y) \geq f(x) > \alpha$; hence $y \in E_\alpha$. It follows that

$$I \cap (x_0, \infty) \subset E_\alpha \subset I \cap [x_0, \infty).$$

Consequently E_α is one of the intervals $I \cap (x_0, \infty)$ or $I \cap [x_0, \infty)$, according to whether or not $f(x_0) > \alpha$ when $x_0 \in I$. In either case E_α is a Borel subset of I.

We have therefore shown that $f^{-1}((\alpha, \infty)) = E_\alpha$ is Borel for every $\alpha \in \mathbb{R}$. By the standard criterion for real-valued functions, f is Borel measurable. □

<u>A Lebesgue measurable set which is not Borel (and incompleteness of $(\mathbb{R}, \mathcal{B}, m)$).</u>

Let $C \subset [0, 1]$ be the Cantor set. Recall that C is closed (hence $C \in \mathcal{B}$) and satisfies:

$$m(C) = 0. \tag{3.1}$$

The Borel σ-algebra $\mathcal{B}$ has cardinality continuum, whereas the power set $\mathcal{P}(C)$ has strictly larger cardinality $2^{\mathfrak{c}}$. Consequently, there exists a subset $A \subset C$ which is *not* a Borel set.

Since $A \subset C$ and $m(C) = 0$, we have $m^*(A) = 0$, where m^* denotes Lebesgue outer measure. In particular, A is Lebesgue measurable (indeed, every subset of a Lebesgue null set is Lebesgue measurable in the completion of m). Thus A is Lebesgue measurable but not Borel.

This shows that the measure space $(\mathbb{R}, \mathcal{B}, m)$ is *not* complete: the null Borel set C contains a subset A which fails to lie in $\mathcal{B}$, even though $m^*(A) = 0$.

Simple functions.

Let $E_i \in S$ and $d_i \in \mathbb{R}$ for $1 \le i \le m$. A function of the form

$$\varphi = \sum_{i=1}^{m} d_i \chi_{E_i}$$

is called a simple function on (X, S). Equivalently, a simple function is a measurable function that takes only finitely many values. When convenient, the sets E_i may be chosen pairwise disjoint.

Notice that:

$$\chi_{E_1}\chi_{E_2} = \chi_{E_1 \cap E_2}, \quad \text{and}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

Hence, without loss of generality, we can assume all of E_i's are pairwise disjoint.

Thus, the canonical representation of a simple function is:

$$\varphi = \sum_{i=1}^{m} d_i \chi_{E_i}, \quad E_i \cap E_j = \emptyset \text{ for all } i \neq j,\ d_i \in \mathbb{R}$$

Simple functions are dense in $L^1(X, S)$.

Why do we need the denseness of simple functions?

Let f be $\mathbb{R}$-integrable on $[0, 1]$. Then:

$$L(P_n f) = \sum_{i=1}^{n} m_i \Delta x_i$$

and:

$$U(P_n f) = \sum_{i=1}^{n} M_i \Delta x_i$$

Write:

$$\varphi_n = \sum_{i=1}^{n} m_i \chi_{[x_{i-1}, x_i)},$$

Then $\varphi_n \uparrow f$ and $\int \varphi_n \, dx = L(P_n f)$ and $\lim \int \varphi_n \, dx = \lim L(P_n f) = \int f \, dx$.

i.e., every $\mathbb{R}$-integrable function is limit of simple functions!

Hence, we can think of a similar conclusion for a measurable function.

Theorem 3.19. *Let $f : (X, \mathcal{S}) \to [0, \infty]$ be measurable. Then there exists a sequence φ_n of simple functions on X such that:*

(i) $\varphi_n \le f$ and $\varphi_n \uparrow f$ pointwise;

(ii) $\varphi_n \uparrow f$ uniformly on any set $A \subset X$ where f is bounded.

Proof. *We first divide the image of f under $[0, 2^n)$ into 2^n disjoint parts.*
See Figure 3.4.

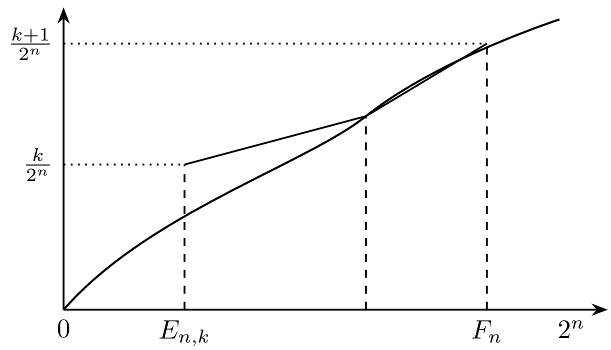


Figure 3.4: A typical step-function approximation to a bounded measurable function.

Let:

$$F_n = \{x : f(x) \ge 2^n\}$$

and:

$$E_{n,k} = \left\{x : \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}\right\}, \qquad (k = 0, 1, \ldots, 2^n - 1)$$

Define:

$$\varphi_n(x) = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \chi_{E_{n,k}}(x) + 2^n \chi_{F_n}(x)$$

where χ_A denotes the indicator function of the set A.

Then $\varphi_n \ge 0$ and the $E_{n,k}$ are disjoint measurable sets in X.

(i) ***Increasing sequence.***

Claim: $\varphi_n(x) \le \varphi_{n+1}(x)$, for all $x \in X$, all $n \in \mathbb{N}$.

If $x \in E_{n,k} = E_{n+1,2k} \cup E_{n+1,2k+1}$:

- *for $x \in E_{n+1,2k}$, $\varphi_n(x) = \frac{k}{2^n} = \frac{2k}{2^{n+1}} = \varphi_{n+1}(x)$.*
- *for $x \in E_{n+1,2k+1}$, $\varphi_n(x) = \frac{k}{2^n} < \frac{2k+1}{2^{n+1}} = \varphi_{n+1}(x)$.*

Now, if $x \in F_n = (F'_{n+1}) \cup F_{n+1}$:

- *For $x \in F_{n+1}$, $\varphi_n(x) = 2^n < 2^{n+1} = \varphi_{n+1}(x)$.*

- *For* $x \in F_n \setminus F_{n+1}$, *we have* $2^n = \frac{2^{n+1}}{2} \leq f(x) < 2^{n+1}$, *so* $x \in E_{n+1,2^{n+1}-1}$ *or* $E_{n+1,2^{n+1}-2}$. *Then:*

$$\varphi_{n+1}(x) \in \left\{ \frac{2^{n+1}-2}{2^{n+1}},\ \frac{2^{n+1}-1}{2^{n+1}} \right\}$$

and $\varphi_n(x) = 2^n = \frac{2^{n+1}}{2} \leq \varphi_{n+1}(x)$.

That is, $\varphi_n \uparrow$ *and* $\varphi_n \leq f$.

(ii) ***Pointwise convergence.***
If $f(x) = \infty$ *for some* $x \in X$, *then for all* n,

$$\varphi_n(x) = 2^n \to \infty = f(x)$$

Now observe that

$$\{x \in X : f(x) < \infty\} = \bigcup_{n \in \mathbb{N}} \{x \in X : f(x) < 2^n\}.$$

If $f(x) < \infty$, *then there exists* $n_0(x) \in \mathbb{N}$, *such that* $f(x) < 2^{n_0} < 2^n$, *for all* $n \geq n_0$. *Thus* $x \in E_{n,k}$ *for some* k *and* $f(x)$ *is in* $[\frac{k}{2^n}, \frac{k+1}{2^n})$.

Therefore,

$$0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}, \text{ for all } n \geq n_0(x)$$

so $\varphi_n \to f$ *pointwise.*

(iii) ***Uniform convergence on bounded sets.***
Let $A = \{x \in X : f(x) < M\}$. *Then there exists* $n_0 \in \mathbb{N}$ *such that* $f(x) < 2^n$ *for all* $n \geq n_0$, $x \in A$. *Hence,*

$$0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}, \text{ for all } n \geq n_0, x \in A$$

and:

$$\sup_{x \in A}[f(x) - \varphi_n(x)] \leq \frac{1}{2^n}, \text{ for all } n \geq n_0$$

which means $\varphi_n \to f$ *uniformly on* A.

Corollary 3.20. *If* $f : (X, \mathcal{S}) \to \mathbb{R}$ *is measurable, then there is a sequence* φ_n *of simple functions on* X *such that* $\varphi_n \to f$ *pointwise and* $|\varphi_n| \uparrow |f|$ *pointwise.*

Proof. Let $f = f^+ - f^-$ *be the decomposition into positive and negative parts. Then* f^+, f^- *are measurable and both map into* $[0, \infty]$. *Thus, by the theorem, there exist* $\varphi_n^+ \uparrow f^+$ *and* $\varphi_n^- \uparrow f^-$ *as simple functions.*

Let $\varphi_n = \varphi_n^+ - \varphi_n^-$. *Then* $\varphi_n \to f$ *pointwise, and:*

$$|\varphi_n| = \varphi_n^+ + \varphi_n^- \uparrow |f|$$

How far is uniform convergence from pointwise convergence?

Example. Let $f_n(t) = t^n$, $t \in [0,1]$. *Then* $f_n(t) \to 0$ *for all* $0 \le t < 1$, *and* $f_n(1) \to 1$.

$$\sup_{t<1} |f_n(t) - f(t)| = 1 \not\to 0 \text{ as } n \to \infty$$

Hence, f_n *is not uniformly convergent. However, for every* $\varepsilon > 0$, $f_n \to 0$ *uniformly on* $[0, 1-\varepsilon]$*; on* $(1-\varepsilon, 1]$*, it can be made smaller than* ε.

In fact, any discontinuous function can be thought of as the limit of a sequence of continuous functions (a consequence of Lusin's theorem; see it later).

Hence, the above exercise can be generalised as follows. This is known as Egorov's theorem.

Theorem 3.21 (Egorov's Theorem)**.** *Let* (X, S, μ) *be a finite measure space. Let* f_n *be a sequence of measurable functions on* X*, which converges to* f *pointwise. Then for each* $\epsilon > 0$*, there exists a measurable set* $E \subset X$ *such that* $\mu(E) < \epsilon$ *and the sequence* f_n *converges to* f *uniformly on* E^c.

Proof. The idea of the proof is to collect all those points where uniform convergence fails. This construction is based on the following observations:

1. For $\epsilon > 0$, $a_n \in \mathbb{R}$, $|a_n - a_k| < \epsilon$ for all $n, k > N$ is equivalent to: for each $K \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_k| < \epsilon$ for all $n, k > n_0$.
2. $f_n(x) \to f(x)$ pointwise means that for any $\epsilon_k > 0$, there exists $n_0 = n_0(x)$ such that $|f_n(x) - f(x)| < \epsilon_k$ for all $n \ge n_0$. For uniform convergence, $\sup_{x \in X} n_0(x) < \infty$.
3. $f_n \to f$ uniformly on X if and only if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$ and all $n \ge n_0$.

Hence, if f_n does not converge to f uniformly on X, then for some $k \in \mathbb{N}$, for all $n_0 \in \mathbb{N}$, there exists $n > n_0$ and $x \in X$ such that $|f_n(x) - f(x)| \ge \frac{1}{k}$. That is, for some $k \in \mathbb{N}$ and for each $n_0 \in \mathbb{N}$, there exists $x \in X$ and $n > n_0$ such that $|f_n(x) - f(x)| \ge \frac{1}{k}$.

Thus, without loss of generality, we collect all points $x \in X$ *such that for some* $k \in \mathbb{N}$ *and for all* $n_0 \in \mathbb{N}$*, there exists* $n > n_0$ *so that* $|f_n(x) - f(x)| \ge \frac{1}{k}$.

Define:

$$E_{m,k} := \bigcup_{n=m}^{\infty} \{x \in X : |f_n(x) - f(x)| \ge \frac{1}{k}\}.$$

For each fixed k, $E_{m,k}$ is a decreasing sequence of measurable sets, and $\mu(E_{m,k}) \le \mu(X) < \infty$. Hence,

$$\lim_{m\to\infty} \mu(E_{m,k}) = \mu\left(\bigcap_{m=1}^{\infty} E_{m,k}\right) = 0.$$

If $x \in \bigcap_{m=1}^{\infty} E_{m,k}$, then for all $j \ge 1$, there exists $n > j$ with $|f_n(x) - f(x)| \ge \frac{1}{k}$, which is a contradiction if $f_n(x) \to f(x)$.

From above, for $\epsilon > 0$, there exists $M_k \in \mathbb{N}$ such that $\mu(E_{M_k,k}) < \epsilon/2^k$ for any k.

Let $E := \bigcup_{k=1}^{\infty} E_{M_k,k}$. Then $\mu(E) < \epsilon$.

Now, for $x \in E^c = \bigcap_{k=1}^{\infty} E^c_{M_k,k}$, for each $k \geq 1$, $x \notin E_{M_k,k}$, so for all $n \geq M_k$, $|f_n(x) - f(x)| < \frac{1}{k}$.

Hence,

$$\sup_{x \in E^c} |f_n(x) - f(x)| \leq \frac{1}{k}, \quad \text{for all } n \geq M_k.$$

Thus, $f_n \to f$ uniformly on E^c.

Remark. The hypothesis that $(X, \mathcal{S}, \mu)$ has finite measure is essential in Egorov's theorem. Consider $f_n : (\mathbb{R}, \mathcal{M}, m) \to \mathbb{R}$ defined by $f_n := \chi_{[n,n+1]}$ for $n \in \mathbb{N}$.

Then, for each $x \in \mathbb{R}$, there exists $n_0 = n_0(x)$ such that $x \leq n_0, n_0 + 1$ and $f_n(x) = 0$ for all $n \geq n_0$.

Thus, $f_n \to 0$ pointwise on $\mathbb{R}$. However, it fails to follow Egorov's theorem. For any set $E \subset [n, n+1]$ with $0 < m(E) < \epsilon$,

$$\sup_{x \in E \subset \mathbb{R}} |f_n(x) - f(x)| = 1 \not\to 0.$$

□

Example 3.22. Show that $f_n = \frac{1}{n}\chi_{(0,n)}$ converges uniformly to 0.

Example 3.23. Show that $f_n = n\chi_{[0,\frac{1}{n}]}$ converges pointwise a.e. to 0, but not uniformly.

Hint: $f_n(0) = n \to \infty$, $f_n(x) = 0$ for $x \geq 1$ for any n. And if $0 < x < 1$, there exists $n_0 \in \mathbb{N}$ with $0 < x < \frac{1}{n_0}$.

Consider $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \chi_{\mathbb{Q}}(x)$ (the Dirichlet function).

Then f is nowhere continuous, but $f|_{\mathbb{R}\setminus\mathbb{Q}} = 0$ and $m(\mathbb{Q}) = 0 < \epsilon$. That is, f is constant on $\mathbb{R} \setminus \mathbb{Q}$ and $m(\mathbb{Q}) = 0 < \epsilon$.

We shall show that every measurable function is nearly continuous (Lusin's theorem).

Lemma 3.24 (Lusin approximation for simple functions)**.** *Let $E \subset \mathbb{R}$ be Lebesgue measurable with $m(E) < \infty$. Let:*

$$\varphi = \sum_{j=1}^{N} c_j \chi_{A_j}$$

be a simple function on E, where the sets $A_1, \ldots, A_N \subset E$ are measurable and pairwise disjoint. Then for every $\varepsilon > 0$ there exist closed sets $F_j \subset A_j$ such that, with:

$$F := \bigcup_{j=1}^{N} F_j \subset E, \qquad m(E \setminus F) < \varepsilon,$$

the restriction $\varphi|_F$ is continuous (indeed, locally constant) with respect to the subspace topology on F.

Proof. Fix $\varepsilon > 0$. Since each A_j is measurable and $m(A_j) \le m(E) < \infty$, by regularity of Lebesgue measure there exists a closed set $F_j \subset A_j$ with:

$$m(A_j \setminus F_j) < \frac{\varepsilon}{N}.$$

Set $F := \bigcup_{j=1}^N F_j$. Then F is closed as a finite union of closed sets, and:

$$m(E \setminus F) \le \sum_{j=1}^N m(A_j \setminus F_j) < \varepsilon.$$

To prove continuity, let $x_k \to x$ in F. Since $F = \bigcup_{j=1}^N F_j$ is a finite union, there exists j_0 and a subsequence (still denoted x_k) with $x_k \in F_{j_0}$ for all k. As F_{j_0} is closed in $\mathbb{R}$, we have $x \in F_{j_0}$. Because the sets F_j are pairwise disjoint, no subsequence of points from F_j with $j \neq j_0$ can converge to x; hence the full sequence is eventually in F_{j_0}. Since φ is constant on F_{j_0}, we conclude $\varphi(x_k) \to \varphi(x)$.

□

Corollary 3.25. *Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$ and let φ be a simple function on E. Then for every $\varepsilon > 0$ there exists a compact set $K \subset E$ such that $\varphi|_K$ is continuous and $m(E \setminus K) < \varepsilon$.*

Proof. Choose $R > 0$ so large that $m(E \cap [-R, R]^c) < \varepsilon/2$. Apply Lemma 3.24 to $E \cap [-R, R]$ with tolerance $\varepsilon/2$ to obtain a closed set $F \subset E \cap [-R, R]$ such that $\varphi|_F$ is continuous and $m((E \cap [-R, R]) \setminus F) < \varepsilon/2$. Then $K := F$ is compact (closed and bounded), and:

$$m(E \setminus K) \le m(E \cap [-R, R]^c) + m((E \cap [-R, R]) \setminus F) < \varepsilon.$$

□

Theorem 3.26 (Lusin's Theorem)**.** *Let $E \subset \mathbb{R}$ be Lebesgue measurable with $m(E) < \infty$, and let $f : E \to \mathbb{R}$ be Lebesgue measurable. Then for every $\varepsilon > 0$ there exists a closed set $F \subset E$ such that $f|_F$ is continuous and $m(E \setminus F) < \varepsilon$. Equivalently, f coincides with a continuous function on E outside a set of arbitrarily small measure.*

Proof. Fix $\varepsilon > 0$. Choose simple functions φ_n on E such that $\varphi_n(x) \to f(x)$ for m-almost every $x \in E$. By Egorov's theorem, there exists a measurable set $E_0 \subset E$ with $m(E \setminus E_0) < \varepsilon/2$ such that $\varphi_n \to f$ uniformly on E_0.

For each $n \in \mathbb{N}$, apply Corollary 3.25 to the simple function φ_n on E_0 with tolerance $\varepsilon/2^{n+2}$. We obtain a compact set $K_n \subset E_0$ such that $\varphi_n|_{K_n}$ is continuous and:

$$m(E_0 \setminus K_n) < \frac{\varepsilon}{2^{n+2}}.$$

Set:

$$F := \bigcap_{n=1}^{\infty} K_n \subset E_0.$$

Then F is closed in $\mathbb{R}$ (as an intersection of compact sets) and:

$$m(E \setminus F) \leq m(E \setminus E_0) + m(E_0 \setminus F) \leq \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+2}} < \varepsilon.$$

On F every φ_n is continuous, and since $\varphi_n \to f$ uniformly on E_0, in particular the convergence is uniform on F. Therefore $f|_F$ is the uniform limit of continuous functions, hence continuous.

□

Question 3.27. Does the converse of Egorov's Theorem hold?

Littlewood's Three Principles:

(i) Every set is nearly finite union of intervals.

(ii) Every function is nearly continuous.

(iii) Every convergent sequence is nearly uniformly convergent.

Here (i) means that if $E \in M(\mathbb{R})$ and $m(E) < \infty$. Then for each $\epsilon > 0$, there exists $O = \bigcup_n I_n$ such that $m(O \Delta E) < \epsilon$.

For $\epsilon > 0$, there exists $O \supset E$ such that $m(O \setminus E) < \epsilon$. But then $m(O) = \sum m(I_n) < \infty$, $O = \bigcup I_n$.

For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} m(I_n) < \epsilon/2$.

Let $O' = \bigcup_1^N I_n$, and $O'' = \bigcup_{n=N}^{\infty} I_n$. Then,

$$m(O' \Delta E) = m(O' \setminus E) + m(E \setminus O') < \epsilon/2 + m(E \setminus O) < \epsilon$$

Chapter 4

Lebesgue integration

This chapter develops integration on a measure space (X, Σ, μ) by turning approximation into a definition. Starting from simple functions, we define the integral of nonnegative measurable functions via monotone limits and then extend the theory to integrable (signed) functions. The central results are the convergence theorems—Beppo–Levi (monotone convergence), Fatou's lemma, and dominated convergence—which explain why the Lebesgue integral behaves well under limiting processes and why "almost everywhere" considerations are natural. We also compare the Lebesgue and Riemann integrals, clarifying the precise sense in which the Lebesgue theory extends the classical one. Throughout, the emphasis is on reusable principles and clean measure-theoretic arguments that apply uniformly across examples.

4.1 Integration of simple functions

Let (X, S, μ) be a measure space, and let

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}, \qquad E_i \in S,\ \alpha_i \in [0, \infty],$$

be a nonnegative simple function. After refining the representation if necessary, we may assume that the sets $E_1, \ldots, E_n$ are pairwise disjoint. We then define

$$\int_X \varphi \, d\mu := \sum_{i=1}^{n} \alpha_i \, \mu(E_i).$$

This definition is independent of the chosen representation of φ.

Moreover,

$$\int_X \varphi \, d\mu = 0 \quad \Longleftrightarrow \quad \varphi = 0 \text{ almost everywhere.}$$

Indeed, if $\sum_{i=1}^{n} \alpha_i \mu(E_i) = 0$ with $\alpha_i \geq 0$, then $\alpha_i \mu(E_i) = 0$ for each i, so $\mu(E_i) = 0$ whenever $\alpha_i > 0$.

For $E \in S$, we define

$$\int_E \varphi \, d\mu := \int_X \varphi \chi_E \, d\mu = \sum_{i=1}^{n} \alpha_i \, \mu(E_i \cap E).$$

Example 4.1. On $(\mathbb{R}, M, m)$, let $f = \chi_{[0,1]}$. Then

$$f = 0 \cdot \chi_{\mathbb{R}\setminus[0,1]} + 1 \cdot \chi_{[0,1]}, \qquad \int_{\mathbb{R}} f \, dm = m([0,1]) = 1.$$

The convention $0 \cdot \infty = 0$ is used whenever it arises in such decompositions.

Let $L^+(X, \mathcal{S}, \mu)$ denote the class of all $\mathcal{S}$-measurable functions $f : X \to [0, \infty]$.

Proposition 4.2. *Let φ, ψ be two simple functions in $L^+(X, \mathcal{S}, \mu)$. Then,*

(i) For $c > 0$, $\int_X c\varphi \, d\mu = c \int_X \varphi \, d\mu$.

(ii) $\int_X (\varphi + \psi) \, d\mu = \int_X \varphi \, d\mu + \int_X \psi \, d\mu$, linearly.

(iii) If $\varphi \le \psi$, then $\int_X \varphi \, d\mu \le \int_X \psi \, d\mu$.

(iv) If $\nu : \mathcal{S} \to [0, \infty]$ be defined by $\nu(A) = \int_A \varphi \, d\mu$ for $A \in \mathcal{S}$, then ν is a measure on $(X, \mathcal{S})$.

Proof. (i): Proof of it is trivial.

(ii) Let $\varphi = \sum_{i=1}^{m} \lambda_i \chi_{E_i}$ and $\psi = \sum_{j=1}^{n} \beta_j \chi_{F_j}$.

Without loss of generality, we can write $X = \bigcup_{i=1}^{m} E_i = \bigcup_{j=1}^{n} F_j$. Then, $E_i = \bigcup_{j=1}^{n} (E_i \cap F_j)$ and $F_j = \bigcup_{i=1}^{m} (E_i \cap F_j)$.

Now,

$$\int_X \varphi \, d\mu + \int_X \psi \, d\mu = \sum_{i=1}^{m} \lambda_i \mu(E_i) + \sum_{j=1}^{n} \beta_j \mu(F_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu(E_i \cap F_j) + \sum_{j=1}^{n} \sum_{i=1}^{m} \beta_j \mu(E_i \cap F_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda_i + \beta_j) \mu(E_i \cap F_j) \tag{4.1}$$

and

$$\int_X (\varphi + \psi) \, d\mu = \sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda_i + \beta_j) \mu(E_i \cap F_j) = \int_X \varphi \, d\mu + \int_X \psi \, d\mu \qquad \text{(by (4.1))}$$

(iii) Since $\varphi \le \psi$, we set $\lambda_i \le \beta_j$ if $E_i \cap F_j \ne \varnothing$ (for this, let $\varphi(x) \le \psi(x)$ implies $x \in E_i, x \in F_j$ for some i, j):

$$\int_X \varphi \, d\mu = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu(E_i \cap F_j) \le \sum_{j=1}^{n} \sum_{i=1}^{m} \beta_j \mu(E_i \cap F_j) = \int_X \psi \, d\mu.$$

(iv) For $A \in \mathcal{S}$, write $\nu(A) = \int_A \varphi \, d\mu$. Then $\nu(\varnothing) = 0$.

If $A, B \in \mathcal{S}$ and $A \cap B = \varnothing$, then

$$\nu(A \cup B) = \int_{A \cup B} \varphi \, d\mu = \int_A \varphi \, d\mu + \int_B \varphi \, d\mu = \nu(A) + \nu(B).$$

Hence, ν is a measure on $(X, \mathcal{S})$.

Let $\{A_k\}_{k=1}^{\infty} \subset \mathcal{S}$ and $A_k \cap A_\ell = \varnothing$, for all $k \neq \ell$, and write $A = \bigcup_{k=1}^{\infty} A_k$. Then,

$$\nu(A) = \int_A \varphi \, d\mu = \sum_{i=1}^{m} \lambda_i \mu(A \cap E_i) = \sum_{i=1}^{m} \lambda_i \left[\sum_{k=1}^{\infty} \mu(A_k \cap E_i) \right]$$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{m} \lambda_i \mu(A_k \cap E_i) = \sum_{k=1}^{\infty} \nu(A_k).$$

Now, it is obvious that if $a, b \in \mathbb{R}$, then

$$\int_X (a\varphi + b\psi) \, d\mu = a \int_X \varphi \, d\mu + b \int_X \psi \, d\mu,$$

for φ and ψ are simple measurable functions on $(X, \mathcal{S}, \mu)$ to $[0, \infty]$.

Notice that if $\varphi \leq \psi$ almost everywhere, then $\int_X \varphi \, d\mu \leq \int_X \psi \, d\mu$.

Let $E = \{x \in X : \varphi(x) > \psi(x)\}$. Then $\mu(E) = 0$.

$$\int_X \varphi = \int_E \varphi + \int_{E^c} \varphi = 0 + \int_{E^c} \psi = \int_X \psi.$$

(i.e., $\int_E \varphi = 0$ if and only if $\varphi = 0$ almost everywhere.)

Next, consider $f \in L^+(X, \mathcal{S}, \mu)$. Then there is a sequence of simple $\varphi_n \nearrow f$ pointwise. Hence $\int_X \varphi_n \, d\mu$ is a sequence in $[0, \infty]$. Thus,

$$\lim_{n \to \infty} \int_X \varphi_n \, d\mu = \sup_{n \geq 1} \int_X \varphi_n \, d\mu.$$

(Important)

We define, for $f \in L^+(X, \mathcal{S}, \mu)$,

$$\int_X f \, d\mu = \sup_{\varphi \leq f} \int_X \varphi \, d\mu, \qquad 0 \leq \varphi \leq f, \ \varphi \text{ is simple and measurable.}$$

If $f, g \in L^+(X, \mathcal{S}, \mu)$ and $f \leq g$, then

$$\int_X f \, d\mu = \sup_{\varphi \leq f} \int_X \varphi \, d\mu \leq \sup_{\psi \leq g} \int_X \psi \, d\mu = \int_X g \, d\mu.$$

Moreover, if $f : (X, \mathcal{S}, \mu) \to \overline{\mathbb{R}} = [-\infty, \infty]$, then $f = f^+ - f^-$, we write

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu,$$

if at least one of $\int_X f^+ \, d\mu$ or $\int_X f^- \, d\mu$ is finite.

□

Lemma 4.3. *Let $f \in L^+(X, \mathcal{S}, \mu)$. Then*

$$\int_X f \, d\mu = 0 \text{ if and only if } f = 0 \quad a.e.\ \mu$$

Proof. Suppose $f = 0$ almost everywhere. If $0 \le \varphi \le f$, φ is a simple measurable function, then $\varphi = 0$ almost everywhere.

Then

$$\int_X f \, d\mu = \sup_{\varphi \le f} \int_X \varphi \, d\mu = 0 \quad \text{(by previous result)}.$$

Next, let $E = \{x \in X : f(x) > 0\}$. Then

$$E = \bigcup_{n=1}^{\infty} \{x \in X : f(x) > \frac{1}{n}\} = \bigcup_{n=1}^{\infty} E_n.$$

Now,

$$\mu(E_n) = \int_{E_n} d\mu \le \int_{E_n} f \, d\mu \le \int_X f \, d\mu = 0.$$

So, $\mu(E_n) = 0$ for all n, hence $\mu(E) = 0$.

Let φ be a simple function on a measure space $(X, \mathcal{S}, \mu)$ to $[0, \infty]$. Then,

$$\varphi = \sum_{i=1}^{m} \lambda_i \chi_{E_i}, \qquad E_i \in \mathcal{S}$$

and $E_i \cap E_j = \varnothing$ if $i \ne j$, $E_i \in \mathcal{S}$. By assigning zero outside $\cup_{i=1}^m E_i$, we may assume that

$$\varphi = \sum_{i=1}^{m} \lambda_i \chi_{E_i}, \qquad \bigcup_{i=1}^{m} E_i = X.$$

Let $L^+(X, \mathcal{S}, \mu)$ be the space of all $\mathcal{S}$-measurable functions $f : X \to [0, \infty]$.

□

Proposition 4.4. *Let φ, ψ be two simple functions in $L^+(X, \mathcal{S}, \mu)$. Then,*

(i) For $c > 0$, $\int_X c\varphi \, d\mu = c \int_X \varphi \, d\mu$.

(ii) $\int_X (\varphi + \psi) \, d\mu = \int_X \varphi \, d\mu + \int_X \psi \, d\mu$, linearly.

(iii) If $\varphi \le \psi$, then $\int_X \varphi \, d\mu \le \int_X \psi \, d\mu$.

(iv) If $\nu : \mathcal{S} \to [0, \infty]$ be defined by $\nu(A) = \int_A \varphi \, d\mu$ for $A \in \mathcal{S}$, then ν is a measure on $(X, \mathcal{S})$.

Proof. (i): Proof of it is trivial.

(ii) Let $\varphi = \sum_{i=1}^m \lambda_i \chi_{E_i}$ and $\psi = \sum_{j=1}^n \beta_j \chi_{F_j}$.

Without loss of generality, we can write $X = \bigcup_{i=1}^m E_i = \bigcup_{j=1}^n F_j$. Then, $E_i = \bigcup_{j=1}^n (E_i \cap F_j)$ and $F_j = \bigcup_{i=1}^m (E_i \cap F_j)$.

Now,

$$\int_X \varphi \, d\mu + \int_X \psi \, d\mu = \sum_{i=1}^m \lambda_i \mu(E_i) + \sum_{j=1}^n \beta_j \mu(F_j)$$

$$= \sum_{i=1}^m \sum_{j=1}^n \lambda_i \mu(E_i \cap F_j) + \sum_{j=1}^n \sum_{i=1}^m \beta_j \mu(E_i \cap F_j) = \sum_{i=1}^m \sum_{j=1}^n (\lambda_i + \beta_j) \mu(E_i \cap F_j) \tag{4.2}$$

and

$$\int_X (\varphi + \psi) \, d\mu = \sum_{i=1}^m \sum_{j=1}^n (\lambda_i + \beta_j) \mu(E_i \cap F_j) = \int_X \varphi \, d\mu + \int_X \psi \, d\mu \qquad \text{(by (4.2))}$$

(iii) Since $\varphi \le \psi$, we set $\lambda_i \le \beta_j$ if $E_i \cap F_j \ne \varnothing$ (for this, let $\varphi(x) \le \psi(x)$ implies $x \in E_i, x \in F_j$ for some i, j):

$$\int_X \varphi \, d\mu = \sum_{i=1}^m \sum_{j=1}^n \lambda_i \mu(E_i \cap F_j) \le \sum_{j=1}^n \sum_{i=1}^m \beta_j \mu(E_i \cap F_j) = \int_X \psi \, d\mu.$$

(iv) For $A \in \mathcal{S}$, write $\nu(A) = \int_A \varphi \, d\mu$. Then $\nu(\varnothing) = 0$.

If $A, B \in \mathcal{S}$ and $A \cap B = \varnothing$, then

$$\nu(A \cup B) = \int_{A \cup B} \varphi \, d\mu = \int_A \varphi \, d\mu + \int_B \varphi \, d\mu = \nu(A) + \nu(B).$$

Hence, ν is a measure on $(X, \mathcal{S})$.

Let $\{A_k\}_{k=1}^\infty \subset \mathcal{S}$ and $A_k \cap A_\ell = \varnothing$, for all $k \ne \ell$, and write $A = \bigcup_{k=1}^\infty A_k$. Then,

$$\nu(A) = \int_A \varphi \, d\mu = \sum_{i=1}^m \lambda_i \mu(A \cap E_i) = \sum_{i=1}^m \lambda_i \left[\sum_{k=1}^\infty \mu(A_k \cap E_i) \right]$$

$$= \sum_{k=1}^\infty \sum_{i=1}^m \lambda_i \mu(A_k \cap E_i) = \sum_{k=1}^\infty \nu(A_k).$$

Now, it is obvious that if $a, b \in \mathbb{R}$, then

$$\int_X (a\varphi + b\psi) \, d\mu = a \int_X \varphi \, d\mu + b \int_X \psi \, d\mu,$$

for φ and ψ are simple measurable functions on $(X, \mathcal{S}, \mu)$ to $[0, \infty]$.

Notice that if $\varphi \le \psi$ almost everywhere, then $\int_X \varphi \, d\mu \le \int_X \psi \, d\mu$.

Let $E = \{x \in X : \varphi(x) > \psi(x)\}$. Then $\mu(E) = 0$.

$$\int_X \varphi = \int_E \varphi + \int_{E^c} \varphi = 0 + \int_{E^c} \psi = \int_X \psi.$$

(i.e., $\int_E \varphi = 0$ if and only if $\varphi = 0$ almost everywhere.)

Next, consider $f \in L^+(X, \mathcal{S}, \mu)$. Then there is a sequence of simple $\varphi_n \nearrow f$ pointwise. Hence $\int_X \varphi_n \, d\mu$ is a sequence in $[0, \infty]$. Thus,

$$\lim_{n\to\infty} \int_X \varphi_n \, d\mu = \sup_{n\geq 1} \int_X \varphi_n \, d\mu.$$

(Important)

We define, for $f \in L^+(X, \mathcal{S}, \mu)$,

$$\int_X f \, d\mu = \sup_{\varphi \leq f} \int_X \varphi \, d\mu, \qquad 0 \leq \varphi \leq f, \ \varphi \text{ is simple and measurable.}$$

If $f, g \in L^+(X, \mathcal{S}, \mu)$ and $f \leq g$, then

$$\int_X f \, d\mu = \sup_{\varphi \leq f} \int_X \varphi \, d\mu \leq \sup_{\psi \leq g} \int_X \psi \, d\mu = \int_X g \, d\mu.$$

Moreover, if $f : (X, \mathcal{S}, \mu) \to \mathbb{R} = [-\infty, \infty]$, then $f = f^+ - f^-$, we write

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu,$$

if at least one of $\int_X f^+ \, d\mu$ or $\int_X f^- \, d\mu$ is finite.

□

Lemma 4.5. *Let $f \in L^+(X, \mathcal{S}, \mu)$. Then*

$$\int_X f \, d\mu = 0 \text{ if and only if } f = 0 \quad a.e. \ \mu$$

Proof. Suppose $f = 0$ almost everywhere. If $0 \leq \varphi \leq f$, φ is a simple measurable function, then $\varphi = 0$ almost everywhere.

Then

$$\int_X f \, d\mu = \sup_{\varphi \leq f} \int_X \varphi \, d\mu = 0 \quad \text{(by previous result).}$$

Next, let $E = \{x \in X : f(x) > 0\}$. Then

$$E = \bigcup_{n=1}^{\infty} \{x \in X : f(x) > \frac{1}{n}\} = \bigcup_{n=1}^{\infty} E_n.$$

Now,

$$\mu(E_n) = \int_{E_n} d\mu \leq \int_{E_n} f \, d\mu \leq \int_X f \, d\mu = 0.$$

So, $\mu(E_n) = 0$ for all n, hence $\mu(E) = 0$.

□

4.2 Monotone Convergence Theorem (MCT)

Let $f_n, f \in L^+(X, \mathcal{S}, \mu)$ be such that $f_n \uparrow f$ pointwise almost everywhere. Then

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu.$$

Proof. Since $f_n \le f_{n+1} \le f$, the limit of $\int_X f_n$ will be bounded above by $\int_X f$. Hence,

$$\lim_{n\to\infty} \int_X f_n \le \int_X f.$$

In order to show the other inequality, it is enough to show that for each $\epsilon > 0$,

$$\lim_{n\to\infty} \int_X f_n \ge (1-\epsilon) \int_X f,$$

or for $\varphi \le f$,

$$\lim_{n\to\infty} \int_X f_n \ge (1-\epsilon) \int_X \varphi.$$

Let $E_n = \{x \in X : f_n(x) \ge (1-\epsilon)\varphi(x)\}$. Since $f_n \uparrow f$, $E_n \subseteq E_{n+1}$. Moreover, $X = \bigcup_{n=1}^{\infty} E_n$.

For, let $x \in X$, then $f_n(x) \uparrow f(x)$, and so for some n, $f_n(x) \ge (1-\epsilon)\varphi(x)$. If not, $f_n(x) < (1-\epsilon)\varphi(x)$ for all n, so $f(x) \le (1-\epsilon)\varphi(x)$, $\varphi \le f$ implies Contradiction.

Let $\nu(E_n) = \int_{E_n} \varphi$. Then ν becomes a measure on $(X, \mathcal{S})$ and $E_n \uparrow X$. Hence,

$$\lim_{n\to\infty} \nu(E_n) = \nu(X).$$

Thus,

$$(1-\epsilon)\int_X \varphi = \lim_{n\to\infty} \int_{E_n} (1-\epsilon)\varphi \le \lim_{n\to\infty} \int_{E_n} f_n \le \lim_{n\to\infty} \int_X f_n.$$

□

Remark 4.6. For f_n converges to f pointwise is necessary in MCT.

Let $f_n : (\mathbb{R}, \mathcal{M}, m) \to \mathbb{R}$ by $f_n = \frac{1}{n} R[0,n]$. Then $f_n \to 0$ pointwise (even $f_n \to 0$ uniformly too), but

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n\, dm = 1 \ne 0 = \int_{\mathbb{R}} \lim_{n\to\infty} f_n\, dm.$$

Exercise 4.7. Verify MCT for $f_n : \mathbb{R} \to [0, \infty]$ given by (i) $f_n = \chi_{(0,n)}$ and (ii) $f_n = n\chi_{(0,\frac{1}{n})}$.

Corollary 4.8. *Let $f_n, f \in L^+(X, \mathcal{S}, \mu)$ be such that $f_n \uparrow f$ pointwise almost everywhere on X, then*

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu.$$

Proof. Let $f_n \to f$ pointwise almost everywhere on $E \subseteq X$. Then $\mu(E^c) = 0$. Hence, $E, E^c \in \mathcal{S}$. By MCT on measure space $(E, \mathcal{S}_E, \mu)$ we get

$$\int_E f\, d\mu = \lim_{n\to\infty} \int_E f_n\, d\mu \text{ implies } \int_X \chi_E f\, d\mu = \lim_{n\to\infty} \int_X \chi_E f_n\, d\mu.$$

Now,

$$\int_X f = \int_X (\chi_E f + \chi_{E^c} f) = \int_X \chi_E f + \int_X \chi_{E^c} f,$$

(by linearity of integration)

$$= \lim_{n\to\infty} \int_X \chi_E f_n + \lim_{n\to\infty} \int_X \chi_{E^c} f_n.$$

□

Remark 4.9. Integration is linear on $L^+(X, \mathcal{S}, \mu)$. That is, $f \mapsto \int_X f \, d\mu$ is a linear map. Let $f, g \in L^+(X, \mathcal{S}, \mu)$.

Then there exists sequences φ_n, ψ_n of simple measurable functions in $L^+(X, \mathcal{S}, \mu)$ such that $\varphi_n \uparrow f$ pointwise and $\psi_n \uparrow g$ pointwise by MCT.

$$\int_X (f+g)\, d\mu = \lim_{n\to\infty} \int_X (\varphi_n + \psi_n)\, d\mu = \lim_{n\to\infty} \int_X \varphi_n \, d\mu + \lim_{n\to\infty} \int_X \psi_n \, d\mu$$

$$= \int_X f \, d\mu + \int_X g \, d\mu$$

Remark 4.10. Let ψ_n be a sequence of simple functions in $L^+(X, \mathcal{S}, \mu)$ such that $\psi_n \uparrow f \in L^+(X, \mathcal{S}, \mu)$. Then $\psi_n = \varphi_n \chi_{E_n,n}$, and $\varphi_n \to f$, whereas $\int_X \psi_n < \infty$.

Consider $f \in L^+(X, \mathcal{S}, \mu)$ and $A \in \mathcal{S}$. Then the set function $E \mapsto \int_E f \, d\mu$ defines a measure on $(X, \mathcal{S})$. This will be followed by the following equivalent statement of MCT, known as Beppo-Levi Theorem.

Theorem 4.11. *Let $f_n \in L^+(X, \mathcal{S}, \mu)$. Then*

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Proof. Notice, $\sum_{k=1}^{n} f_k \uparrow \sum_{k=1}^{\infty} f_k$. Hence,

$$\int_X \left(\sum_{k=1}^{n} f_k \right) d\mu = \int_X (\lim_{n\to\infty} \sum_{k=1}^{n} f_k) d\mu = \lim_{n\to\infty} \int_X \left(\sum_{k=1}^{n} f_k \right) d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \int_X f_k d\mu$$

(by MCT).

Now, let $\nu(E) = \int_E f \, d\mu$. Then $\nu(\varnothing) = 0$. If $E = \bigcup_{n=1}^{\infty} E_n$, $E_n \in \mathcal{S}$, then $f_E = \sum_{n=1}^{\infty} f_{E_n,n}$.

By Beppo-Levi theorem,

$$\nu(E) = \sum_{n=1}^{\infty} \int_X \chi_{E_n} f \, d\mu = \sum_{n=1}^{\infty} \nu(E_n)$$

Hence, ν is a measure on $(X, \mathcal{S})$.

Recall that monotone convergence theorem (MCT) allows us to commute lim and $\int$ when sequence $f_n \uparrow f$ and non-negative integrands. However, other cases still need to be addressed.

For example, if $f_n = \frac{1}{n}\chi_{(n,n+1)}$ on $(\mathbb{R}, \mathcal{M}, m)$, then $f_n \to 0$ but f_n is not monotone. We have

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm = 0 = \int_{\mathbb{R}} \lim_{n\to\infty} f_n \, dm,$$

If $f_n = \frac{1}{n}\chi_{(0,n)} \to 0$ but $\int f_n \to 1 > 0 = \int \lim f_n$.

Hence, "equality" need not be the case for arbitrary sequence, but we can compare both the limits.

□

4.3 Fatou's Lemma

Let $f_n \in L^+(X, \mathcal{S}, \mu)$. Then

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Proof. Since $\liminf_{n\to k} f_n \le f_j$, for all $j \ge k$,

$$\int_X \liminf_{n\to k} f_n \, d\mu \le \int_X f_j \, d\mu, \quad \text{for all } j \ge k.$$

$$\int_X \liminf_{n\to k} f_n \, d\mu \le \inf_{j\ge k} \int_X f_j \, d\mu \tag{4.3}$$

Define, for each $k \in \mathbb{N}$,

$$g_k := \inf_{n\ge k} f_n = \inf\{f_k, f_{k+1}, \dots\}.$$

Then (g_k) is increasing and

$$\lim_{k\to\infty} g_k = \sup_{k\in\mathbb{N}} g_k = \liminf_{n\to\infty} f_n.$$

Hence, by Monotone Convergence Theorem, it follows that

$$\int_X \lim_{k\to\infty} g_k \, d\mu = \lim_{k\to\infty} \int_X g_k \, d\mu \le \lim_{k\to\infty} \inf_{j\ge k} \int_X f_j \, d\mu = \liminf_{k\to\infty} \int_X f_k \, d\mu.$$

□

Remark 4.12. (i) If $\lim_{k\to\infty} f_k$ exists, then

$$\int_X \lim_{k\to\infty} f_k \, d\mu \le \lim_{k\to\infty} \int_X f_k \, d\mu.$$

(ii) The inequality in Fatou's lemma can be strict. To see this, consider $f_n = \frac{1}{n}\chi_{[0,n]}$ on $(\mathbb{R}, \mathcal{M}, m)$. Then we obtain

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n \, d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n \, d\mu = 0 < 1 = \liminf_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu.$$

(iii) Fatou's lemma need not be true beyond non-negative functions. For example, $f_n = -\frac{1}{n}\chi_{[n,2n]}$ on $(\mathbb{R}, \mathcal{M}, m)$.

Hence, $\liminf_{k\to\infty} f_n = \lim_{k\to\infty}(-\frac{1}{k}) = 0$, however,

$$\int_X \liminf_{k\to\infty} f_n \, d\mu = 0 > -1 = \liminf_{k\to\infty} \int_X f_k \, d\mu.$$

Example 4.13. Let $f_n \in L^+(X, \mathcal{S}, \mu)$ and $f = \lim f_n$ with $f_n \leq f$ for all $n \geq 1$. Show that

$$\int_X \lim f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

(Hint: use Fatou's lemma for $f_n + f - f_n \geq 0$ both.)

Example 4.14. Let $f_n \in L^+(X, \mathcal{S}, \mu)$ be given by

$$f_n(x) = \begin{cases} \chi_E(x) & \text{if } n \text{ is odd} \\ 1 - \chi_E(x) & \text{if } n \text{ is even} \end{cases}$$

for some $E \in \mathcal{S}$. Verify Fatou's lemma for f_n.

(Hint: use $\int_X f = \int_E f + \int_{E^c} f$.)

4.4 Integrable Functions

Let $f : (X, \mathcal{S}, \mu) \to \mathbb{R}$ be measurable. Then f^+, f^- both are measurable and $f = f^+ - f^-$. $|f| = f^+ + f^-$.

Define f as said to be *integrable* on $(X, \mathcal{S}, \mu)$ if both $\int_X f^+ \, d\mu$ and $\int_X f^- \, d\mu$ are finite. In this case, we write

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

As $|f| = f^+ + f^-$, it follows that $\int_X |f| \, d\mu$ is finite if and only if $\int_X f \, d\mu$ is finite.

Denote

$$L^1(X, \mathcal{S}, \mu) := \left\{ f : X \to \mathbb{R} \text{ measurable}, \int_X |f| \, d\mu < \infty \right\}.$$

Notice that we also use the symbol $L^1(X)$ or $L^1(X, \mathcal{S})$.

We can see that $R[0,1] \subsetneq L^1([0,1], \mathcal{M}, m)$ since for

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ -1 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \end{cases}$$

we have $f \notin R[0,1]$ but $f \in L^1([0,1], \mathcal{M}, m)$.

Moreover, if $E \in \mathcal{S}$ and $f \in L^1(X, \mathcal{S}, \mu)$, then

$$\int_E f \, d\mu = \int_X f\chi_E \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$$

Notice that $L^1(X,\mathcal{S},\mu)$ is a linear space. We define a norm on $L^1(X,\mathcal{S},\mu)$ by setting

$$\|f\|_1 = \int_X |f|\,d\mu, \qquad f \in L^1(X,\mathcal{S},\mu).$$

To see this, we need to recognize $f=0$ a.e. to be $f=0$ since $\int_X |f|\,d\mu = 0$ if and only if $f=0$ a.e.

Lemma 4.15. *Let $f \in L^1(X)$.*

(i) $|\int_X f\,d\mu| \le \int_X |f|\,d\mu$ *(Conjugation of $f \mapsto \int_X f$)*

(ii) $\int_E f\,d\mu = 0$ *for all* $E \in \mathcal{S}$ *if and only if* $\int_X |f|\,d\mu = 0$ *if and only if* $f = 0$ *a.e.*

(iii) $\{x \in X : f(x) \neq 0\}$ *is a σ-finite set.*

(iv) $\mu\{x \in X : |f(x)| = \infty\} = 0$

Proof. (i)

$$\left|\int_X f\,d\mu\right| = \left|\int_X f^+\,d\mu - \int_X f^-\,d\mu\right| \le \int_X f^+\,d\mu + \int_X f^-\,d\mu = \int_X |f|\,d\mu.$$

(ii)Suppose $\int_E f\,d\mu = 0$ for all $E \in \mathcal{S}$. Let $E_0 = \{x \in X : f(x) > 0\}$. Then $E_0 \in \mathcal{S}$ and

$$\int_X |f|\,d\mu = \int_{E_0} f\,d\mu + \int_{E_0^c} f\,d\mu = \int_{E_0} f\,d\mu - \int_{E_0^c} f\,d\mu = \int_{E_0} f\,d\mu = 0.$$

If $\int_X |f| d\mu = 0$, then for $E \subseteq S$,

$$\left|\int_E f\right| = \int_E |f| \le \int_X |f| = 0.$$

(iii)$\{x \in X : |f(x)| \neq 0\} = \bigcup_{n=1}^\infty \{x \in X : |f(x)| \ge \frac{1}{n}\} = \bigcup_{n=1}^\infty E_n$, where $E_n = \{x : |f(x)| \ge \frac{1}{n}\}$.

–Now, $\mu(E_n) \cdot \frac{1}{n} \le \int_{E_n} |f(x)| d\mu \le \int_X |f(x)| d\mu < \infty$.

–Since $\int_X |f| d\mu < \infty$.

–

(iv)$\{x \in X : |f(x)| = \infty\} = \bigcap_{n=1}^\infty \{x : |f(x)| \ge n\} = \bigcap_{n=1}^\infty F_n$, where $F_n = \{x : |f(x)| \ge n\}$.

–But $\mu(F_n) \le \frac{1}{n}\int_X |f| d\mu$ for all $n \ge 1$ implies $\mu(F_n) \to 0$.

–$\mu(\{x \in X : |f(x)| = \infty\}) \le \mu(F_n) \le \frac{1}{n}\|f\|_{L^1} \to 0$.

Note that in proving (iii) & (iv), we have proved the following interesting result.

□

4.5 Chebyshev's Inequality

Let $f \in L^1(X, S, \mu)$ and $\alpha > 0$. Then

$$\mu(\{x \in X : |f(x)| \geq \alpha\}) \leq \frac{1}{\alpha}\|f\|_{L^1}.$$

So far we have shown that $L^1(X, S, \mu)$ is a **normed linear space**. For $f \in L^1(X, S, \mu)$,

$$\|f\|_{L^1} = \int_X |f| d\mu < \infty, \quad \text{and} \quad \|f\|_{L^1} = 0 \text{ implies } f = 0 \text{ a.e.}$$

Next, we shall show that $L^1(X, S, \mu)$ is a complete space. To see this, we need a wonderful result known as DCT.

4.6 Dominated Convergence Theorem (DCT)

Let f_n be a sequence of measurable functions on (X, S, μ) such that:

(i) $f_n \to f$ pointwise on X

(ii) $|f_n| \leq g$, where $g \in L^1(X, S, \mu)$.

Then

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Proof. Since $\lim f_n = f$ and $|f_n| \leq g \in L^1$, it follows that $f_n, f \in L^1$ (by monotone convergence).

Now,

$$0 \leq g + f_n \xrightarrow{ptwise} g + f$$
$$0 \leq g - f_n \xrightarrow{ptwise} g - f$$

Then by Fatou's Lemma,

$$\int_X (g+f) d\mu = \int_X \lim(g + f_n) d\mu \leq \lim \int_X (g + f_n) d\mu$$

$$\text{implies } \int_X f d\mu \leq \liminf \int_X f_n d\mu \quad (\text{since } \int_X g < \infty).$$

Similarly,

$$\int_X (g - f) d\mu = \lim \int_X (g - f_n) d\mu$$

$$\text{implies } \limsup \int_X f_n d\mu \leq \int_X f d\mu.$$

Hence, we have

$$\lim_{n\to\infty}\int_X f_n d\mu = \int_X f d\mu = \lim_{n\to\infty}\int_X f_n d\mu.$$

□

Corollary 4.16. *Let f_n be a sequence of measurable functions on (X, S, μ) such that:*

(i) $f_n \to f$ pointwise a.e.

(ii) $|f_n| \le g$ a.e. for some $g \in L^1(X, S, \mu)$.

Then

$$\int_X f d\mu = \lim_{n\to\infty}\int_X f_n d\mu.$$

In the Dominated Convergence Theorem, we require $f_n \to f$ pointwise a.e. and $|f_n| \le g$ a.e., where $g \in L^1$. This completes

$$|f_n - f| \to 0 \text{ a.e.} \quad \text{and} \quad |f_n - f| \le 2g \in L^1.$$

Hence,

$$\left|\int_X f_n d\mu - \int_X f d\mu\right| \le \int_X |f_n - f| d\mu \to 0 \text{ as } |f_n - f| \to 0.$$

This shows that the map $f \mapsto \int_X f d\mu$ is "continuous" on $L^1(X, S, \mu)$.

Thus, we can think of fundamental theorems of calculus for Lebesgue integrable functions.

Theorem 4.17. *Let $f \in L^1(\mathbb{R}, \mathcal{M}, m)$. Define*

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{(-\infty,x]} f dm_{|(-\infty,x]}$$

Then F is continuous on $\mathbb{R}$ to $\mathbb{R}$.

Proof. Let $x_n \to x$ in $\mathbb{R}$, then $f \cdot \chi_{(-\infty,x_n]} \to f \cdot \chi_{(-\infty,x]}$ pointwise.

By DCT, we have

$$\int_{\mathbb{R}} f \cdot \chi_{(-\infty,x_n]} dm = \lim_{n\to\infty}\int_{\mathbb{R}} f \cdot \chi_{(-\infty,x_n]} dm$$

implies $F(x) = \lim_{n\to\infty} F(x_n)$.

Example: For $f_n : (\mathbb{R}, \mathcal{M}, m) \to \mathbb{R}$ and given n,

(i) $f_n = n\chi_{(0,\frac{1}{n}]}$

(ii) $f_n = \frac{1}{n}\chi_{[n,n+1]}$

(iii) For $x \in [n, n+1]$, $f_n \to 0$ pointwise, and

$$\int_{\mathbb{R}} f_n dm = 1 > 0 = \int_{\mathbb{R}} \lim f_n dm$$

Hence, $|f_n| < g \in L^1$ in the statement of DCT is necessary.

Now, to prove $L^1(X, S, \mu)$ is complete, we all need to show that every absolutely convergent series in $L^1(X, S, \mu)$ is convergent.

□

Theorem 4.18. *Let $\{f_n\}$ be a sequence of integrable functions on (X, S, μ) such that*

$$\sum_{n=1}^{\infty} \int_X |f_n| < \infty.$$

Then $\sum_{n=1}^{\infty} f_n$ converges pointwise almost everywhere to some $f \in L^1(X, S, \mu)$, and

$$\int_X \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_X f_n.$$

Proof. Assume that $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$. By the Monotone Convergence Theorem,

$$\int_X \sum_{n=1}^{\infty} |f_n| \, d\mu = \sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty.$$

Thus the function

$$g := \sum_{n=1}^{\infty} |f_n|$$

belongs to $L^1(X, S, \mu)$.

Since $g < \infty$ almost everywhere, the numerical series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for almost every $x \in X$. Define

$$f(x) := \sum_{n=1}^{\infty} f_n(x)$$

on that full-measure set, and redefine f arbitrarily on the remaining null set.

Let $s_k := \sum_{n=1}^{k} f_n$. Then $|s_k| \leq g$ almost everywhere for every k, and $s_k \to f$ almost everywhere. In particular, f is measurable and $|f| \leq g$ almost everywhere, so $f \in L^1(X, S, \mu)$.

By the Dominated Convergence Theorem,

$$\int_X f \, d\mu = \lim_{k \to \infty} \int_X s_k \, d\mu = \lim_{k \to \infty} \sum_{n=1}^{k} \int_X f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Moreover,

$$\|f - s_k\|_1 \leq \int_X \sum_{n>k} |f_n| \, d\mu = \sum_{n>k} \|f_n\|_1 \to 0,$$

so the partial sums converge to f in L^1.

Therefore every absolutely convergent series in $L^1(X, S, \mu)$ converges in L^1, and hence $L^1(X, S, \mu)$ is complete.

□

Corollary 4.19. *If $f \in L^1(X,S,\mu)$, then the set function $E \mapsto \int_E f d\mu$ is countably additive.*

Proof. Let $\nu(E) = \int_E f d\mu$. Then $\nu(E) \in \mathbb{R}$ because

$$|\nu(E)| = \left|\int_E f d\mu\right| \le \int_E |f| d\mu \le \int_X |f| d\mu < \infty.$$

Next, let $E = \dot\bigcup_{n=1}^{\infty} E_n$, where $E_n \in S$. Then

$$(*) \quad \nu(E) = \int_X f\chi_E d\mu = \int_X \sum_{n=1}^{\infty} f\chi_{E_n} d\mu$$

Similarly,

$$\sum_{n=1}^{\infty} \int_X |f\chi_{E_n}| d\mu = \int_E |f| d\mu \le \int_X |f| d\mu < \infty.$$

Hence, by previous theorem,

$$\int_X \sum_{n=1}^{\infty} f\chi_{E_n} d\mu \text{ converges to a finite number.}$$

Thus, $\nu(E) \in \mathbb{R}$ and once again by the previous theorem,

$$\nu(E) = \sum_{n=1}^{\infty} \int_X f\chi_{E_n} d\mu = \sum_{n=1}^{\infty} \nu(E_n).$$

Note that $\nu : S \to \mathbb{R} = (-\infty, \infty)$, which satisfies $\nu(\emptyset) = 0$, and

$$\nu\left(\dot\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n).$$

Such set functions are called *signed measures*, which we shall see later.

Absolute Continuity of the Integral

If $f \in L^1(X,S,\mu)$, we have seen that $\nu(E) = \int_E f d\mu$ defines a set function which is countably additive on (X,S).

$$\text{Moreover, } |\nu(E)| < \int_E |f| d\mu \le \int_X |f| d\mu.$$

This shows that $\nu(E)$ is small if $\mu(E)$ is small. Thus, we can prove the following result.

□

Theorem 4.20. *Let $f \in L^1(X,S,\mu)$. Then for $\varepsilon > 0$, there exists $\delta > 0$ and a set $E \in S$ such that*

$$\mu(E) < \delta \text{ implies } \int_E |f| d\mu < \varepsilon \quad (\varepsilon > 0 \text{ depends on } f).$$

Proof. For one N, we have:

$$\int_X |f|d\mu = \int_{\{x:|f(x)|\leq N\}} |f|d\mu + \int_{\{x:|f(x)|>N\}} |f|d\mu < \infty.$$

We only need to prove that the second integral on the right-hand side is small, while N is large.

Let $E_n = \{x \in X : |f(x)| > n\}$. Then $E_n \subset E$, where $E = \{x \in X : |f(x)| = \infty\}$. Since $f \in L^1$, $\mu(E_n) = \mu(E) = 0$.

Next,

$$\lim_{n\to\infty}(\chi_{E_n}(x) - \chi_E(x)) = \lim_{n\to\infty} \chi_{E_n\setminus E}(x) = 0.$$

Claim: If $x \notin E$, then there exists $m \in \mathbb{N}$ such that $x \notin E_m$, for all $m > n$.

Hence, $f\chi_{E_n} \to f\chi_E = 0$ almost everywhere on X.

Since $|f\chi_{E_n}| \leq |f| \in L^1$, by Dominated Convergence Theorem, it follows that

$$\lim_{n\to\infty} \int_X f\chi_{E_n} d\mu = 0.$$

That is,

$$\lim_{\mu(E_n)=0} \int_{E_n} |f|d\mu = 0.$$

Hence, for all $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\mu(E_n) < \delta$ implies $\int_{E_n} |f|d\mu < \varepsilon$. In particular, $\mu(E_{n_0}) < \delta$ implies $\int_{E_{n_0}} |f|d\mu < \varepsilon$.

□

Theorem 4.21 (Bounded Convergence Theorem (BCT)). *Let (X, S, μ) be a finite measure space. If f_n and f are measurable functions on (X, S, μ) such that*

(i) $|f_n(x)| \leq M$, for all n and for all $x \in X$, and

(ii) $f_n \to f$ pointwise on X,

Then

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Proof. Since $M\mu(X) < \infty$, $\int_X |f_n| \leq M\mu(X) < \infty$.

Hence, f_n is dominated by $M \in L^1(X)$. By Dominated Convergence Theorem,

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Now, with the help of Monotone Convergence Theorem, Fatou's Lemma, Dominated Convergence Theorem and Bounded Convergence Theorem, we shall compare Riemann Integral with Lebesgue Integral.

□

4.7 Comparison of the Riemann and Lebesgue integrals

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$.

Let $\Delta x_i = x_i - x_{i-1}$, $M = \sup f(x)$, and

$$m_i = \inf_{x_{i-1}<x<x_i} f(x), \quad M_i = \sup_{x_{i-1}<x<x_i} f(x)$$

Write $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i, \quad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$.

Since f is bounded, there exists $m, M > 0$ such that $m \leq f(x) \leq M$, for all $x \in [a, b]$.

Hence,

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a) \tag{4.4}$$

The function f is said to be Riemann Integrable (for $f \in R[a, b]$) if

$$\inf_P U(P, f) = \sup_P L(P, f)$$

$$\text{implies } \inf_P \omega(P, f) = \inf_P \{U(P, f) - L(P, f)\} = 0$$

Hence, for each $\epsilon > 0$, there exists a partition P such that $\omega(P, f) < \epsilon$. Also for $\epsilon = 1/n, n \in \mathbb{N}$, there exists a partition P_n such that $\omega(P_n, f) < \frac{1}{n}$, for all $n \in \mathbb{N}$.

Hence $\lim_{n\to\infty} \omega(P_n, f) = 0$.

Theorem 4.22. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in R[a, b]$ if and only if there exists a sequence of partitions P_n such that $\lim_{n\to\infty} \omega(P_n, f) = 0$.*

Proof. We have already established the forward implication. Conversely, assume that there exists a sequence of partitions P_n such that $\omega(P_n, f) \to 0$. Given $\epsilon > 0$, choose n_0 such that $\omega(P_{n_0}, f) < \epsilon$. Then

$$0 \leq \inf_P \omega(P, f) \leq \omega(P_{n_0}, f) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\inf_P \omega(P, f) = 0$. Recall that $\omega(P, f) = U(P, f) - L(P, f)$ and $U(P, f) \geq L(P, f)$ for every partition P. Hence

$$\inf_P U(P, f) - \sup_P L(P, f) = 0,$$

so $\inf_P U(P, f) = \sup_P L(P, f)$, i.e. f is Riemann integrable on $[a, b]$.

□

Example 4.23. Let $f : [0, 1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & x = 1/2 \\ 0 & x \neq 1/2 \end{cases}$

Then f is bounded and for $P_n = \left\{\frac{i}{n} : i = 0, 1, 2, \dots, n\right\}$,

$$\omega(P_n, f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i < 2 \max_{q<i\leq n} \Delta x_i < 2\frac{1}{n} \to 0.$$

(Hints: $1/2 \in [2k-1, 2k]/[2k, 2k+1]$)

If $f : [a, b] \to \mathbb{R}$ is continuous then $f \in R[a, b]$.

Since f is continuous, it is uniformly continuous on $[a, b]$. For each $\epsilon > 0$, there exists $\delta > 0$ such that $|s - t| < \delta$ implies $|f(s) - f(t)| < \epsilon$.

Choose a partition P such that $\Delta x_i < \delta$. Then for any $s, t \in [x_{i-1}, x_i]$, $-\epsilon < f(s) - f(t) < \epsilon$

So, by taking sup and then inf we get:

$$-\epsilon < M_i - m_i < \epsilon, \quad i = 1, 2, \dots, n$$

Hence,

$$\omega(P, f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \leq \epsilon \sum_{i=1}^{n} \Delta x_i = \epsilon(b - a)$$

Notice that if $P_1 \subset P_2$, then $L(P_1, f) \geq L(P_2, f)$ and $U(P_1, f) \leq U(P_2, f)$. Hence,

$$\omega(P_1, f) \geq \omega(P_2, f)$$

Using this fact, it is enough to look at $\lim_{n\to\infty} \omega(P_n, f) = 0$ while P_n are an increasing sequence.

Theorem 4.24. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in R[a, b]$ if and only if there exists an increasing sequence of partitions P_n of $[a, b]$ such that $\lim_{n\to\infty} \omega(P_n, f) = 0$.*

Proof. Since $f \in R[a, b]$, by previous theorem, there exists a sequence of partitions P_n such that $\lim_{n\to\infty} \omega(P_n, f) = 0$.

Now, let $Q_1 = P_1$, $Q_n = P_1 \cup P_2 \cup \cdots \cup P_n$. Then $\omega(Q_n, f) \leq \omega(P_n, f) \to 0$.

Converse is obvious from previous theorem.

□

Example 4.25. Let $f : [0, 1] \to \mathbb{R}$ be such that $f(x) = 1/n$ if $x = 1/n$, 0 otherwise. Then $f \in R[a, b]$ and $\int_0^1 f(x)dx = 0$.

Let $\epsilon > 0$. Since $1/n \to 0$, there exists $n_0 \in \mathbb{N}$ such that $1/n_0 \in [0, \epsilon]$ and $1/n \in [0, \epsilon]$, for all $n \geq n_0$.

Hence, only finitely many y_n's are in $[\epsilon, 1]$.

See Figure 4.1.

We can cover the points $\{y_i, 1/n_0 - 1, \dots, 1\}$ by intervals $[x_i, x_{i+1}]$, $[x_3, x_4], \dots, [x_{m-1}, x_m]$ such that

$$\sum_{i=2}^{m} \Delta x_i < \epsilon$$

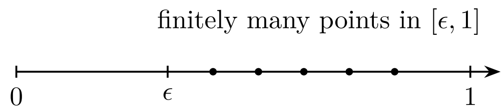


Figure 4.1: Only finitely many values of $1/n$ lie in the interval $[\epsilon, 1]$.

Then the partition $P = \{0, \epsilon, x_1, x_2, \ldots, x_m = 1\}$ as desired, and

$$\omega(P, f) = \sum_{i=0}^{m} (M_i - m_i)\Delta x_i = M_0 \Delta x_0 + \sum_{i=1}^{m} M_i \Delta x_i < 1 \cdot \epsilon + \epsilon = 2\epsilon \cdot 1$$

Hence, $f \in R[a, b]$ and $\int_0^1 f(x)dx = \sup_P L(P, f) = 0$.

Theorem 4.26. *Let $f \in R[a, b]$. Then $f \in L^1[a, b]$ and*

$$\int_{[a,b]} f(x)\, dm(x) = \int_a^b f(x)\, dx,$$

where the left-hand side denotes the Lebesgue integral and the right-hand side denotes the Riemann integral.

Proof. Let $I = [a, b]$. Since $f \in R[a, b]$, the function f is bounded on I, so there exists $K > 0$ such that $|f(x)| \le K$ for all $x \in I$. In particular, once measurability is noted (see the remark below), we will have $f \in L^1(I)$.

Fix $n \in \mathbb{N}$. Choose a partition

$$P_n : \; a = x_0 < x_1 < \cdots < x_{k_n} = b$$

such that

$$U(P_n, f) - L(P_n, f) < \frac{1}{n}.$$

Define the lower and upper step functions

$$s_n := \sum_{i=1}^{k_n} m_i \, \chi_{(x_{i-1}, x_i]}, \qquad m_i := \inf_{(x_{i-1}, x_i]} f,$$

and

$$t_n := \sum_{i=1}^{k_n} M_i \, \chi_{(x_{i-1}, x_i]}, \qquad M_i := \sup_{(x_{i-1}, x_i]} f.$$

Then s_n and t_n are measurable, $s_n \le f \le t_n$, and

$$\int_I s_n \, dm = L(P_n, f), \qquad \int_I t_n \, dm = U(P_n, f).$$

Consequently,

$$\int_I (t_n - s_n)\, dm = U(P_n, f) - L(P_n, f) < \frac{1}{n}.$$

By monotonicity of the Lebesgue integral and the bounds $s_n \le f \le t_n$, we obtain

$$L(P_n, f) = \int_I s_n\, dm \le \int_I f\, dm \le \int_I t_n\, dm = U(P_n, f).$$

Letting $n \to \infty$ and using the defining property of Riemann integrability (namely $U(P_n, f) \to \int_a^b f(x)\, dx$ and $L(P_n, f) \to \int_a^b f(x)\, dx$ along a suitable choice of partitions), the squeeze principle yields

$$\int_I f\, dm = \int_a^b f(x)\, dx.$$

Remark on measurability. It is a standard result (Lebesgue's criterion for Riemann integrability) that every Riemann integrable function on $[a, b]$ is Lebesgue measurable. Hence the Lebesgue integral $\int_I f\, dm$ is well-defined.

□

Remark 4.27. If we assign a norm on $R[a, b]$ through $\|f\|_1 = \int |f(x)| dx$. Then by the previous result,

$$(R[a, b], \|\cdot\|_1) \subset (L[a, b], \|\cdot\|_1)$$

The inclusion is proper: the function $f(x) = \frac{1}{\sqrt{x}}$ for $x \neq 0$ and $f(0) = 0$ on $[0, 1]$ is not Riemann integrable (it is unbounded near 0), but it is Lebesgue integrable.

For $n \in \mathbb{N}$, if we write $f_n = f\chi_{[1/n,1]}$, then f_n increases to f pointwise. Hence, by (*) MCT,

$$\int_{[0,1]} f\, dm = \lim \int_{[\frac{1}{n},1]} f\, dm = \lim \int_{1/n}^{1} f(x) dx = 2.$$

Thus, $f \in L^1([0, 1])$. In fact, the space $C_R([0, 1], \|\cdot\|_1)$ is an incomplete one, because for $m > n$,

$$\|f_n - f_m\|_1 = \int_{1/m}^{1/n} \frac{1}{\sqrt{x}} dx = 2\left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{n}}\right)$$

It implies $\{f_n\}$ is a Cauchy sequence in $R([0, 1])$, but $\lim f_n(x) = f(x),\ f \notin R([0, 1])$. However,

$$R([0, 1]) \subset L^1([0, 1]), \text{ that we see later.}$$

Remark 4.28. Observation made in $\mathbb{R}^n$ is wider and can be generalized to the following result.

Theorem 4.29. *Let $f : L^1([a, b]) \to \mathbb{R}$ be such that*

(i) $f \in R([a, b])$, for all $b > a$ and $b \in \mathbb{R}$.

(ii) $\int_{[a,b]} |f|\, dm \le M < \infty$, for all $b > a$

(M is independent of b)

Then $f \in L^1([a, \infty))$ and

$$\int_{[a,\infty)} f\, dm = \lim_{b\to\infty} \int_{[a,b]} f\, dm = \lim_{b\to\infty} \int_a^b f(x) dx.$$

Proof. Let $f_n = f\chi_{[a,n]}$. Then $f_n \to f$ pointwise on $I = [a, \infty)$. Notice that $\int_I |f_n| dm$ is an increasing sequence which is bounded above by M. Hence

$$\lim \int_I |f_n| dm \leq M.$$

Since $|f_n| \uparrow |f|$ pointwise, by MCT,

$$\int_I |f| dm = \lim \int_I |f_n| dm \leq M < \infty.$$

Hence, $f \in L^1(I)$. Moreover, $|f_n| < |f| \in L^1(I)$, by BCT

$$\int_I f\, dm = \lim_{n\to\infty} \int_I f_n\, dm = \lim_{n\to\infty} \int_{[a,n]} f\, dx.$$

□

Example 4.30. $f_n : \mathbb{R} \to \mathbb{R}$ by

(i) $f_n(t) = \frac{1}{n} e^{-t^2}$

(ii) $f_n(t) = \frac{1}{1+nt^2}$

(iii) $f_n(t) = e^{-\frac{t^2}{n}} S(0, n)$

(iv) $f_n(t) = c - \frac{t^2}{n}$

If $f_n \to f$, check for $f \in L^1(\mathbb{R}, M, m)$, and commutation of limit and integral.

Characterization of R-integrable functions.

Theorem 4.31. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in R([a, b])$ if and only if f is continuous on $[a, b]$ almost everywhere.*

Proof. Let $f \in R([a, b])$. Then there is a sequence of partitions P_n of $[a, b]$ such that

$$\varphi_n = \sum_{i=1}^{n} m_i \chi_{(x_{i-1},x_i]} \uparrow f \Delta m = \sum_{i=1}^{n} M_i \chi_{(x_{i-1},x_i]}$$

pointwise, almost everywhere on $[a, b]$ (see previous theorem). Consider

$$\varphi_n \leq f \leq \psi_n \quad \text{for all } n, \text{ for all } x_i \text{ provided on } A^c \subset [a, b] = I.$$

Then $m(A) = 0$. Note that partitioning bits of I could also be points of discontinuity. Hence, let $D = A \cup \left(\bigcup_{n=1}^{\infty} D_p^n\right)$; if $x \in I \setminus D$, then for each $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$|\varphi_n(x) - f(x)| < \epsilon, \quad |\psi_n(x) - f(x)| < \epsilon, \quad n \geq n_0.$$

Since φ_n and ψ_n are simple functions, $x \in (x_{i-1}, x_i] \subset P_n$ for some n, n_0 such that

$$x \in (x_{i-1}, x_i] \;\&\; M_i - f(x) < \epsilon \;\&\; f(x) - m_i < \epsilon.$$

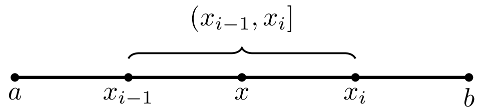


Figure 4.2: A typical atom $(x_{i-1}, x_i]$ of a partition used to compare f with upper and lower simple-function bounds.

See Figure 4.2.

For $y \in (x_{i-1}, x_i]$ we get

$$-\epsilon < m_i - f(x) < f(y) - f(x) < M_i - f(x) < \epsilon.$$

Hence, f is continuous and $x \in I \setminus D$, $m(D) = 0$.
Thus, f is continuous a.e. on I.

Conversely, suppose f is a.e. continuous on I. Let $x_t \in (a, b)$ be a point of continuity of f. Then for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(x_t - \epsilon) < f(y) < f(x_t + \epsilon) < f(x) + \epsilon \text{ on } |x - y| < \delta. \tag{4.5}$$

Suppose $\|P_n\| \to 0$, then for $\epsilon > 0$, there exists $N \in \mathbb{N}$

such that $\|P_n\| < \delta$, for all $n \geq N_o$.

$$a \;\mid\; x_{i-1} \;\mid\; x \;\mid\; x_i \;\mid\; x_{i+1} \;\mid\; b$$

From (4.5),

$$f(x) - \epsilon < (\inf f(y)), \quad y \in [x_{i-1}, x_i] \leq f(x) + \epsilon$$

Therefore,

$$f(x) - \epsilon \leq m_i \leq f(x) + \epsilon$$

Since $x \in (x_{i-1}, x_i)$, it follows that

$$f(x) - \epsilon \leq \varphi_n(x) \leq f(x) + \epsilon, \quad \text{for all } n, \eta, \eta_0.$$

implies $\varphi_n(x) \to f(x)$ for all $x \in I \setminus D$.

Similarly, $\psi_n(x) \to f(x)$.

But $m_n \leq \varphi_n(x) \leq f(x) \leq \psi_n(x) \leq M$, then by BCT, we get

$$\int_I f\,dm = \lim \int_I \varphi_n\,dm = \lim U(P_n, f)$$

$$\int_I f\,dm = \lim \int_I \psi_n\,dm = \lim L(P_n, f)$$

Hence, $f \in R[a,b]$.

$\square$

Remark 4.32. We have observed that

$$C[a,b] \subset R[a,b] \subset L^1[a,b]$$

However, we will later show that $C[a,b] = L^1[a,b]$ and hence $R[a,b] = L^1[a,b]$. Thus, a Lebesgue category function is limit of R-int. functions on $[a,b]$.

4.8 Improper Riemann integration

Let $f : [a,\infty) \to \mathbb{R}$ be such that:

1. $f \in R[a,b]$ for all $b > a$,
2. $\lim_{b\to\infty} \int_a^b f(x)\,dx$ exists (finite).

Then we say that f is improper R-integrable, and its improper integral

$$\int_a^\infty f\,dx = \lim_{b\to\infty} \int_a^b f\,dx$$

Remark 4.33. An improper Riemann-integrable function need not be Lebesgue integrable.

Example 4.34. Consider

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^1 \frac{\sin x}{x} dx + \int_1^\infty \frac{\sin x}{x} dx$$

Since $\frac{\sin x}{x}$ is bounded on $[0,1]$, if we write

$$g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

then $g \in R[0,1]$.

For $a > 1$,

$$\int_a^q \frac{\sin x}{x} dx = \left[\frac{-\cos x}{x}\right]_a^q - \int_a^q \frac{\cos x}{x^2} dx$$

$$= \cos a - \cos q - \int_a^q \frac{\cos x}{x^2} dx$$

Thus,

$$\lim_{q\to\infty}\int_a^q \frac{\sin x}{x}dx = \cos a - 0 - \lim_{q\to\infty}\int_a^q \frac{\cos x}{x^2}dx$$

Hence, $f \in IR[0,\infty)$. Moreover,

$$\int_{[0,\infty)} |f|\, dm = \sum_{n=1}^{\infty}\int_{(n-1)\pi}^{n\pi}\left|\frac{\sin x}{x}\right| dx = \sum_{n=1}^{\infty}\frac{1}{n\pi}\int_0^{\pi}|\sin x|dx = \sum_{n=1}^{\infty}\frac{2}{n\pi}$$

(by Beppo-Levi Theorem).

Hence, $f \notin L^1[0,\infty)$.

Example 4.35. Show that $\sin x$ on $[0,\infty)$ (or $\mathbb{R}$) is not improperly Riemann-integrable.

Theorem 4.36. *Let $f : [a,\infty) \to \mathbb{R}$ be such that:*

(i) $f \in R[a,b]$ for all $b > a$,

(ii) $\int_a^b |f(x)|dx \le M$, for all $b > a$ (M is independent of b),

then both f and $|f|$ are improper R-integrable on (a,∞) and $f \in L^1(a,\infty)$ with

$$\int f\, dm = \int_a^{\infty} f(x)dx$$

Proof. Since $\int_a^n |f(x)|dx$ is an increasing sequence and bounded above,

$$\lim_{n\to\infty}\int_a^n |f(x)|dx < \infty,$$

hence $f \in IR[a,\infty)$.

Now, $||f(x)| - f(x)| \le 2|f(x)|$,

$$\int_a^b ||f(x)| - f(x)|dx \le 2\int_a^b |f(x)|dx \le 2M$$

By previous case, $|f| - f \in IR[a,\infty)$. Hence $f = (|f| - (|f| - f))$.

∎

4.9 Further Results

If $f \in R[a,b]$, then $f \in L^1[a,b]$ for all $b > a$, and hence $\int_{[a,b]} |f|\, dm \le M$ for all $b > a$. By previous theorem on page 129, it follows that $f \in L^1([a,\infty))$, and

$$\int_{[a,\infty)} f\, dm = \lim_{b\to\infty}\int_{[a,b]} f\, dm = \lim_{b\to\infty}\int_a^b f(x)dx = \int_a^{\infty} f(x)dx$$

Example 4.37. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1+x^2}$.

Then

$$\int_a^b f(x)dx = \tan^{-1} b - \tan^{-1} a \le \pi, \quad \text{for all } a, b \in \mathbb{R},$$

So $f \in L^1(\mathbb{R})$ and

$$\int f\,dm = \lim_{a\to-\infty}\lim_{b\to\infty}\int_a^b f(x)dx = \pi$$

Theorem 4.38. *Let $f:[0,a]\to\mathbb{R}$ be such that:*

(i) $f \in L^1[\epsilon,a]$ *for all* $\epsilon > 0$,

(ii) $\int_{[\epsilon,a]}|f|\,dm \le M$ *for all* $\epsilon > 0$,

then $f \in L^1[0,a]$ *and*

$$\int_{[0,a]} f\,dm = \lim_{\epsilon\to 0}\int_{[\epsilon,a]} f\,dm$$

Proof. Let $f_n = R[\frac{1}{n},a]f$. Then $f_n \to f$ pointwise.

$|f_n| \le |f|$ pointwise. By Monotone Convergence Theorem,

$$\int_{[0,a]}|f|\,dm = \lim_{n\to\infty}\int_{[\frac{1}{n},a]}|f|\,dm < \infty$$

Hence $f \in L^1[0,a]$. Now $|f_n| \le |f| \in L^1[0,a]$ and $f_n \to f$ pointwise, by Dominated Convergence Theorem,

$$\int_{[0,a]} f\,dm = \lim_{n\to\infty}\int_{[\frac{1}{n},a]} f$$

□

Theorem 4.39. *Let $f:[a,\infty)\to\mathbb{R}$ be such that $f \in R([a,b])$ for every $b > a$. Then $f \in L^1([a,\infty))$ if and only if $|f|$ is improperly Riemann integrable on $[a,\infty)$.*

Proof. Let $f \in L^1([a,\omega))$. Then $f_n = \chi_{[a,n)}f$ converges pointwise to f, and $|f_n| < |f| \in L^1$. By Dominated Convergence Theorem,

$$\infty > \int|f|\,dm = \lim\int|f_n|\,dm = \lim\int_a^n|f(x)|\,dx = \int_a^\omega|f(x)|\,dx.$$

i.e. $f \in R[a,\omega)$.

Conversely, suppose $|f| \in \mathbb{R}[a,\omega)$. Then for $g_n = \chi_{[a,n)}|f|$, $g_n \uparrow |f|$. By Monotone Convergence Theorem,

$$\int|f|\,dm = \lim\int|f_n|\,dm = \lim\int_a^n|f(x)|\,dx = \int_a^\omega|f(x)|\,dx.$$

Hence $f \in L^1([a,\omega))$.

□

4.10 Absolute continuity and the Lebesgue fundamental theorem of calculus

The fundamental theorem of calculus admits a sharp formulation in the Lebesgue setting once one replaces classical differentiability hypotheses by *absolute continuity.* We work on an interval $[a,b] \subset \mathbb{R}$ and write m for Lebesgue measure.

4.11 Absolute continuity on an interval

Definition 4.40. A function $F : [a,b] \to \mathbb{R}$ is said to be *absolutely continuous* on $[a,b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite collection of pairwise disjoint open intervals $\{(x_k, y_k)\}_{k=1}^N \subset (a,b)$,

$$\sum_{k=1}^{N}(y_k - x_k) < \delta \quad \text{implies} \quad \sum_{k=1}^{N}|F(y_k) - F(x_k)| < \varepsilon.$$

We denote the class of absolutely continuous functions on $[a,b]$ by $\mathrm{AC}[a,b]$.

Remark 4.41. Every $F \in \mathrm{AC}[a,b]$ is uniformly continuous, hence continuous. Moreover, $F \in \mathrm{AC}[a,b]$ is of bounded variation and is differentiable m-almost everywhere on (a,b).

4.12 The Lebesgue differentiation theorem

For $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ we define the (centered) Hardy–Littlewood maximal function

$$Mf(x) := \sup_{r>0} \frac{1}{2r}\int_{x-r}^{x+r} |f(t)|\, dt.$$

Lemma 4.42 (Vitali covering lemma for intervals)**.** *Let $\mathcal{I}$ be a family of bounded open intervals in $\mathbb{R}$. Then there exists a countable pairwise disjoint subfamily $\{I_j\}_{j\geq 1} \subset \mathcal{I}$ such that*

$$\bigcup_{I\in\mathcal{I}} I \subset \bigcup_{j\geq 1} 5I_j,$$

where $5I_j$ denotes the interval with the same center as I_j and five times its length.

Proof. This is a standard one-dimensional form of the Vitali selection argument. Choose $I_1 \in \mathcal{I}$ with maximal length. Having chosen $I_1, \ldots, I_n$, choose $I_{n+1} \in \mathcal{I}$ with maximal length among those intervals disjoint from $I_1, \ldots, I_n$. The resulting family is pairwise disjoint. If an interval $I \in \mathcal{I}$ meets some I_j with $|I| \leq |I_j|$, then $I \subset 5I_j$. If I meets none of the chosen intervals, then it would have been eligible at some stage and hence intersect some selected I_j of length at least $|I|$. Therefore every $I \in \mathcal{I}$ is contained in $\bigcup_j 5I_j$.

□

Theorem 4.43 (Hardy–Littlewood maximal inequality, weak $(1,1)$)**.** *Let $f \in L^1(\mathbb{R})$ and $\alpha > 0$. Then*

$$m(\{x \in \mathbb{R} : Mf(x) > \alpha\}) \le \frac{2}{\alpha} \|f\|_{L^1(\mathbb{R})}.$$

Proof. Let $E_\alpha := \{x : Mf(x) > \alpha\}$. For each $x \in E_\alpha$ choose $r_x > 0$ such that

$$\frac{1}{2r_x} \int_{x-r_x}^{x+r_x} |f(t)|\, dt > \alpha,$$

and set $I_x := (x - r_x, x + r_x)$. By Lemma 4.42 we can select pairwise disjoint intervals $\{I_{x_j}\}_{j\ge1}$ such that $E_\alpha \subset \bigcup_j 5I_{x_j}$. Hence,

$$m(E_\alpha) \le \sum_j m(5I_{x_j}) = 5 \sum_j m(I_{x_j}).$$

On the other hand, by the choice of I_{x_j},

$$\alpha\, m(I_{x_j}) < \int_{I_{x_j}} |f(t)|\, dt.$$

Summing over j and using disjointness gives

$$\alpha \sum_j m(I_{x_j}) < \int_{\bigcup_j I_{x_j}} |f(t)|\, dt \le \|f\|_{L^1(\mathbb{R})}.$$

Combining the last two estimates yields $m(E_\alpha) \le \frac{5}{\alpha}\|f\|_1$. A minor refinement of the covering argument (working with a maximal disjoint subfamily) improves the constant to 2; the precise constant is immaterial for our purposes.

□

Theorem 4.44 (Lebesgue differentiation theorem)**.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then for m-almost every $x \in \mathbb{R}$,*

$$\lim_{r\downarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t) - f(x)|\, dt = 0. \tag{4.6}$$

Equivalently, for m-almost every x,

$$\lim_{r\downarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} f(t)\, dt = f(x).$$

Proof. By localization it suffices to assume $f \in L^1(\mathbb{R})$. Fix $\eta > 0$. Choose $g \in C_c(\mathbb{R})$ such that $\|f - g\|_{L^1(\mathbb{R})} < \eta$. For each x and $r > 0$ we have

$$\frac{1}{2r} \int_{x-r}^{x+r} |f(t) - f(x)| \le \frac{1}{2r} \int_{x-r}^{x+r} |f(t) - g(t)|\, dt + \frac{1}{2r} \int_{x-r}^{x+r} |g(t) - g(x)|\, dt + |g(x) - f(x)|.$$

Since g is continuous, the middle term tends to 0 as $r \downarrow 0$ for every x. Thus the main task is to control the first and the last term. Define $h := f - g \in L^1(\mathbb{R})$. Then

$$\sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |h(t)|\, dt = Mh(x).$$

By Theorem 4.43, for any $\alpha > 0$,

$$m(\{x : Mh(x) > \alpha\}) \le \frac{C}{\alpha}\|h\|_1 \le \frac{C\eta}{\alpha}.$$

Also, by Markov's inequality,

$$m(\{x : |h(x)| > \alpha\}) \le \frac{\eta}{\alpha}.$$

Letting $\eta \downarrow 0$ along a sequence and applying the Borel–Cantelli lemma yields a full-measure set on which both $Mh(x)$ and $h(x)$ are arbitrarily small. On that set, taking $r \downarrow 0$ gives (4.6).

□

4.13 Lebesgue FTC

Theorem 4.45 (Lebesgue fundamental theorem of calculus)**.** *Let $f \in L^1([a,b])$ and define*

$$F(x) := \int_a^x f(t)\,dt, \qquad x \in [a,b]. \tag{4.7}$$

Then $F \in \mathrm{AC}[a,b]$ and $F'(x) = f(x)$ for m-almost every $x \in (a,b)$.

Conversely, if $F \in \mathrm{AC}[a,b]$, then $F' \in L^1([a,b])$ and

$$F(x) = F(a) + \int_a^x F'(t)\,dt, \qquad x \in [a,b]. \tag{4.8}$$

Proof. Assume $f \in L^1([a,b])$ and define F by (4.7). To see $F \in \mathrm{AC}[a,b]$, fix $\varepsilon > 0$. Choose $\delta > 0$ such that $m(A) < \delta$ implies $\int_A |f|\,dm < \varepsilon$; this is possible by absolute continuity of the integral for L^1-functions. If $\{(x_k, y_k)\}$ are pairwise disjoint with $\sum_k (y_k - x_k) < \delta$, then

$$\sum_k |F(y_k) - F(x_k)| = \sum_k \left| \int_{x_k}^{y_k} f(t)\,dt \right| \le \int_{\bigcup_k (x_k, y_k)} |f(t)|\,dt < \varepsilon.$$

Hence $F \in \mathrm{AC}[a,b]$.

For the derivative, fix $x \in (a,b)$ such that Theorem 4.44 holds for the function $f\chi_{[a,b]}$ at x. Then

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

for h small enough with $x + h \in [a,b]$. Applying the differentiation theorem to the averages over $(x, x+h)$ as $h \to 0$ yields $F'(x) = f(x)$ for almost every x.

Conversely, let $F \in \mathrm{AC}[a,b]$. By Remark 4.41, F is differentiable almost everywhere. Set $g := F'$ on the set where the derivative exists and $g := 0$ elsewhere. A standard covering argument (or the Lebesgue differentiation theorem applied to g) shows $g \in L^1([a,b])$ and that F admits the representation (4.8).

□

Chapter 5

L^p spaces

Integration naturally leads to function spaces measured by the size of $|f|^p$. This chapter studies the spaces $L^p(X,\Sigma,\mu)$ for $1 \le p \le \infty$, emphasizing the functional-analytic structure that becomes central in advanced analysis. We establish the basic inequalities (Hölder and Minkowski), interpret L^p as a normed space modulo equality almost everywhere, and discuss completeness and the role of essential supremum in L^∞. Along the way we connect analytic estimates with measure-theoretic control of level sets, and we build intuition for the different modes of convergence that arise in practice (pointwise, almost everywhere, in measure, and in L^p).

5.1 Basic properties and inequalities

Let (X,Σ,μ) be a measure space and let $1 \le p < \infty$. We define

$$L^p(X,\Sigma,\mu) := \left\{ f : X \to \mathbb{R} \text{ measurable} \;\middle|\; \int_X |f|^p \, d\mu < \infty \right\}.$$

As usual, functions that agree almost everywhere are identified. With this convention $L^p(X,\Sigma,\mu)$ is a vector space. Indeed, if $f, g \in L^p$, then

$$|f+g|^p \le (|f|+|g|)^p \le 2^{p-1}(|f|^p + |g|^p),$$

so $f+g \in L^p$, and similarly $\alpha f \in L^p$ for every scalar α.

On an infinite-measure space there is, in general, no inclusion relation between L^p and L^q when $p \ne q$.

Example 5.1. On $(0,1)$, the function $f(x) = x^{-1/2}$ belongs to $L^1(0,1)$ but not to $L^2(0,1)$. On $\mathbb{R}$, the function $g(x) = \frac{1}{1+|x|}$ belongs to $L^2(\mathbb{R})$ but not to $L^1(\mathbb{R})$.

The second claim follows from

$$\int_{\mathbb{R}} \frac{1}{(1+|x|)^2} \, dx < \infty, \qquad \int_{\mathbb{R}} \frac{1}{1+|x|} \, dx = \infty.$$

Thus neither $L^1(\mathbb{R}) \subset L^2(\mathbb{R})$ nor $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$.

A different phenomenon occurs on finite-measure spaces.

Proposition 5.2. *Let (X, Σ, μ) be a finite measure space, and let $1 \le p < q \le \infty$. Then $L^q(X, \Sigma, \mu) \subseteq L^p(X, \Sigma, \mu)$. More precisely, if $q < \infty$, then*

$$\|f\|_p \le \mu(X)^{\frac{1}{p}-\frac{1}{q}} \|f\|_q, \qquad f \in L^q(X, \Sigma, \mu),$$

and if $q = \infty$, then

$$\|f\|_p \le \mu(X)^{1/p} \|f\|_\infty.$$

Proof. If $q = \infty$, then $|f| \le \|f\|_\infty$ almost everywhere, so

$$\int_X |f|^p \, d\mu \le \mu(X) \|f\|_\infty^p.$$

Now assume $q < \infty$ and set $r = q/p > 1$. Let r' be the conjugate exponent of r. Applying Hölder's inequality to $|f|^p$ and 1, we obtain

$$\int_X |f|^p \, d\mu \le \left(\int_X |f|^{pr} \, d\mu \right)^{1/r} \left(\int_X 1^{r'} \, d\mu \right)^{1/r'} = \left(\int_X |f|^q \, d\mu \right)^{p/q} \mu(X)^{1-p/q}.$$

Taking pth roots yields the estimate. ☐

The fundamental inequalities for L^p spaces begin with Young's inequality.

Theorem 5.3 (Young's inequality)**.** *Let $1 < p < \infty$ and let q be the conjugate exponent, so that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $a, b \ge 0$,*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Equality holds if and only if $a^p = b^q$.

Proof. Consider the convex function $\phi(t) = t^p/p$ on $[0, \infty)$. Its Legendre transform is $\phi^*(s) = s^q/q$. The general inequality $st \le \phi(t) + \phi^*(s)$ yields

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Equality holds exactly when $b = \phi'(a) = a^{p-1}$, equivalently when $a^p = b^q$. ☐

Theorem 5.4 (Hölder's inequality)**.** *Let $1 < p < \infty$ and let q be the conjugate exponent. If $f \in L^p(X, \Sigma, \mu)$ and $g \in L^q(X, \Sigma, \mu)$, then $fg \in L^1(X, \Sigma, \mu)$ and*

$$\int_X |fg| \, d\mu \le \|f\|_p \, \|g\|_q.$$

For $p = 1$ and $q = \infty$, one has

$$\int_X |fg| \, d\mu \le \|f\|_1 \, \|g\|_\infty.$$

Proof. The case $p = 1$, $q = \infty$ is immediate from $|fg| \le \|g\|_\infty |f|$ almost everywhere. Assume now that $1 < p < \infty$ and that f and g are not identically zero. Apply Young's inequality pointwise to

$$a = \frac{|f(x)|}{\|f\|_p}, \qquad b = \frac{|g(x)|}{\|g\|_q},$$

to obtain

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \le \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}.$$

Integrating over X gives

$$\frac{1}{\|f\|_p\|g\|_q} \int_X |fg| \, d\mu \le \frac{1}{p} + \frac{1}{q} = 1.$$

Hence $\int_X |fg| \, d\mu \le \|f\|_p \|g\|_q$. □

Theorem 5.5 (Minkowski's inequality)**.** *Let* $1 \le p \le \infty$. *If* $f, g \in L^p(X, \Sigma, \mu)$, *then* $f + g \in L^p(X, \Sigma, \mu)$ *and*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Proof. For $p = \infty$ the inequality follows from

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty$$

almost everywhere. For $p = 1$, it is just the triangle inequality for the integral. Assume $1 < p < \infty$ and let q be the conjugate exponent. Then

$$\|f + g\|_p^p = \int_X |f + g| \, |f + g|^{p-1} \, d\mu \le \int_X |f| \, |f + g|^{p-1} \, d\mu + \int_X |g| \, |f + g|^{p-1} \, d\mu.$$

By Hölder's inequality,

$$\|f + g\|_p^p \le \|f\|_p \, \||f + g|^{p-1}\|_q + \|g\|_p \, \||f + g|^{p-1}\|_q.$$

Since $(p - 1)q = p$, we have

$$\||f + g|^{p-1}\|_q = \left(\int_X |f + g|^{(p-1)q} \, d\mu \right)^{1/q} = \left(\int_X |f + g|^p \, d\mu \right)^{1/q} = \|f + g\|_p^{p/q}.$$

Therefore

$$\|f + g\|_p^p \le (\|f\|_p + \|g\|_p)\|f + g\|_p^{p/q}.$$

If $f + g = 0$ almost everywhere there is nothing to prove; otherwise divide by $\|f + g\|_p^{p/q}$ to obtain the claim. □

The next theorem shows that L^p is not merely a normed space but a Banach space.

Theorem 5.6. *For every* $1 \le p \le \infty$, *the space* $L^p(X, \Sigma, \mu)$ *is complete. Moreover, if* $1 \le p < \infty$ *and* $f_n \to f$ *in* L^p, *then some subsequence* (f_{n_k}) *converges to* f *almost everywhere.*

Proof. We first treat the case $1 \le p < \infty$. Let (f_n) be a Cauchy sequence in L^p. Choose a subsequence (f_{n_k}) such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p < 2^{-k} \qquad (k \geq 1).$$

Set

$$g(x) := |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

By Minkowski's inequality, the partial sums of g are uniformly bounded in L^p, so $g \in L^p(X)$ by monotone convergence. In particular, $g(x) < \infty$ for almost every x, and therefore the numerical series

$$f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

converges for almost every x. Denote its sum by $f(x)$. Then $f_{n_k}(x) \to f(x)$ almost everywhere and $|f(x)| \leq g(x)$ almost everywhere.

To show that $f_{n_k} \to f$ in L^p, note that

$$|f - f_{n_k}| \leq \sum_{j=k}^{\infty} |f_{n_{j+1}} - f_{n_j}| \leq g + |f_{n_1}|.$$

Since the right-hand side belongs to L^p and the left-hand side tends to 0 almost everywhere, dominated convergence gives $\|f - f_{n_k}\|_p \to 0$. Because (f_n) is Cauchy, it follows that $f_n \to f$ in L^p.

Now assume $f_n \to f$ in L^p. By passing inductively to a subsequence we may arrange that

$$\|f_{n_{k+1}} - f_{n_k}\|_p < 2^{-k}.$$

The argument above then shows that (f_{n_k}) converges almost everywhere to some measurable function g. Since $f_{n_k} \to f$ in L^p as well, uniqueness of the L^p limit implies $g = f$ almost everywhere.

For $p = \infty$, let (f_n) be Cauchy in L^∞. Choose representatives so that

$$\|f_n - f_m\|_\infty \to 0.$$

Then $(f_n(x))$ is Cauchy for almost every x, hence converges almost everywhere to some measurable function f. The uniform essential estimate passes to the limit, and one checks directly that $\|f_n - f\|_\infty \to 0$. □

Lemma 5.7. *For $1 \leq p < \infty$, the simple integrable functions are dense in $L^p(X, \Sigma, \mu)$.*

Proof. Let $f \in L^p(X, \Sigma, \mu)$. Since $|f|^p \in L^1(X, \Sigma, \mu)$, there exists a sequence of simple measurable functions (φ_n) such that $\varphi_n \to f$ almost everywhere and $|\varphi_n| \leq |f|$ almost everywhere. Then

$$|f - \varphi_n|^p \leq 2^p |f|^p,$$

and the right-hand side is integrable. By dominated convergence,

$$\|f - \varphi_n\|_p^p = \int_X |f - \varphi_n|^p \, d\mu \to 0.$$

Hence simple functions are dense in L^p. □

This density result is often the first step in approximating L^p functions by more regular classes of functions.

Definition 5.8 (Support)**.** Let X be a topological space and let $f : X \to \mathbb{R}$. The *support* of f is the closure of the set on which f is nonzero:

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

If $\operatorname{supp}(f)$ is compact, then f is said to be *compactly supported*. We write $C_c(X)$ for the space of continuous compactly supported functions on X.

On a locally compact Hausdorff space, Urysohn's lemma guarantees a large supply of such functions: if $K \subset O$ with K compact and O open, then there exists $u \in C_c(X)$ such that $0 \leq u \leq 1$, $u = 1$ on K, and $u = 0$ on O^c.

Theorem 5.9. *For $1 \leq p < \infty$, the space $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R}, \mathcal{M}, m)$.*

Proof. Let $f \in L^p(\mathbb{R})$ and let $\varepsilon > 0$. By Theorem 5.7, choose a simple function

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

with $m(E_i) < \infty$ and $\|f - \varphi\|_p < \varepsilon/2$. For each i, regularity of Lebesgue measure gives compact and open sets $K_i \subset E_i \subset O_i$ such that

$$m(O_i \setminus K_i) < \left(\frac{\varepsilon}{2n \max(1, \sum_{j=1}^{n} |\alpha_j|)} \right)^p.$$

By Urysohn's lemma, there exists $g_i \in C_c(\mathbb{R})$ such that $0 \leq g_i \leq 1$, $g_i = 1$ on K_i, and $g_i = 0$ on O_i^c. Put

$$g := \sum_{i=1}^{n} \alpha_i g_i \in C_c(\mathbb{R}).$$

Since $|\chi_{E_i} - g_i| \leq 1$ and the difference vanishes outside $O_i \setminus K_i$, we have

$$\|\chi_{E_i} - g_i\|_p \leq m(O_i \setminus K_i)^{1/p}.$$

Therefore

$$\|\varphi - g\|_p \leq \sum_{i=1}^{n} |\alpha_i| \, \|\chi_{E_i} - g_i\|_p < \frac{\varepsilon}{2}.$$

Hence

$$\|f - g\|_p \leq \|f - \varphi\|_p + \|\varphi - g\|_p < \varepsilon,$$

which proves the density. ☐

Theorem 5.10. *Let $(X, \mathscr{S}, \mu)$ be a regular measure space on a locally compact Hausdorff space X. Then $C_c(X)$ is dense in $L^p(X, \mathscr{S}, \mu)$ for every $1 \le p < \infty$.*

Proof. The argument is the same as in the Lebesgue-measure case. First approximate an L^p function by simple functions. Then use inner and outer regularity of μ to approximate each measurable set of finite measure by a compact set from inside and an open set from outside, and finally apply Urysohn's lemma to replace characteristic functions by continuous compactly supported functions. ☐

A useful comparison with classical integration is the following.

Theorem 5.11. *Let $f : [a, \infty) \to \mathbb{R}$ be such that $f \in R([a, b])$ for every $b > a$. Then $f \in L^1([a, \infty))$ if and only if $|f|$ is improperly Riemann integrable on $[a, \infty)$.*

Proof. Truncate f on $[a, n]$, compare the Lebesgue and Riemann integrals on each finite interval, and pass to the limit using monotone convergence in one direction and dominated convergence in the other. ☐

5.2 Functions vanishing at infinity

A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to *vanish at infinity* if

$$\lim_{|x| \to \infty} f(x) = 0.$$

Equivalently, for every $\varepsilon > 0$ there exists $R > 0$ such that $|f(x)| < \varepsilon$ whenever $|x| > R$. We write

$$C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) : \lim_{|x| \to \infty} f(x) = 0\}.$$

Equipped with the supremum norm

$$\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|,$$

this is a normed vector space.

Proposition 5.12. *The space $(C_0(\mathbb{R}), \| \cdot \|_\infty)$ is complete.*

Proof. Let (f_n) be a Cauchy sequence in $C_0(\mathbb{R})$ with respect to $\| \cdot \|_\infty$. Then (f_n) converges uniformly on $\mathbb{R}$ to some bounded continuous function f. It remains to show that f also vanishes at infinity.

Fix $\varepsilon > 0$. Choose n_0 such that $\|f - f_{n_0}\|_\infty < \varepsilon/2$. Since $f_{n_0} \in C_0(\mathbb{R})$, there exists $R > 0$ such that $|f_{n_0}(x)| < \varepsilon/2$ for all $|x| > R$. Hence, for $|x| > R$,

$$|f(x)| \le |f(x) - f_{n_0}(x)| + |f_{n_0}(x)| < \varepsilon.$$

Thus $f \in C_0(\mathbb{R})$. □

Proposition 5.13. *The space $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ with respect to the supremum norm.*

Proof. Let $f \in C_0(\mathbb{R})$ and let $\varepsilon > 0$. Choose $R > 0$ so that $|f(x)| < \varepsilon$ whenever $|x| > R$. Let $K = [-R, R]$ and choose an open set O containing K with compact closure. By Urysohn's lemma, there exists $u \in C_c(\mathbb{R})$ such that $0 \leq u \leq 1$, $u = 1$ on K, and $u = 0$ on O^c. Define $h := fu$. Then $h \in C_c(\mathbb{R})$.

If $x \in K$, then $h(x) = f(x)$. If $x \notin K$, then

$$|f(x) - h(x)| = |f(x)|(1 - u(x)) \leq |f(x)| < \varepsilon.$$

Therefore $\|f - h\|_\infty < \varepsilon$, proving the claim. □

The space $C_0(\mathbb{R})$ is important because it sits naturally between $C_c(\mathbb{R})$ and $L^\infty(\mathbb{R})$: compactly supported continuous functions are dense in $C_0(\mathbb{R})$, and every function in $C_0(\mathbb{R})$ is bounded.

5.3 $L^\infty(X, S, \mu)$

Let (X, Σ, μ) be a measure space. A measurable function $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is called *essentially bounded* if there exists $M \geq 0$ such that

$$\mu(\{x \in X : |f(x)| > M\}) = 0.$$

In other words, $|f(x)| \leq M$ for almost every $x \in X$. The *essential supremum* of f is defined by

$$\|f\|_\infty := \inf\{M \geq 0 : |f(x)| \leq M \text{ for almost every } x \in X\} = \operatorname*{ess\,sup}_{x \in X} |f(x)|.$$

If no such M exists, we write $\|f\|_\infty = \infty$.

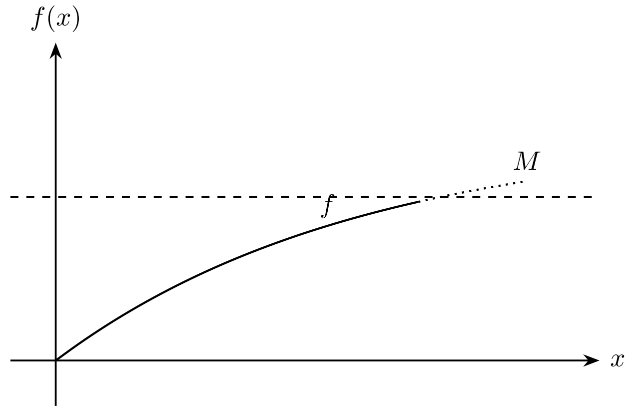
By definition, one always has

$$\mu(\{x \in X : |f(x)| > \|f\|_\infty\}) = 0.$$

Moreover,

$$\|f\|_\infty \leq \sup_{x \in X} |f(x)|,$$

but equality need not hold.



Example 5.14. Let $f = \chi_{\mathbb{Q}}$ on $\mathbb{R}$ with Lebesgue measure. Then $f = 0$ almost everywhere, so

$$\|f\|_\infty = 0,$$

whereas $\sup_{x\in\mathbb{R}} |f(x)| = 1$. Thus the essential supremum and the pointwise supremum can be different.

On the infinite-measure space $(\mathbb{R}, \mathcal{M}, m)$, the constant function $f \equiv 1$ belongs to $L^\infty(\mathbb{R})$ but does not belong to $L^p(\mathbb{R})$ for any $1 \le p < \infty$. Hence, in general, $L^\infty(X, \Sigma, \mu) \not\subseteq L^p(X, \Sigma, \mu)$.

If $\mu(X) < \infty$ and $f \in L^\infty(X, \Sigma, \mu)$, then

$$\int_X |f|^p \, d\mu \le \mu(X) \, \|f\|_\infty^p,$$

so

$$\|f\|_p \le \mu(X)^{1/p} \, \|f\|_\infty.$$

Therefore, on a finite-measure space one has the continuous inclusion

$$L^\infty(X, \Sigma, \mu) \subseteq L^p(X, \Sigma, \mu), \qquad 1 \le p < \infty.$$

Proposition 5.15. *If $f \in L^\infty(X, \Sigma, \mu)$ and $0 < \alpha < \|f\|_\infty$, then*

$$\mu(\{x \in X : |f(x)| > \alpha\}) > 0.$$

Proof. If this measure were 0, then $|f| \le \alpha$ almost everywhere, which would imply $\|f\|_\infty \le \alpha$, a contradiction. □

Theorem 5.16. $L^\infty(X, \Sigma, \mu)$ *is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $L^\infty(X, \Sigma, \mu)$. Passing to a subsequence if necessary, we may assume

$$\|f_{n_{k+1}} - f_{n_k}\|_\infty < 2^{-k}, \qquad k \ge 1.$$

For each k, choose a null set N_k such that

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| < 2^{-k}, \qquad x \in X \setminus N_k.$$

Set $N := \bigcup_{k\geq 1} N_k$. Then $\mu(N) = 0$, and for every $x \in X \setminus N$ the series

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

converges. Hence $(f_{n_k}(x))$ is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$) for each $x \in X \setminus N$.

Define

$$f(x) := \lim_{k\to\infty} f_{n_k}(x), \qquad x \in X \setminus N,$$

and set $f(x) = 0$ on N. Since f is the almost-everywhere limit of measurable functions, f is measurable.

For $m > k$ and $x \in X \setminus N$,

$$|f_{n_m}(x) - f_{n_k}(x)| \leq \sum_{j=k}^{m-1} 2^{-j},$$

and letting $m \to \infty$ gives

$$|f(x) - f_{n_k}(x)| \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1}$$

for almost every x. Therefore,

$$\|f - f_{n_k}\|_\infty \leq 2^{-k+1} \to 0.$$

Since the original sequence (f_n) is Cauchy and a subsequence converges to f in L^∞, the whole sequence converges to f in L^∞. Thus $L^\infty(X, \Sigma, \mu)$ is complete.

□

Theorem 5.17. *The bounded measurable simple functions are dense in* $L^\infty(X, \Sigma, \mu)$.

Proof. Let $f \in L^\infty(X, \Sigma, \mu)$ and fix $\varepsilon > 0$. Choose $M > \|f\|_\infty$ such that $|f| \leq M$ almost everywhere. Partition the interval $[-M, M]$ into finitely many subintervals of length at most ε, and on each subinterval choose one representative value. Define a simple function φ by assigning to $f(x)$ the representative corresponding to the interval that contains $f(x)$ whenever $|f(x)| \leq M$, and put $\varphi(x) = 0$ on the null set where $|f(x)| > M$. Then φ is measurable, bounded, and satisfies

$$|f(x) - \varphi(x)| \leq \varepsilon$$

for almost every x. Hence $\|f - \varphi\|_\infty \leq \varepsilon$. □

Proposition 5.18. *If* $\mu(X) < \infty$ *and* $f \in L^\infty(X, \Sigma, \mu)$*, then*

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Proof. For every $1 \leq p < \infty$,

$$\|f\|_p \leq \mu(X)^{1/p}\, \|f\|_\infty.$$

Hence

$$\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty.$$

Now fix $\varepsilon > 0$ and set

$$E_\varepsilon := \{x \in X : |f(x)| > \|f\|_\infty - \varepsilon\}.$$

By Proposition 5.15, $\mu(E_\varepsilon) > 0$. Therefore,

$$\|f\|_p^p = \int_X |f|^p \, d\mu \ge \int_{E_\varepsilon} |f|^p \, d\mu \ge (\|f\|_\infty - \varepsilon)^p \mu(E_\varepsilon).$$

Taking pth roots and letting $p \to \infty$, we obtain

$$\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty.$$

Combining the two inequalities proves the claim. □

Theorem 5.19. *$L^\infty(\mathbb{R}, \mathcal{M}, m)$ is not separable.*

Proof. For each $t \in (0,1)$, define $f_t := \chi_{[0,t)}$. If $s \neq t$, then $|f_s - f_t| = 1$ on the symmetric difference of $[0,s)$ and $[0,t)$, which has positive measure. Hence

$$\|f_s - f_t\|_\infty = 1.$$

Therefore the family of open balls

$$\mathcal{B} := \{B_{1/2}(f_t) : t \in (0,1)\}$$

is uncountable and pairwise disjoint.

Assume, toward a contradiction, that $A \subset L^\infty(\mathbb{R}, \mathcal{M}, m)$ is a countable dense subset. Every ball in $\mathcal{B}$ must meet A, and distinct balls in $\mathcal{B}$ must contain distinct points of A. This yields an injection from an uncountable set into a countable one, which is impossible. Hence $L^\infty(\mathbb{R}, \mathcal{M}, m)$ is not separable. □

A preliminary duality observation.

Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{R}$ or $\mathbb{C}$. A linear functional $T : X \to \mathbb{C}$ is called *bounded* if there exists $M \ge 0$ such that

$$|T(x)| \le M\|x\|, \qquad x \in X.$$

Its operator norm is defined by

$$\|T\| := \sup_{\|x\|\le 1} |T(x)|.$$

Equivalently,

$$\|T\| = \inf\{M \geq 0 : |T(x)| \leq M\|x\| \text{ for all } x \in X\} = \sup_{x\neq 0} \frac{|T(x)|}{\|x\|}.$$

Theorem 5.20 (Canonical embedding of L^q into $(L^p)^*$)**.** *Let $1 \leq p \leq \infty$, and let q be the conjugate exponent (with the convention $q = 1$ when $p = \infty$). For $g \in L^q(X, \Sigma, \mu)$ define*

$$T_g(f) := \int_X f g \, d\mu.$$

Then T_g is a bounded linear functional on $L^p(X, \Sigma, \mu)$ and

$$\|T_g\| \leq \|g\|_q.$$

Moreover, equality holds whenever either $1 < p < \infty$ or $p = \infty$.

Proof. Linearity is immediate. By Hölder's inequality,

$$|T_g(f)| \leq \|f\|_p \, \|g\|_q,$$

so $T_g \in (L^p)^*$ and $\|T_g\| \leq \|g\|_q$.

If $1 < p < \infty$, let

$$f_0 := \frac{|g|^{q-1} \operatorname{sgn}(g)}{\|g\|_q^{q-1}} \qquad (g \neq 0).$$

Then $f_0 \in L^p$ and $\|f_0\|_p = 1$, while

$$T_g(f_0) = \int_X \frac{|g|^q}{\|g\|_q^{q-1}} \, d\mu = \|g\|_q.$$

Hence $\|T_g\| = \|g\|_q$.

If $p = \infty$ and $q = 1$, take $f_0 = \operatorname{sgn}(g)$. Then $\|f_0\|_\infty = 1$ and

$$T_g(f_0) = \int_X |g| \, d\mu = \|g\|_1,$$

so again $\|T_g\| = \|g\|_q$. □

Remark 5.21. The converse statement — namely, that every bounded linear functional on L^p is given by integration against a function in L^q — is subtler. It will be proved later for $1 < p < \infty$, and also for $p = 1$ under the additional assumption that the measure is σ-finite; see Theorem 6.19.

Remark 5.22. The dual of L^∞ is, in general, strictly larger than L^1. A precise description involves finitely additive measures and lies beyond the scope of these notes.

Chapter 6

Signed measures and the Radon–Nikodym theorem

We extend the basic theory of measures to signed measures. *After the Hahn and Jordan decompositions we introduce total variation and the associated variation measure. The central result is the Radon–Nikodym theorem, which identifies absolutely continuous measures via a density and yields the Lebesgue decomposition into absolutely continuous and singular parts. We also record standard consequences, including a useful chain rule and the representation of bounded linear functionals on L^p by integration.*

6.1 Signed measures and decompositions

A set function $\nu : S \to [-\infty, \infty]$ on a measurable space (X, S) is called a signed measure if

(i) $\nu(\emptyset) = 0$;

(ii) ν does not take both values $+\infty$ and $-\infty$;

(iii) for every pairwise disjoint sequence $\{E_i\}_{i\geq 1} \subset S$,

$$\nu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \nu(E_i),$$

where the series is interpreted in the extended real sense.

Example 6.1. If μ_1 and μ_2 are measures on (X, S) and at least one of them is finite, then $\nu := \mu_1 - \mu_2$ is a signed measure.

If f is measurable and at least one of $\int f^+ \, d\mu$ or $\int f^- \, d\mu$ is finite, then

$$\nu(E) := \int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$$

defines a signed measure on (X, S).

Later we shall see that every signed measure admits a Hahn–Jordan decomposition, so these examples are representative rather than exceptional.

Lemma 6.2. *Let ν be a signed measure on a measure space (X, S).*

(i) If $E_n \uparrow E$ in S, then $\nu(E) = \lim_{n\to\infty} \nu(E_n)$.

(ii) If $E_n \downarrow E$ in S and $\nu(E_1)$ is finite, then $\nu(E) = \lim_{n\to\infty} \nu(E_n)$.

Proof. For (i), define $F_1 := E_1$ and $F_n := E_n \setminus E_{n-1}$ for $n \geq 2$. Then $\{F_n\}$ is pairwise disjoint, $E_n = \dot\bigcup_{k=1}^{n} F_k$, and $E = \dot\bigcup_{k=1}^{\infty} F_k$. Hence, by countable additivity of the signed measure,

$$\nu(E_n) = \sum_{k=1}^{n} \nu(F_k) \quad \text{and} \quad \nu(E) = \sum_{k=1}^{\infty} \nu(F_k).$$

Taking the limit in the finite partial sums gives $\nu(E) = \lim_{n\to\infty} \nu(E_n)$.

For (ii), set $F_n := E_1 \setminus E_n$. Then $F_n \uparrow E_1 \setminus E$. By part (i),

$$\nu(E_1 \setminus E) = \lim_{n\to\infty} \nu(F_n).$$

Since $E_n \subset E_1$ and $\nu(E_1)$ is finite, we may write

$$\nu(F_n) = \nu(E_1) - \nu(E_n) \quad \text{and} \quad \nu(E_1 \setminus E) = \nu(E_1) - \nu(E).$$

Therefore

$$\nu(E_1) - \nu(E) = \lim_{n\to\infty} \big(\nu(E_1) - \nu(E_n)\big),$$

and subtraction of the finite constant $\nu(E_1)$ yields $\nu(E) = \lim_{n\to\infty} \nu(E_n)$. □

Definition 6.3. Let ν be a signed measure on (X, S). A set $E \in S$ is said to be positive set (resp. negative set, null set)

if $\nu(F) > 0$ (or $\nu(F) < 0$, $\nu(F) = 0$) for each $F \subseteq E$ and $F \subseteq S$.

Example. If $\nu(E) = \int_E f d\mu$ where μ is a positive measure, and at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite, then E is a positive, negative, or null set if $f > 0$, $f < 0$ or $f = 0$ a.e. μ on E.

Lemma 6.4. *Union of any countable family of positive sets is a positive set.*

Proof. Let $P_1, P_2, \ldots$ be positive sets for ν. Write $Q_n = P_n \setminus \bigcup_{i=1}^{n-1} P_i$, $n \geq 2$. Then

$$\bigcup Q_n \text{ and } \bigcup Q_n = \bigcup P_n$$

If $E \subseteq \bigcup P_i$, then

$$\nu(E) = \nu\left(\bigcup (Q_n \cap E)\right) = \sum \nu(Q_n \cap E) > 0.$$

Next, we see that any set X can be written as disjoint union of positive and negative sets.

□

Theorem 6.5 (Hahn Decomposition Theorem)**.** *Let ν be a signed measure on (X, S). Then there exists a positive set P and a negative set N such that $X = P \cup N$.*

Proof. Without loss of generality, we can assume that ν does not take value $+\infty$ (otherwise, we consider $-\nu$).

Let $m = \sup\{\nu(E) : E \text{ is a positive set}\}$.
Since class of all positive sets is nonempty (as it contains the empty set). Hence,

$$-\infty < m < \infty.$$

Moreover, there exists a sequence P_i of positive sets such that $\nu(P_i) \to m$.

Let $P = \bigcup P_i$. Then P is a positive set, and $P_i \subseteq P$. Hence

$$\nu(P_i) \leq \nu(P) \leq m.$$

Also,

$$\nu(P) = \nu\left(\bigcup P_i\right) = \sum \nu(P_i) \to m.$$

$$\text{implies } \nu(P) = m.$$

Let $N = X \setminus P$. We show that N is a negative set.

Notice that N cannot contain any nonnull positive sets. Indeed, if $E \subseteq N$ is a positive set and $\nu(E) > 0$, then $E \cup P$ is a positive set, and

$$\nu(E \cup P) = \nu(E) + \nu(P) > m,$$

a contradiction.

On the other hand, if $A \subseteq N$ and $\nu(A) > 0$, then there exists $B \subseteq A$ with $\nu(B) > \nu(A)$. This is possible because A cannot be a positive set, and for all $C \subseteq A$ with $\nu(C) < 0$. Let $B = A \setminus C$, then

$$\nu(B) = \nu(A) - \nu(C) > \nu(A).$$

On contrary, suppose N is not a positive set. Then we can find least positive integer n such that

$$\frac{1}{n} = \max\left\{\frac{1}{k} : 1 \in N, \text{ there exists } B \subseteq N, B \in S \text{ with } \nu(B) > \frac{1}{n}\right\}.$$

That is, $1/n$ is the least positive integer such that there exists $B \subseteq N$ and $\nu(B) > 1/n$ (i.e., $\not\exists$there exists $B \subseteq N$ such that $\nu(B) \leq 1/(n-1)$).

But B_1 cannot be a positive set, hence there exists least positive integer n_2 and $B_2 \subseteq B_1$ such that

$$\nu(B_2) > \nu(B_1) + \frac{1}{n_2}.$$

By induction, there exists $B_i \subseteq B_{i-1}$ such that

$$\nu(B_i) > \nu(B_{i-1}) + \frac{1}{n_i}, \text{ for all } i \geq 2.$$

Let $B = \bigcap_{i\geq 1} B_i$. Then

$$\infty > \nu(B) = \lim \nu(B_i) \geq \sum_{i\geq 1} \frac{1}{n_i} > 0.$$

(Because $\nu(B_i) > \nu(B_{i-1}) + \frac{1}{n_{i-1}} + \frac{1}{n_i}$, etc.)
implies $n_i \to \infty$ is possible (i.e., the process is endless).

Notice that $0 < \nu(B) < \infty$. But B cannot be a positive set. Hence, there exists $C \subseteq B$ such that $\nu(C) > \nu(B)$.

But then, we can find a large n_i such that $\nu(C) > \nu(B) + \frac{1}{n_{i-1}}$.
This contradicts the construction of n_i (i.e., n_i was least as defined by $(*)$). Remark: If P' and N' is another decomposition of X. Then $P \cap P' \subset P$ and $P \cap P' \subset N$ implies $P \cap P'$ is both the +ve, hence $P \cap P'$ is null set. Similarly $N \cap N'$ is null set. Thus

$$P \cap P' = N \cap N' = \text{null set}.$$

$X = P \cup N$ is known as Hahn decomposition for ν. It is not unique (cut, ν-null set can be transferred from P to N or from N to P). However, it leads to a canonical decomposition of ν as the dif and only iference of two positive measures.

To see this, we need the following concept:

□

Definition 6.6. Two signed measures μ and ν on (X, S) are said to be mutually singular (or ν is singular with respect to. μ) if there exists $E, F \in S$ such that $E \cap F = \varnothing$, $E \cup F = X$, E is null for ν and F is null for μ, and we write $\mu \perp \nu$.

Next, we decompose signed measure into two positive measures.

Theorem 6.7 (Jordan decomposition theorem)**.** *If ν is a signed measure, then there exists ! two measures ν^+ and ν^- such that*

$$\nu = \nu^+ - \nu^- \quad \textit{and} \quad \nu^+ \perp \nu^-.$$

Proof. Let $X = P \cup N$ be a Hahn decomposition of ν, and let

$$\nu^+(E) = \nu(E \cap P)$$

and

$$\nu^-(E) = -\nu(E \cap N).$$

Then

$$\nu(E) = \nu(E \cap X) = \nu(E \cap P) + \nu(E \cap N)$$

$$\text{implies } \nu(E) = \nu^+(E) - \nu^-(E).$$

Obviously, $\nu^+ \perp \nu^-$.

If $\nu = \mu^+ - \mu^-$ *and* $\mu^+ \perp \mu^-$, *let* $E, F \in S$ *such that* $E \cap F = \varnothing$, $E \cup F = X$, *and* $\mu^+(F) = 0$, $\nu^-(E) = 0$. *Then* $X = E \cup F$ *is another Hahn decomposition of* ν. *Hence,* $E \Delta P$ *is a null set.*

Now, for $A \in S$,

$$\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P).$$

$$\text{implies } \mu^+(A) = \nu^+(A) \text{ implies } \mu^+ = \nu^+.$$

Similarly, $\mu^- = \nu^-$.

The measures ν^+ *and* ν^- *are called positive and negative variation of* ν *respectively. This is similar to functions of bounded variation as dif and only iference of two increasing functions.*

Also, $|\nu| = \nu^+ + \nu^-$ *is called total variation of* ν.

Remarks: (i) If ν *does not take value* $+\infty$, *then* $\mu^+(X) = \nu(P) < \infty$. *In particular, if the range of* ν *is contained in* $\mathbb{R}$, *then* ν *is bounded.*

(ii) $\nu(E) = \int_E f d\mu$, *where* $\mu = |\nu|$, $f = \chi_P - \chi_N$, $X = P \cup N$, *a Hahn decomposition for* ν.

We write

$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-) = L^1(|\nu|),$$

and for $f \in L^1(\nu)$,

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-.$$

Note that ν *is called finite (or* σ*-finite) if* $|\nu|$ *is finite (or* σ*-finite).*

Example 6.8. (i) $E \in S$ is null set for ν if and only if $|\nu|(E) = 0$.

(ii) $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Example 6.9. If $f \in L^1(\nu)$, then

$$\left| \int_X f d\nu \right| \leq \int |f| d|\nu|.$$

Example 6.10. $E \in S$, then

$$|\nu|(E) = \sup \left\{ \left| \int_E f d\nu \right| : |f| \leq 1 \right\} = \alpha \text{ (say)}.$$

$$\left| \int_E f d\nu \right| \leq \int_E |f| d|\nu| \leq |\nu|(E) \quad (\text{since } |f| \leq 1)$$

implies $\alpha \leq |\nu|(E)$. On the other hand, let $f_0 = \chi_P - \chi_N$, $P \cup N = X$ is a Hahn decomposition.

$$\alpha \geq \left|\int_E f_0 d\nu\right| = \left|\int_E f_0 d\nu^+ - \int_E f_0 d\nu^-\right| = \nu^+(E \cap P) + \nu^-(E \cap N) = |\nu|(E).$$

implies $\alpha = |\nu|(E)$.

Example 6.11. (i) $\nu^+(E) = \sup\{\nu(F) : F \subset S, F \subset E\}$

(ii) $\nu^-(E) = -\inf\{\nu(F) : F \subset S, F \subset E\}$

(iii) $|\nu|(E) = \sup\left\{\sum_{i=1}^n |\nu(E_i)| : \bigcup_{i=1}^n E_i = E\right\}$, E_i are disjoint sets.

$$\sum_{i=1}^n |\nu(E_i)| \leq \sum_{j=1}^m \Big(\nu^+(F_j) + \nu^-(F_j)\Big) = \nu^+(E) + \nu^-(E) = |\nu|(E).$$

implies $\alpha \leq |\nu|(E)$.

On the other hand, let $E_1 \cup E_2 = E$ be a Hahn decomposition of E. Then

$$\alpha \geq |\nu(E_1)| + |\nu(E_2)| = \nu^+(E_1) + \nu^-(E_2) = \nu^+(E_1 \cup E_2) + \nu^-(E_1 \cup E_2) = |\nu|(E)$$

implies $\alpha = |\nu|(E)$.

6.2 Absolute continuity, the Radon–Nikodym theorem, and Lebesgue decomposition

Throughout this section (X, Σ) denotes a measurable space, and μ denotes a (positive) measure on Σ.

Definition 6.12 (Absolute continuity and singularity)**.** Let ν be a signed measure on (X, Σ) and let μ be a measure on (X, Σ).

(i) We say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if

$$\mu(E) = 0 \text{ implies } \nu(E) = 0, \qquad E \in \Sigma.$$

(ii) We say that ν is *singular* with respect to μ, and write $\nu \perp \mu$, if there exists $N \in \Sigma$ such that $\mu(N) = 0$ and $|\nu|(X \setminus N) = 0$.

Remark 6.13. One has $\nu \ll \mu$ if and only if $|\nu| \ll \mu$, equivalently $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Proposition 6.14 (ε–δ characterization)**.** *Let ν be a finite signed measure and μ a finite measure on (X, Σ). Then $\nu \ll \mu$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\mu(E) < \delta \textit{ implies } |\nu(E)| < \varepsilon, \qquad E \in \Sigma.$$

Proof. Since $|\nu(E)| \le |\nu|(E)$ and $\nu \ll \mu$ if and only if $|\nu| \ll \mu$, it suffices to prove the statement for finite *positive* measures. Assume $\nu \ll \mu$. If the ε–δ condition fails, then there exist $\varepsilon_0 > 0$ and sets $E_n \in \Sigma$ such that $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \ge \varepsilon_0$ for all n. Let $F_k := \bigcup_{n\ge k} E_n$ and $F := \bigcap_{k\ge 1} F_k$. Then $\mu(F) \le \mu(F_k) \le \sum_{n\ge k} 2^{-n} = 2^{1-k} \to 0$, hence $\mu(F) = 0$. But $\nu(F_k) \ge \varepsilon_0$ for each k, and by continuity from above,

$$\nu(F) = \lim_{k\to\infty} \nu(F_k) \ge \varepsilon_0,$$

contradicting $\nu \ll \mu$.

Conversely, assume the ε–δ condition. If $\mu(E) = 0$, then $\mu(E) < \delta$ for every $\delta > 0$, hence $|\nu(E)| < \varepsilon$ for every $\varepsilon > 0$, so $\nu(E) = 0$. Thus $\nu \ll \mu$.

□

6.3 Radon–Nikodym theorem

Theorem 6.15 (Radon–Nikodym (finite, positive case))**.** *Let μ and ν be finite measures on (X, Σ) with $\nu \ll \mu$. Then there exists a measurable function $f \ge 0$ such that*

$$\nu(E) = \int_E f \, d\mu, \qquad E \in \Sigma. \tag{6.1}$$

Moreover, f is unique up to μ-almost everywhere equality and $f \in L^1(X, \mu)$ with $\int_X f \, d\mu = \nu(X)$.

Proof. Consider the family

$$\mathcal{F} := \Big\{ g : X \to [0, \infty] \text{ measurable} : \int_E g \, d\mu \le \nu(E) \text{ for all } E \in \Sigma \Big\}.$$

It is nonempty (since $0 \in \mathcal{F}$) and is closed under pointwise maxima: if $g, h \in \mathcal{F}$ then $\max\{g, h\} \in \mathcal{F}$. Let

$$a := \sup_{g\in\mathcal{F}} \int_X g \, d\mu \le \nu(X) < \infty.$$

Choose $g_n \in \mathcal{F}$ with $\int_X g_n \, d\mu \to a$ and set $h_n := \max\{g_1, \ldots, g_n\}$ and $f := \sup_n h_n$. Then $h_n \uparrow f$ pointwise and, by monotone convergence, for every $E \in \Sigma$,

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E h_n \, d\mu \le \nu(E),$$

so $f \in \mathcal{F}$ and $\int_X f \, d\mu = a$.

Define a set function λ on Σ by

$$\lambda(E) := \nu(E) - \int_E f \, d\mu, \qquad E \in \Sigma.$$

Then λ is a finite measure and $\lambda \ge 0$ by construction. If $\lambda(X) > 0$, then for each $n \in \mathbb{N}$ consider the signed measure $\lambda - \frac{1}{n}\mu$ and let $X = P_n \cup N_n$ be a Hahn decomposition. Set $P := \bigcup_{n\ge1} P_n$ and $N := X \setminus P = \bigcap_{n\ge1} N_n$. On N we have $\lambda(E) \le \frac{1}{n}\mu(E)$ for all $E \subset N$

and all n, hence $\lambda(N) = 0$. Therefore $\lambda(P) = \lambda(X) > 0$. If $\mu(P_n) = 0$ for every n, then $\mu(P) = 0$ and hence $\lambda(P) = 0$ (since $\lambda \ll \mu$), a contradiction. Thus $\mu(P_{n_0}) > 0$ for some n_0. For every $E \subset P_{n_0}$, the Hahn decomposition gives $\lambda(E) \geq \frac{1}{n_0}\mu(E)$. Pick $\varepsilon := \frac{1}{2n_0}$ and define $f_\varepsilon := f + \varepsilon\,\chi_{P_{n_0}}$. For any $E \in \Sigma$,

$$\int_E f_\varepsilon\,d\mu = \int_E f\,d\mu + \varepsilon\,\mu(E \cap P_{n_0}) \leq \nu(E) - \lambda(E \cap P_{n_0}) + \varepsilon\,\mu(E \cap P_{n_0}) \leq \nu(E),$$

since $\lambda(E \cap P_{n_0}) \geq \frac{1}{n_0}\mu(E \cap P_{n_0}) \geq \varepsilon\,\mu(E \cap P_{n_0})$. Hence $f_\varepsilon \in \mathcal{F}$, and

$$\int_X f_\varepsilon\,d\mu = \int_X f\,d\mu + \varepsilon\,\mu(P_{n_0}) > a,$$

contradicting the definition of a. Therefore $\lambda(X) = 0$, i.e. $\nu(E) = \int_E f\,d\mu$ for all E.

For uniqueness, if f, g both satisfy (6.1), then $\int_E (f - g)\,d\mu = 0$ for every E. Taking $E = \{f > g\}$ and $E = \{g > f\}$ gives $f = g$ μ-almost everywhere.

□

Theorem 6.16 (Radon–Nikodym (signed and σ-finite)). *Let μ be a σ-finite measure on (X, Σ) and let ν be a σ-finite signed measure with $\nu \ll \mu$. Then there exists $f \in L^1_{\mathrm{loc}}(X, \mu)$ such that*

$$\nu(E) = \int_E f\,d\mu, \qquad E \in \Sigma,$$

and f is unique up to μ-almost everywhere equality. We write $f = \frac{d\nu}{d\mu}$.

Proof. First assume ν is a σ-finite *positive* measure. Choose $X = \bigcup_{k \geq 1} X_k$ with $\mu(X_k) < \infty$ and $\nu(X_k) < \infty$. Apply Theorem 6.15 to the restricted measures on each X_k to obtain densities $f_k \geq 0$ such that $\nu(E \cap X_k) = \int_{E \cap X_k} f_k\,d\mu$. By uniqueness the densities agree μ-a.e. on overlaps, hence define f by $f = f_k$ on X_k. Then $\nu(E) = \int_E f\,d\mu$ for all E.

For a signed measure ν, apply the preceding argument to ν^+ and ν^- and set $f = f^+ - f^-$.

□

6.4 Lebesgue decomposition and chain rule

Theorem 6.17 (Lebesgue decomposition). *Let μ be a σ-finite measure on (X, Σ) and let ν be a σ-finite signed measure on (X, Σ). Then there exist unique σ-finite signed measures ν_a and ν_s such that*

$$\nu = \nu_a + \nu_s, \qquad \nu_a \ll \mu, \qquad \nu_s \perp \mu.$$

Moreover, ν_a admits a Radon–Nikodym derivative $f = \frac{d\nu_a}{d\mu}$, i.e. $\nu_a(E) = \int_E f\,d\mu$.

Proof. We sketch the standard construction. First treat the case where ν is a positive measure. Let $\mathcal{N} := \{E \in \Sigma : \mu(E) = 0\}$ and set

$$\nu_s(E) := \sup\{\nu(F) : F \subseteq E,\ F \in \mathcal{N}\}, \qquad E \in \Sigma.$$

One checks that ν_s is a measure with $\nu_s \perp \mu$ and that $\nu_a := \nu - \nu_s$ satisfies $\nu_a \ll \mu$. Then apply Theorem 6.16 to ν_a to obtain the density. For signed ν, apply the construction to ν^+ and ν^-. Uniqueness follows from the fact that if $\eta \ll \mu$ and $\eta \perp \mu$ then $\eta = 0$.

□

Proposition 6.18 (Chain rule)**.** *Let μ, λ be σ-finite measures and ν a σ-finite signed measure on (X, Σ). If $\nu \ll \lambda$ and $\lambda \ll \mu$, then $\nu \ll \mu$ and*

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu} \qquad \mu\text{-a.e.}$$

Proof. For $E \in \Sigma$ and $g := \frac{d\nu}{d\lambda}$ we have $\nu(E) = \int_E g\, d\lambda$. Writing $d\lambda = \frac{d\lambda}{d\mu}\, d\mu$ and applying Theorem 6.16 to $\lambda \ll \mu$ gives

$$\nu(E) = \int_E g\, \frac{d\lambda}{d\mu}\, d\mu,$$

which identifies the Radon–Nikodym derivative of ν with respect to μ.

□

6.5 Duality of L^p spaces

Let (X, Σ, μ) be a measure space. For $1 \le p < \infty$ and q with $\frac{1}{p} + \frac{1}{q} = 1$, the bilinear form

$$\langle f, g \rangle := \int_X fg\, d\mu$$

is well-defined whenever $f \in L^p$ and $g \in L^q$ (Hölder's inequality).

Theorem 6.19 (Duality of L^p)**.** *Assume $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Then every bounded linear functional $T \in (L^p(X, \mu))^*$ is of the form*

$$T(f) = \int_X fg\, d\mu \qquad (f \in L^p)$$

for a unique $g \in L^q(X, \mu)$. Moreover, $\|T\| = \|g\|_q$.

If μ is σ-finite, then the same conclusion holds for $p = 1$ with $q = \infty$.

Proof. For $g \in L^q$, Hölder's inequality gives

$$|T_g(f)| := \left| \int_X fg\, d\mu \right| \le \|f\|_p\, \|g\|_q,$$

so $T_g \in (L^p)^*$ with $\|T_g\| \le \|g\|_q$; equality follows by testing against $f = \mathrm{sgn}(g)|g|^{q-1}\chi_{\{|g|>0\}}$ when $1 < p < \infty$.

Conversely, let $T \in (L^p)^*$ and first assume $\mu(X) < \infty$. Define a signed measure ν on Σ by $\nu(E) := T(\chi_E)$. If E_j are pairwise disjoint, then $\chi_{\cup_{j=1}^n E_j} = \sum_{j=1}^n \chi_{E_j}$ in L^p, hence $\nu(\cup_{j=1}^n E_j) = \sum_{j=1}^n \nu(E_j)$. Moreover, if $E = \cup_{j\ge1} E_j$ is a disjoint union, then $\chi_{\cup_{j=1}^n E_j} \to \chi_E$

in L^p because

$$\left\|\chi_E - \chi_{\cup_{j=1}^n E_j}\right\|_p = \mu\Big(\bigcup_{j>n} E_j\Big)^{1/p} \to 0,$$

so by continuity of T we get countable additivity. If $\mu(E) = 0$, then $\chi_E = 0$ in L^p, hence $\nu(E) = 0$ and $\nu \ll \mu$. By Theorem 6.16, there exists $g \in L^1(\mu)$ such that $\nu(E) = \int_E g\,d\mu$. By linearity and density of simple functions, $T(f) = \int_X fg\,d\mu$ for all simple f; by continuity this extends to all $f \in L^p$.

Finally, $g \in L^q$ and $\|T\| = \|g\|_q$ follow from the boundedness of T and a standard extremal argument (again using $f = \operatorname{sgn}(g)|g|^{q-1}$ when $1 < p < \infty$). If μ is merely σ-finite, decompose $X = \cup_k X_k$ with $\mu(X_k) < \infty$ and apply the previous argument to the restriction of T to $L^p(X_k)$; the resulting densities patch together by uniqueness.

□

Let (X, S, μ) and (Y, T, ν) be measure spaces. For $A \in S$ and $B \in T$, the set $A \times B$ is called a measurable rectangle, and we define the product σ-algebra by

$$S \otimes T := \sigma\big(\{A \times B : A \in S,\ B \in T\}\big).$$

When μ and ν are σ-finite, there exists a unique measure $\mu \times \nu$ on $S \otimes T$ such that

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B), \qquad A \in S,\ B \in T.$$

This measure is called the *product measure*.

For a set $E \subset X \times Y$ and points $x \in X$, $y \in Y$, define the sections

$$E_x := \{y \in Y : (x, y) \in E\}, \qquad E^y := \{x \in X : (x, y) \in E\}.$$

See Figure 6.1.

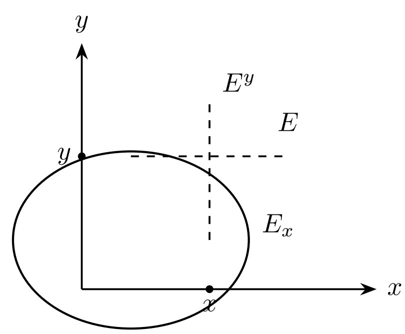


Figure 6.1: Sections and rectangles used in product-measure arguments.

Theorem 6.20 (Monotone class theorem)**.** *If $\mathcal{A}$ is an algebra of subsets of X, then the monotone class generated by $\mathcal{A}$ coincides with the σ-algebra generated by $\mathcal{A}$.*

Proof. Let $\mathcal{M}$ be the monotone class generated by $\mathcal{A}$. Since every σ-algebra is a monotone class, we automatically have $\mathcal{M} \subset \sigma(\mathcal{A})$.

To prove the reverse inclusion, it is enough to show that $\mathcal{M}$ is itself a σ-algebra. A standard Dynkin-system argument gives that $\mathcal{M}$ is closed under relative complements of nested sets, and since $\mathcal{A}$ is an algebra, one then shows that $\mathcal{M}$ is an algebra.

Once $\mathcal{M}$ is an algebra, closure under increasing unions implies closure under arbitrary countable unions via the decomposition

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \Big(E_n \setminus \bigcup_{k<n} E_k\Big),$$

where the sets on the right are pairwise disjoint and the partial unions increase. Hence $\mathcal{M}$ is a σ-algebra containing $\mathcal{A}$.

Therefore $\sigma(\mathcal{A}) \subset \mathcal{M}$, and so $\mathcal{M} = \sigma(\mathcal{A})$.

□

Theorem 6.21. *Let (X, S, μ) and (Y, T, ν) be σ-finite measure spaces, and let $E \in S \otimes T$. Then*

(i) $E_x \in T$ for every $x \in X$, and $E^y \in S$ for every $y \in Y$.

(ii) The functions $x \mapsto \nu(E_x)$ on X and $y \mapsto \mu(E^y)$ on Y are measurable.

(iii)

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y).$$

Proof. Let $\mathcal{C}$ be the class of all $E \in S \otimes T$ for which (i)–(iii) hold. For a rectangle $E = A \times B$ we have

$$E_x = \begin{cases} B, & x \in A, \\ \varnothing, & x \notin A, \end{cases} \qquad E^y = \begin{cases} A, & y \in B, \\ \varnothing, & y \notin B, \end{cases}$$

so (i)–(iii) are immediate and

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B) = \int_X \nu((A \times B)_x)\, d\mu(x).$$

Thus every measurable rectangle belongs to $\mathcal{C}$.

The class $\mathcal{C}$ is a monotone class: if $E_n \uparrow E$ or $E_n \downarrow E$, then the sections satisfy $(E_n)_x \uparrow E_x$ or $(E_n)_x \downarrow E_x$ for each x (and similarly for E^y). The measurability and integral identities pass to the limit by the Monotone Convergence Theorem; for decreasing sequences one first reduces to finite-measure rectangles and then uses σ-finiteness.

Since rectangles form an algebra generating $S \otimes T$, the monotone class theorem yields $\mathcal{C} = S \otimes T$.

□

Example 6.22. Let $f : X \to \mathbb{R}$ measurable. Then we can define

$$\varphi : X \times \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}$$

by

$$\varphi(x, y) = (f(x), y) \text{ and } \psi(z, \eta) = z - \eta$$

$$\psi \circ \varphi : (a, b) \times (c, d) \to \{(x, y) \in X \times \mathbb{R} : \varphi(x, y) \in (a, b) \times (c, d)\}$$

$$= \{(x, y) : a < f(x) < b, c < y < d\}$$

$= \varphi^{-1}((a, b) \times (c, d))$ is a measurable subset of $X \times \mathbb{R}$.
Hence, $\varphi \circ \psi$ is measurable.

Consider:

$$(\psi \circ \varphi)^{-1}(0) = \{(x, y) \in X \times \mathbb{R} : (\psi \circ \varphi)(x, y) = 0\}$$

$$= \{(x, y) : \psi(f(x), y) = 0\}$$

$$= \{(x, y) : y = f(x), x \in X\}$$

$= G_f$, the graph of f.
Hence, the graph of a measurable function is measurable.

Theorem 6.23 (Tonelli)**.** *Let (X, S, μ) and (Y, T, ν) be σ-finite measure spaces.*

Let $f : X \times Y \to [0, \infty]$ be a $S \otimes T$-measurable function. Then for fixed $(x_0, y_0) \in X \times Y$,

(i) $x_1 \mapsto f(x_1, y_0)$ and $y_1 \mapsto f(x_0, y_1)$ are measurable functions on (X, S) and (Y, T) respectively.

(ii) $y \mapsto \int_X f(x, y)d\mu(x)$ and $x \mapsto \int_Y f(x, y)d\nu(y)$ are measurable.

(iii)

$$\int_{X\times Y} f(x, y)d(\mu \times \nu)(x, y) = \int_Y \left(\int_X f(x, y)d\mu(x)\right) d\nu(y)$$
$$= \int_X \left(\int_Y f(x, y)d\nu(y)\right) d\mu(x)$$

***Proof**. Since f is $S \otimes T$-measurable on $X \times Y$, there exists a sequence $\{\varphi_n\}$ of simple functions that increases to f pointwise. Hence,*

$$\lim_{n\to\infty} \varphi_n(x, y) = f(x, y) \qquad \left\{ \begin{array}{c} y \text{ fixed} \\ x \text{ fixed} \end{array} \right\}$$

are measurable.

Now, by the Monotone Convergence Theorem (MCT),

$$y \longmapsto \int_X f(x, y)\, d\mu(x) = \lim_{n\to\infty} \int_X \varphi_n(x, y)\, d\mu(x) = \lim_{n\to\infty} \sum_{j=1}^{k_n} a_j^n \nu(E_j^y), \tag{6.2}$$

where $\varphi_n = \sum_{j=1}^{k_n} a_j^n \chi_{E_j^n}$.

This proves (ii).

Moreover, $(\varphi_n)_y = \sum a_j \nu(E_j^y)$ *is an increasing sequence.*

By applying the Monotone Convergence Theorem to (1), we get

$$\begin{aligned}\int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y) &= \lim_{n\to\infty} \int_Y \left(\int_X \varphi_n(x,y)\, d\mu(x) \right) d\nu(y) \\ &= \lim_{n\to\infty} \int_{X\times Y} \varphi_n(x,y)\, d(\mu\times\nu)(x,y).\end{aligned}$$

Similarly, the other equality follows.

Chapter 7

Product measures and Fubini–Tonelli theory

This chapter develops measure and integration on Cartesian products. We construct the product σ-algebra and (under standard hypotheses) the product measure, and we prove Tonelli's and Fubini's theorems. These results justify iterated integration and provide the natural framework for multi-parameter problems in analysis and probability.

7.1 Fubini's Theorem

Let (X, S, μ) and (Y, T, ν) be two σ-finite measure spaces. If $f \in L^1(\mu \times \nu)$, then

(i) $x \mapsto f(x, y)$ and $y \mapsto f(x, y)$ are a.e. integrable on X and Y respectively.

(ii) $y \mapsto \int_X f(x, y) d\mu(x)$ and $x \mapsto \int_Y f(x, y) d\nu(y)$ are integrable on Y and X, respectively.

(iii)

$$\begin{aligned}\int_{X\times Y} f(x,y)d(\mu\times\nu)(x,y) &= \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu(y) \\ &= \int_X \left(\int_Y f(x,y)d\nu(y)\right) d\mu(x)\end{aligned}$$

Proof. If $f = f^+ - f^-$, then f^+, f^- are in $L^1(\mu \times \nu)$ and are non-negative. Hence, by linearity of integral on L^1,

$$\int_{X\times Y} f d(\mu\times\nu) = \int_{X\times Y} f^+ d(\mu\times\nu) - \int_{X\times Y} f^- d(\mu\times\nu).$$

Hence by Tonelli's theorem,

$$\int_Y \left(\int_X |f(x,y)|d\mu(x)\right) d\nu(y) = \int_X \left(\int_Y |f(x,y)|d\nu(y)\right) d\mu(x) < \infty.$$

Therefore,

$$\int_Y \left(\int_X f^+(x,y) d\mu(x) \right) d\nu(y) \text{ and } \int_X \left(\int_Y f^+(x,y) d\nu(y) \right) d\mu(x)$$

are finite a.e. with respect to ν and μ respectively, and integrable with respect to μ and ν respectively. This proves (i) and (ii). Hence by Tonelli's theorem and the above, we get (iii).

Remark 7.1. If the measure spaces (X, S, μ) and (Y, T, ν) are complete, their product $(X \times Y, S \otimes T, \mu \times \nu)$ need not be complete.

Suppose $A \subset X$, $A \neq \varnothing$, $\mu(A) = 0$. Let $B \subset Y$, with $B \in T$. Then $A \times B \subset X \times Y$, but $\mu^*(A \times B) = \nu^*(A \times Y) = \mu(A)\nu(Y) = 0$. However, $A \times B \in S \otimes T$.

Example 7.2. Let m_1^* and m_2^* denote the usual Lebesgue outer measures on $\mathbb{R}$ and $\mathbb{R}^2$ respectively, and m_1, m_2 are their corresponding Lebesgue measures. Then $m_1 \times m_1$ is *not* a complete measure, however m_2 is complete, though the completion of $m_1 \times m_1$ is m_2.

Let $\mathcal{R}_2 = \{(a,b) \times (c,d) : a,b,c,d \in \mathbb{R}\}$, and $\mathcal{B}_2$ is the σ-algebra generated by $\mathcal{R}_2$. Then $\mathcal{B}_2$ is nothing but the Borel σ-algebra on $\mathbb{R}^2$.

Since $\mathcal{R}_2 \subset M_1 \otimes M_1$, it follows that $\mathcal{B}_2 \subset M_1 \otimes M_1$ (since $M_1 \otimes M_1$ is a σ-algebra).

Furthermore, $\mathcal{R}_2 \subset M_2$ and M_2 is the smallest σ-algebra containing $\mathcal{R}_2$, hence

$$\mathcal{B}_2 \subset M_1 \otimes M_1 \subset M_2.$$

But completion of $\mathcal{B}_2$ is M_2. So, if $E \in M_2$, then there exists $F, G \in \mathcal{B}_2$ with $F \subset E \subset G$ such that $m_2(G \setminus F) = 0$. Thus,

$$(m_1 \times m_1)(E \setminus F) \leq m_1 \times m_1(G \setminus F) = m_2(G \setminus F) = 0.$$

$(m_1 \times m_1)(E) = (m_1 \times m_1)(F) = m_2(F) = m_2(G)$. Since $m_2(F) \leq m_2(E) \leq m_2(G)$, implies $(m_1 \times m_1)(E) = m_2(E)$.

Example 7.3. Let $B(\mathbb{R}^2)$ be the σ-algebra generated by open sets (or open rectangles in $\mathbb{R}^2$), then $B(\mathbb{R}^2) = B(\mathbb{R}) \otimes B(\mathbb{R})$.

Since $(a,b) \times (c,d) \subset B(\mathbb{R}) \otimes B(\mathbb{R})$,

It follows that

$$\mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$$

On the other hand, $(a,b) \times (c,d) \subset \mathcal{B}(\mathbb{R}^2)$

By varying (a,b) and fixing (c,d), we see

$$\mathcal{B}(\mathbb{R}) \times (c,d) \subset \mathcal{B}(\mathbb{R}^2)$$

$$\text{implies } \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^2)$$

But then,

$$\sigma(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})) \subseteq \mathcal{B}(\mathbb{R}^2)$$

That is,

$$\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R}^2)$$

(since $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$).

Thus,

$$\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}).$$

Chapter 8

Regular Borel measures and the Riesz representation theorem

On compact metric spaces, Borel measures admit powerful regularity properties. We prove the Riesz representation theorem, identifying positive linear functionals on $C(K)$ with integration against a unique regular Borel measure. The chapter also develops auxiliary tools—Urysohn functions, capacities on open sets, and an induced outer measure—that streamline the construction and clarify the geometric content of the representation.

8.1 Regular Borel measures on compact metric spaces

Let (K, d) be a compact metric space and let $\mathcal{B}(K)$ denote the Borel σ-algebra.

Definition 8.1 (Regular Borel measure)**.** A finite Borel measure μ on $(K, \mathcal{B}(K))$ is called *regular* if for every Borel set $E \subset K$,

$$\mu(E) = \inf\{\mu(U) : E \subset U,\ U \text{ open}\} \qquad \text{and} \qquad \mu(E) = \sup\{\mu(F) : F \subset E,\ F \text{ closed}\}.$$

8.2 Urysohn functions in metric spaces

Lemma 8.2 (Urysohn lemma for metric spaces)**.** *If $F \subset U \subset K$ with F closed and U open, then there exists $\varphi \in C(K)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on F, and $\varphi \equiv 0$ on $K \setminus U$.*

Proof. Set

$$\varphi(x) := \frac{d(x, K \setminus U)}{d(x, K \setminus U) + d(x, F)}, \qquad x \in K.$$

The distance to a closed set is continuous, hence φ is continuous. If $x \in F$ then $d(x, F) = 0$ and $d(x, K \setminus U) > 0$, so $\varphi(x) = 1$. If $x \in K \setminus U$ then $d(x, K \setminus U) = 0$ and $d(x, F) > 0$, so $\varphi(x) = 0$.

□

8.3 Riesz representation theorem

Let $C(K)$ denote the Banach space of real-valued continuous functions on K with the supremum norm. A linear functional $L : C(K) \to \mathbb{R}$ is called *positive* if $L(f) \geq 0$ whenever $f \geq 0$ on K.

8.4 A capacity on open sets

Definition 8.3. Let $L : C(K) \to \mathbb{R}$ be a positive linear functional. For each open set $U \subset K$ define

$$\mu_0(U) := \sup\Big\{L(\varphi) : \varphi \in C(K),\ 0 \leq \varphi \leq 1,\ \operatorname{supp}(\varphi) \subset U\Big\}.$$

Lemma 8.4. *The set function μ_0 is monotone and finitely subadditive on open sets: if $U, V \subset K$ are open, then*

$$\mu_0(U \cup V) \leq \mu_0(U) + \mu_0(V).$$

Proof. Monotonicity is immediate from the definition. For subadditivity, fix $\psi \in C(K)$ with $0 \leq \psi \leq 1$ and $\operatorname{supp}(\psi) \subset U \cup V$. Set $F := \operatorname{supp}(\psi) \cap (K \setminus V)$. Then F is closed and $F \subset U$. By Lemma 8.2, choose $\eta \in C(K)$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on F, and $\eta \equiv 0$ on $K \setminus U$. Define

$$\psi_1 := \eta\,\psi, \qquad \psi_2 := (1-\eta)\,\psi.$$

Then $0 \leq \psi_1, \psi_2 \leq 1$, $\operatorname{supp}(\psi_1) \subset U$, and $\operatorname{supp}(\psi_2) \subset V$. By linearity and positivity,

$$L(\psi) = L(\psi_1) + L(\psi_2) \leq \mu_0(U) + \mu_0(V).$$

Taking the supremum over all admissible ψ with support in $U \cup V$ yields the claim.

□

8.5 An outer measure

Define $\mu^* : \mathcal{P}(K) \to [0, \infty]$ by

$$\mu^*(A) := \inf\left\{\sum_{n=1}^{\infty} \mu_0(U_n) : A \subset \bigcup_{n=1}^{\infty} U_n,\ U_n \text{ open}\right\}. \tag{8.1}$$

Lemma 8.5. *The set function μ^* is an outer measure on K. Moreover, for every open set $U \subset K$ one has $\mu^*(U) = \mu_0(U)$.*

Proof. That μ^* is an outer measure is standard: $\mu^*(\emptyset) = 0$, monotonicity is immediate, and countable subadditivity follows by concatenating covers.

To show $\mu^*(U) = \mu_0(U)$ for open U, note first that $\mu^*(U) \le \mu_0(U)$ by taking the single-set cover $\{U\}$ in (8.1). Conversely, let $U \subset \bigcup_n U_n$ be an open cover. Fix $\varphi \in C(K)$ with $0 \le \varphi \le 1$ and $\operatorname{supp}(\varphi) \subset U$. Since $\operatorname{supp}(\varphi)$ is compact and contained in $\bigcup_{n\in\mathbb{N}} U_n$, it is covered by finitely many $U_{n_1}, \dots, U_{n_m}$. Repeatedly applying Lemma 8.4 gives

$$L(\varphi) \le \mu_0(U_{n_1} \cup \cdots \cup U_{n_m}) \le \sum_{j=1}^{m} \mu_0(U_{n_j}) \le \sum_{n=1}^{\infty} \mu_0(U_n).$$

Taking the supremum over such φ yields $\mu_0(U) \le \sum_n \mu_0(U_n)$, and then taking the infimum over all covers of U yields $\mu_0(U) \le \mu^*(U)$.

□

8.6 Borel measurability and the associated measure

A fundamental theorem of Carathéodory states that, for any outer measure μ^* on a metric space, all Borel sets are μ^*-measurable. We therefore define

$$\mu := \mu^*|_{\mathcal{B}(K)}.$$

Then μ is a Borel measure on K. Outer regularity is immediate from (8.1) and Lemma 8.5. Inner regularity follows from compactness and the Urysohn lemma (one first proves inner regularity for open sets by approximating from within by compact sets, and then extends to all Borel sets by standard arguments).

8.7 Identification of L with integration

Theorem 8.6 (Riesz representation theorem on compact metric spaces)**.** *Let K be a compact metric space and let $L : C(K) \to \mathbb{R}$ be a positive linear functional. Then there exists a unique regular Borel measure μ on $(K, \mathcal{B}(K))$ such that*

$$L(f) = \int_K f\, d\mu, \qquad f \in C(K). \tag{8.2}$$

Moreover, $\|L\| = \mu(K) = L(1)$.

Proof. Let μ be the Borel measure constructed above from L.

Step 1: $L(\varphi) \le \int \varphi\, d\mu$ for $0 \le \varphi \le 1$. Fix $\varphi \in C(K)$ with $0 \le \varphi \le 1$. For each $t \in (0,1)$ the set $U_t := \{x : \varphi(x) > t\}$ is open. By Lemma 8.2, for each t and each closed $F \subset U_t$ we may find $\psi \in C(K)$ with $0 \le \psi \le 1$, $\psi \equiv 1$ on F, and $\operatorname{supp}(\psi) \subset U_t$. By definition of μ_0 and Lemma 8.5,

$$L(\psi) \le \mu_0(U_t) = \mu(U_t).$$

Approximating φ from above by Riemann sums of the form $\sum_j (t_j - t_{j-1})\chi_{U_{t_{j-1}}}$ and using positivity of L yields the inequality $L(\varphi) \le \int \varphi\, d\mu$.

Step 2: Reverse inequality for $0 \le \varphi \le 1$. Apply Step 1 to the constant function 1 to obtain

$$L(1) \le \int_K 1\,d\mu = \mu(K).$$

On the other hand, $\mu(K) = \mu_0(K)$ by Lemma 8.5, and $\mu_0(K) = L(1)$ because $0 \le \psi \le 1$ implies $L(\psi) \le L(1)$ by positivity. Hence $L(1) = \mu(K)$.

Now let $0 \le \varphi \le 1$. Since $1 - \varphi \in C(K)$ and $0 \le 1 - \varphi \le 1$, Step 1 gives

$$L(1-\varphi) \le \int_K (1-\varphi)\,d\mu.$$

Using linearity and $L(1) = \mu(K)$, we obtain

$$L(\varphi) = L(1) - L(1-\varphi) \ge \mu(K) - \int_K (1-\varphi)\,d\mu = \int_K \varphi\,d\mu.$$

Combining Steps 1 and 2 gives (8.2) for $0 \le \varphi \le 1$. By linearity and homogeneity we obtain (8.2) for all $f \in C(K)$.

Finally, for a positive functional on $C(K)$ we have $\|L\| = L(1)$, and

$$L(1) = \int_K 1\,d\mu = \mu(K).$$

Uniqueness follows because if two regular Borel measures agree on $C(K)$, then they agree on open sets by approximation via Lemma 8.2, hence on all Borel sets.

□

Remark 8.7. The Riesz–Markov–Kakutani theorem extends this result to locally compact Hausdorff spaces X by identifying positive linear functionals on $C_c(X)$ (or $C_0(X)$) with Radon measures on X.

Problem Sets

Problem Set 1: Outer measures and Carathéodory measurability

1. State TRUE or FALSE giving proper justification for each of the following statements.

   (a) There exists an unbounded subset A of $\mathbb{R}$ such that $m^*(A) = 5$.

   (b) There exists an open subset A of $\mathbb{R}$ such that $[\frac{1}{2}, \frac{3}{4}] \subset A$ and $m^*(A) = \frac{1}{4}$.

   (c) There exists an open subset A of $\mathbb{R}$ such that $m^*(A) < \frac{1}{5}$ but $A \cap (a, b) \neq \emptyset$ for all $a, b \in \mathbb{R}$ with $a < b$.

   (d) If A and B are open subsets of $\mathbb{R}$ such that $A \subsetneq B$, then it is necessary that $m^*(A) < m^*(B)$.

   (e) There is a Lebesgue measurable set $A \subset \mathbb{R}$ such that $m(A) = 0$ but $m(\partial A) = \infty$.

   (f) If A and $A \cup B$ are Lebesgue measurable subsets of $\mathbb{R}$, then B is necessarily Lebesgue measurable subset of $\mathbb{R}$.

   (g) There exists a non-zero finite measure μ on $\mathcal{M}(\mathbb{R})$, which is constant on every bounded open interval (a, b) with $a < b$.

   (h) Let E be a subset of $\mathbb{R}$ such that $m^*(E) = 0$. Then it imply that E is contained in a Borel set B with $m^*(B) = 0$.

   (i) For $A \subseteq \mathbb{R}$, define $\mu_o(A) = \begin{cases} 0 & \text{if } A \text{ is a compact subset of } \mathbb{R}, \\ 1 & \text{if } A \text{ is a non-compact subset of } \mathbb{R}. \end{cases}$ Then μ_o is necessarily a pre-measure on $\mathcal{B}(\mathbb{R})$.

2. Examine whether $\mathcal{A}$ is a σ-algebra of subsets of $\mathbb{R}$, where

   (a) $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) = 0 \text{ or } m^*(\mathbb{R} \setminus A) = 0\}$.

   (b) $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) < +\infty \text{ or } m^*(\mathbb{R} \setminus A) < +\infty\}$.

   (c) $\mathcal{A} = \{A \subset \mathbb{R} : A \text{ or } \mathbb{R} \setminus A \text{ is an open subset of } \mathbb{R}\}$.

3. Let X be an uncountable set. Show that the class $\{\{x\} : x \in X\}$ generates the σ-algebra $\{A \subset X : A \text{ is countable or } X \setminus A \text{ is countable}\}$.

4. Let $\mathcal{S}$ be a class of subsets of a nonempty set X and let $A \subset X$. Show that $\sigma(\mathcal{S} \cap A) = \sigma(\mathcal{S}) \cap A$, where for each class $\mathcal{C}$ of subsets of X, $\mathcal{C} \cap A = \{C \cap A : C \in \mathcal{C}\}$.

5. Let X, Y be nonempty sets and let $f : X \to Y$. If $\mathcal{S}$ is a class of subsets of Y, then show that $\sigma(f^{-1}(\mathcal{S})) = f^{-1}(\sigma(\mathcal{S}))$, where for each class $\mathcal{C}$ of subsets of Y, $f^{-1}(\mathcal{C}) = \{f^{-1}(C) : C \in \mathcal{C}\}$.

6. If $\mathcal{S}$ is a class of subsets of a nonempty set X and if $A \in \sigma(\mathcal{S})$, then show that there exists a countable subclass $\mathcal{S}_0$ of $\mathcal{S}$ such that $A \in \sigma(\mathcal{S}_0)$.

7. Prove that every infinite σ-algebra on an infinite set is uncountable.

8. Show that $\mathcal{B}(\mathbb{R})$ is generated by each of the following classes. (a) $\{(a, +\infty) : a \in \mathbb{R}\}$ (b) $\{(-\infty, a] : a \in \mathbb{Q}\}$ (c) $\{[a, b) : a, b \in \mathbb{Q}, a < b\}$ (d) $\{A \subset \mathbb{R} : A \text{ is compact}\}$

9. Let E be a Borel subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. Show that $x + E$ is a Borel subset of $\mathbb{R}$.

10. Examine whether μ^* is an outer measure on $\mathbb{R}$, where for each $A \subset \mathbb{R}$,

    (a) $\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is bounded,} \\ 1 & \text{if } A \text{ is unbounded.} \end{cases}$

    (b) $\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \text{ is nonempty and bounded,} \\ +\infty & \text{if } A \text{ is unbounded.} \end{cases}$

11. Consider the outer measure μ^* on $\mathbb{R}$, where for each $A \subset \mathbb{R}$, $\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ 1 & \text{if } A \text{ is uncountable.} \end{cases}$ Determine all the μ^*-measurable subsets of $\mathbb{R}$.

12. If $\mathcal{S} = \{\emptyset, [1, 2]\}$ and if $\mu(\emptyset) = 0$, $\mu([1, 2]) = 1$, then determine the outer measure μ^* on $\mathbb{R}$ induced by the set function $\mu : \mathcal{S} \to [0, +\infty)$. Also, determine all the μ^*-measurable subsets of $\mathbb{R}$.

13. Let $(X, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \to [0, +\infty]$ be finitely additive with $\mu(\emptyset) = 0$. Show that μ is a measure on $\mathcal{A}$ if either of the following conditions is satisfied.

    (a) For every increasing sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{A}$, $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$.

    (b) For every decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{A}$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$, $\lim_{n\to\infty} \mu(A_n) = 0$.

14. Let μ^* be an outer measure generated by a finite premeasure μ_o on an algebra $\mathcal{A}$ on a nonempty set X. Show that $E \subseteq X$ is μ^*-measurable iff for each $\varepsilon > 0$, there exists a μ^*-measurable set G containing E such that $\mu^*(G \setminus F) < \varepsilon$.

15. Let μ^* be an outer measure on a non-empty set X. Let $F = \{A \subseteq \mathbb{R} : \ \mu^*(A) = 0\}$. Find the σ-algebra generated by F.

16. Let $\mathcal{A}$ be a σ-algebra of on $\mathbb{R}$. Write $\bar{\mathcal{A}} = \{E \cup N : E \in \mathcal{A} \text{ and } N \subseteq F \in \mathcal{A} \text{ with } m(F) = 0\}$. Show that $\bar{\mathcal{A}}$ is a σ-algebra. Moreover, deduce that $\overline{B(\mathbb{R})} = \mathcal{M}(\mathbb{R})$.

17. Let μ^* be an outer measure generated by a pre-measure μ_o on an algebra $\mathcal{A}$ on $\mathbb{R}$. Let E be μ^* measurable (in Caratheodory's sense). Show that there exists a set $G \in \sigma(\mathcal{A})$ such that $\mu^*(E) = \mu^*(G)$.

18. Let (X, τ) be a topological space. Let $\mathcal{B}(X)$ be the σ-algebra generated by τ. Let μ^* be the outer measure generated by a σ-finite pre-measure μ_o on $\mathcal{B}(X)$. Show that $E \in M_{\mu^*}$ if and only if there exists $G \in \mathcal{B}(X)$ such that $\mu^*(G \setminus E) = 0$.

19. Let $f : [0, 2) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1, \\ 3 - x & \text{if } 1 < x < 2. \end{cases}$ Find $m^*(A)$, where $A = f^{-1}((\frac{9}{16}, \frac{5}{4})) = \{x \in [0, 2) : f(x) \in (\frac{9}{16}, \frac{5}{4})\}$.

20. Let $B \subset A \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A \setminus B) = m^*(A)$.

21. Let $A \subset \mathbb{R}$ such that $m^*(A) > 0$. Show that there exists $B \subset A$ such that B is bounded and $m^*(B) > 0$.

22. If G is a nonempty open subset of $\mathbb{R}$, then show that $m^*(G) > 0$.

23. Let A be a countable subset of $\mathbb{R}$ and let $B \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A + B) = 0$.

24. Prove or disprove: A subset E of $\mathbb{R}$ is Lebesgue measurable iff $m^*(A \cup B) = m^*(A) + m^*(B)$ for each $A \subset E$ and for each $B \subset \mathbb{R} \setminus E$.

25. Let $E = \{x \in [0,1] :$ The decimal representation of x does not contain the digit $5\}$. Show that $m(E) = 0$.

26. Let $A_n \subset \mathbb{R}$ for $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} m^*(A_n) < \infty$. If $E = \{x \in \mathbb{R} : x \in A_n$ for infinitely many $n\}$, then show that $m(E) = 0$.

27. Show that a subset E of $\mathbb{R}$ is Lebesgue measurable iff $m^*(I) = m^*(I \cap E) + m^*(I \setminus E)$ for every bounded open interval I of $\mathbb{R}$.

28. Let $A \subset E \subset B \subset \mathbb{R}$ such that A, B are Lebesgue measurable and $m(A) = m(B) < \infty$. Show that E is Lebesgue measurable. More generally, let $A \subset B \subset \mathbb{R}$ such that A is Lebesgue measurable and $m^*(B) = m(A) < \infty$. Show that B is Lebesgue measurable.

29. Let $A, B \subset \mathbb{R}$ such that $m^*(A) = 0$ and $A \cup B$ is Lebesgue measurable. Show that B is Lebesgue measurable.

30. Let $A, B \subset \mathbb{R}$ such that A is Lebesgue measurable and $m^*(A \triangle B) = 0$. Show that B is Lebesgue measurable.

31. Let $A, B \subset \mathbb{R}$ be such that $A \cap B$ is Lebesgue measurable and $m^*(A \triangle B) = 0$. Show that A and B are Lebesgue measurable and $m(A) = m(B)$.

32. Let $A \subset \mathbb{R}$ such that $A \cap B$ is Lebesgue measurable for every bounded subset B of $\mathbb{R}$. Show that A is Lebesgue measurable.

33. Let A and B be subsets of $[0,1]$ which satisfy $m^*(A \cup B) = m^*(A) + m^*(B)$. If $A \triangle B$ is Lebesgue measurable then prove that A and B are Lebesgue measurable.

34. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and let $A \subset \mathbb{R}$. Show that $m^*(E \cap A) + m^*(E \cup A) = m^*(E) + m^*(A)$.

35. Let A be a subset of $\mathbb{R}$ such that $m^*(A \cup B) = m^*(A) + m^*(B)$ for every subset B of $\mathbb{R}$. Show that A is Lebesgue measurable. Moreover, if $m(A) < \infty$, then show that $m(A) = 0$.

36. Let I and J be disjoint open intervals in $\mathbb{R}$ and let $A \subset I$, $B \subset J$. Show that $m^*(A \cup B) = m^*(A) + m^*(B)$.

37. Let A be a subset of $\mathbb{R}$ with $0 < m^*(A) < \infty$. Show that for each $\epsilon > 0$ there exist an open set O containing A and a compact set $K \subset \mathbb{R}$ such that $m(O \setminus K) < \epsilon$.

Problem Set 2: Lebesgue measurable sets and regularity

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) $\{x \in \mathbb{R} : x^6 - 6x^4 \text{ is irrational}\}$ is a Lebesgue measurable subset of $\mathbb{R}$.
   (b) If A is a Lebesgue measurable subset of $\mathbb{R}$ and if B is a Lebesgue non-measurable subset of $\mathbb{R}$ such that $B \subset A$, then it is necessary that $m^*(A \setminus B) > 0$.
   (c) Whether the set $E = \bigcup_{x \in \mathbb{R}} (x + \mathbb{Q})$ is Lebesgue measurable?
   (d) Let C be the Cantor set in $[0,1]$ and $a, b \in \mathbb{R}$ with $a < b$. Then the set $C + (a, b)$ is Borel measurable.
   (e) If A and B are disjoint subsets of $\mathbb{R}$ such that A is Lebesgue measurable and B is Lebesgue non-measurable, then it is possible that $m^*(A \cup B) < m^*(A) + m^*(B)$.
   (f) If A is subset of $\mathbb{R}$ with $m^*(A) < \infty$, then $m^*(A^2) < \infty$.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $F \subset \mathbb{R}$ be a countable set. Show that $E + F$ is Lebesgue measurable.

3. If $A \subset \mathbb{R}$, then show that there exists a Lebesgue measurable subset E of $\mathbb{R}$ such that $m^*(A) = m(E)$.

4. Let $A \subset [0,1]$ be Lebesgue measurable with $m(A) = 1$. If $B \subset [0,1]$, then show that $m^*(A \cap B) = m^*(B)$.

5. For $i = 1, ..., n$, let $E_i \subset (0,1)$ be Lebesgue measurable such that $\sum\limits_{i=1}^{n} m(E_i) > n - 1$. Show that $m(\bigcap\limits_{i=1}^{n} E_i) > 0$.

6. Let $\{E_i\}$ be a decreasing sequence of Lebesgue measurable sets in $[0,1]$ which satisfying $\sum\limits_{i=1}^{n} m(E_i) > n - \frac{1}{n}$. Show that $m\left(\bigcap\limits_{i=1}^{\infty} E_i\right) = 1$.

7. Let $A \subset \mathbb{R}$ such that $m^*(A) > 0$. Show that there exist $x, y \in A$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$.

8. Let A and B be Lebesgue measurable subsets of $(0, 1)$ such that $m(A) > \frac{1}{2}$ and $m(B) > \frac{1}{2}$. Prove that there exist $a \in A$ and $b \in B$ such that $a + b = 1$.

9. Suppose F is a closed subset of $[0, 1]$ such that $F \cap (a, b) \neq \emptyset$ for all $a, b \in [0, 1]$ with $a < b$. Show that $m(F) = 1$.

10. Let A be an unbounded Lebesgue measurable subset of $\mathbb{R}$ such that $m(A) < \infty$. Show that for each $\varepsilon > 0$, there exists a bounded Lebesgue measurable set B in $\mathbb{R}$ such that $B \subset A$ and $m(A \setminus B) < \varepsilon$.

11. For $n \in \mathbb{N}$, write $E = \bigcup\limits_{n=1}^{\infty} \left[n, n + \frac{1}{n^{3/2}}\right]$. Show that $m(E) < \infty$ and $m^*(\{x^2 : x \in E\}) = \infty$.

12. If $A \subset \mathbb{R}$ such that $m^*(A) = 0$, then show that $m^*(\{x^2 : x \in A\}) = 0$.

13. Let A be a closed subset of $[0, 1]$ that satisfies $A \cap (\alpha, \beta) \neq \emptyset$ for all $\alpha, \beta \in [0, 1]$ with $\alpha < \beta$. Show that $m(A \smallsetminus A^2) = 0$.

14. Let $A, B \subset \mathbb{R}$ such that $A \cup B$ is Lebesgue measurable and $m(A \cup B) = m^*(A) + m^*(B) < \infty$. Show that both A and B are Lebesgue measurable.

15. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}$ and let $\{E_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}$ such that $A_n \subset E_n$ for each $n \in \mathbb{N}$. Show that $m^*(\bigcup\limits_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty} m^*(A_n)$.

16. Let $E \subset \mathbb{R}$ and let $\alpha \in \mathbb{R}$. If $\alpha E = \{\alpha x : x \in E\}$, then show that $m^*(\alpha E) = |\alpha| m^*(E)$. Also, show that if E is Lebesgue measurable, then αE is Lebesgue measurable.

17. If E is a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < +\infty$ and if $f(x) = m(E \cap (-\infty, x])$ for all $x \in \mathbb{R}$, then show that $f : \mathbb{R} \to \mathbb{R}$ is continuous.

18. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < \infty$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = m\{E \cap (-\infty, x^2)\}$. Show that f is differentiable at 0 and $f'(0) = 0$.

19. Let $E \subset \mathbb{R}$ and $m^*(E) > 0$. Then for each $0 < \alpha < 1$, there exists an open interval I such that $m^*(E \cap I) \geq \alpha m(I)$.

20. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $m(E) < \infty$. Then there exist a sequence of compact set (K_n) contained in E and a set N Lebesgue measure zero such that $E = F \cup N$, where $F = \cup_{n=1}^{\infty} K_n$.

21. Let $E \subset \mathbb{R}$ be Lebesgue measurable and $m(E) < \infty$. Show that for each $\epsilon > 0$, there exist compact set K and open set O with $K \subseteq E \subseteq O$ such that $m(O \setminus K) < \epsilon$.

22. Let $m^*(A) > 0$. Then show that there exists at least one closed set $F \subset \mathbb{R}$ with $m(F) < \infty$ such that $A \cap F \neq \emptyset$.

23. Let μ be a finite measure on $\mathcal{M}(\mathbb{R})$. Suppose for each closed set $F \subset \mathbb{R}$ with $m(F) < \infty$, implies $\mu(F) = 0$. Then show that $\mu = 0$.

24. Let E be a measurable subset of $\mathbb{R}$ with $m(E) < \infty$ and $m\{E \cap (n, n+1)\} < \frac{1}{2^{|n|+2}} m(E)$, for all $n \in \mathbb{Z}$. Show that $m(E) = 0$.

25. Let $\{E_n\}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ such that $\sum\limits_{n=1}^{\infty} m(E_n) < \infty$. Show that $m\left(\bigcap\limits_{n=1}^{\infty} E_n\right) = 0$.

26. Let $A \subset \mathbb{R}$ be a closed set with $m(A) = 0$. Show that A is nowhere dense in $\mathbb{R}$. But does this conclusion hold true when A is not closed?

27. Let $[-1, 1] \cap \mathbb{Q} = \{r_1, r_2, \ldots\}$. For a Lebesgue measurable set $E \subset [0, 1]$ with $m(E) > 0$, define $E_n = E + r_n$; $n \in \mathbb{N}$. Show that all of E_n's can not be pairwise disjoint. Moreover, deduce that there exist $x, y \in E$ such that $x - y \in \mathbb{Q}$.

28. Let $\tilde{M}$ be the class of all Lebesgue measurable subset of $[0, 1]$. If $N \notin \tilde{M}$. Prove/disprove $N \cap (\mathbb{R} \setminus \mathbb{Q}) \in \tilde{M}$.

29. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) = \infty$. Show that there exists a sequence $\{E_n\}$ of pairwise disjoint measurable subsets of E such that $m(E_n) < \infty$, for all n and $E = \bigcup\limits_{n=1}^{\infty} E_n$.

30. Let F be a closed subset of $\mathbb{R}$ with $m(F) = 0$. Then for any $A \subset F$, show that $m^*\{x \in \mathbb{R} : d(x, A) = 0\} = 0$.

31. Let A be a bounded subset of $\mathbb{R}$. Show that $m\left(\overline{A}\right) < \infty$.

32. Let K be a compact subset of $\mathbb{R}$ and $O_n = \left\{x \in \mathbb{R} : d(x, K) < \frac{1}{n}\right\}$. Show that each of O_n is Lebesgue measurable and $\lim_{n\to\infty} m(O_n) = m(K)$.

33. Let (X, S, μ) be a finite measure space. For a sequence of sets $A_n \in S$, if we define $\overline{\lim} A_n = \cap_{k\geq 1}(\cup_{n\geq k} A_n)$, then show that $\mu(\overline{\lim} A_n) \geq \overline{\lim}\, \mu(A_n)$.

Problem Set 3: Measurable functions and approximation

1. State TRUE or FALSE giving proper justification for each of the following statements.

   (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous m-a.e. on $\mathbb{R}$, then there must exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ m-a.e. on $\mathbb{R}$.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and if $g : \mathbb{R} \to \mathbb{R}$ is such that $f = g$ m-a.e. on $\mathbb{R}$, then g must be continuous m-a.e. on $\mathbb{R}$.

   (c) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous such that $f = g$ m-a.e. on $\mathbb{R}$, then it is necessary that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

   (d) An almost everywhere vanishing Lebesgue measurable function need not be continuous.

   (e) There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f = \chi_{[0,1]}$ m-a.e. on $\mathbb{R}$.

   (f) Let $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 1$. Then f is Borel measurable on $\mathbb{R}$.

   (g) For $n \in \mathbb{N}$, define $f_n = \chi_{(n,n+1)}$. Does there exist a measurable set E in $\mathbb{R}$ with $m(E) = \infty$ such that f_n converges to 0 uniformly on E?

   (h) Let $f, g : \mathbb{R} \to [0, \infty)$ be Lebesgue measurable such that $m\{x \in \mathbb{R} : f\,g \neq 0\} = 0$. Does it imply that $\max\{f, g\} = f + g$?

   (i) Let $\text{supp} h = \{x \in \mathbb{R} : h(x) \neq 0\}$. Suppose f, $g : \mathbb{R} \to [0, \infty)$ are such that $\text{supp} f \cap \text{supp} g = \emptyset$. Does it imply that $\max\{f, g\} = f + g$?

2. If $(X, \mathcal{A})$ is a measurable space and $A \subset X$, then show that $\chi_A : X \to \mathbb{R}$ is $\mathcal{A}$-measurable iff A is $\mathcal{A}$-measurable.

3. If $(X, \mathcal{A})$ is a measurable space, then show that $f : X \to [-\infty, +\infty]$ is $\mathcal{A}$-measurable iff $\{x \in X : f(x) > r\} \in \mathcal{A}$ for each $r \in \mathbb{Q}$.

4. Let D be a dense subset of $\mathbb{R}$. Show that $f : \mathbb{R} \to \bar{\mathbb{R}}$ is a Lebesgue measurable function if and only if $\{x \in \mathbb{R} : f(x) > r\}$ is a Lebesgue measurable set for each $r \in D$.

5. Let $f : \mathbb{R} \to [0, \infty]$ be such that $m^*(\{x \in \mathbb{R} : f(x) \geq 2^n\}) < \frac{1}{2^n}$, whenever $n \in \mathbb{N}$. Show that $\{x \in \mathbb{R} : f(x) = \infty\}$ is Lebesgue measurable.

6. Let $f_n,\ f$ be real valued measurable functions on $\mathbb{R}$. Let $E = \{x \in \mathbb{R} : \lim f_n(x) = f(x)\}$. Show that E is Lebesgue measurable.

7. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} f(x) & \text{if } |f(x)| \le 5, \\ 0 & \text{if } |f(x)| > 5. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

8. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} 0 & \text{if } f(x) \in \mathbb{Q}, \\ 1 & \text{if } f(x) \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

9. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} -2 & \text{if } f(x) < -2, \\ f(x) & \text{if } -2 \le f(x) \le 3, \\ 3 & \text{if } f(x) > 3. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

10. Let $f : [0,\ 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$ Find the Lebesgue measure of the set $\{x \in \mathbb{R} : f(x) \ge 0\}$.

11. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then show that $g \circ f$ is $\mathcal{A}$-measurable.

12. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If G is an open subset of $\mathbb{R}^2$, then show that $\{x \in X : (f(x), g(x)) \in G\}$ is $\mathcal{A}$-measurable.

13. If $f : \mathbb{R} \to \mathbb{R}$ is continuous m-a.e. on $\mathbb{R}$, then show that f is Lebesgue measurable.

14. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, then show that $f' : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x,\ .)$ and $f(.,\ y)$ are continuous then f is Lebesgue measurable.

16. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x,\ .)$ is measurable and $f(.,\ y)$ is continuous. Show that f is Lebesgue measurable.

17. Let $f, g : (X, \mathcal{A}) \to \mathbb{R}$. Define $\varphi(x) = (f(x), g(x))$. Then show that f and g are $\mathcal{A}$-measurable if and only if φ is $\mathcal{A}$-measurable.

18. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$ and let $f : X \to \mathbb{R}$ be measurable. Let $A_n = \{x \in X : |f(x)| > n\}$. Show that A_n is $\mathcal{A}$-measurable and $\lim \mu(A_n) = 0$.

19. Let $f : X \to \overline{\mathbb{R}}$ be an almost finite measurable function on a finite measure space (X, S, μ). Let $A_n = \{x \in X : \ |f(x)| > n\}$. Show that $\lim \mu(A_n) = 0$.

20. Let $f : [a, b] \to \mathbb{R}$ be Lebesgue measurable. Let $N = \{x \in [a, b] : f(x) = 0\}$. Show that $g = \chi_N + \frac{1}{f}\chi_{N^c}$ is Lebesgue measurable.

21. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose for each $\epsilon > 0$ there exists an open set O such that $m(O) < \epsilon$ and f is constant on $\mathbb{R} \setminus O$. Show that f is Lebesgue measurable.

22. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous one-one and onto map. Then show that f sends Borel sets onto Borel sets.

23. Let $\mathbb{Q}$ denotes set of rationals. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = \begin{cases} 1 & \text{if } x + y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

    and $g(x, y) = \begin{cases} 1 & \text{if } \frac{x}{y} \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$ Show that f and g are Lebesgue measurable.

24. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Show that $\{x \in \mathbb{R} : f \text{ is continuous at x }\}$ is Lebesgue measurable.

25. Let C be the Cantor ternary set. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in C \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$

    Show that f is Lebesgue measurable. By letting C has a non-Borel measurable subset, construct a Lebesgue measurable function which is not Borel measurable.

26. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and E be Lebesgue measurable $E \subset [a, b]$. Show that $m(E) = 0$, implies $m(f(E)) = 0$ if and only if for every Lebesgue measurable subset $A \subset [a, b]$ the set $f(A)$ is Lebesgue measurable.

27. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sup\{|x + y| : \ y \in [0, 1]\}$. Show that f is Borel measurable.

28. Let $f : (X, S, \mu) \to \mathbb{R}$ be measurable and $\mathcal{B}(\mathbb{R})$ denotes the Borel sigma algebra on $\mathbb{R}$. Define a set function $\mu_f : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ by $\mu_f(B) = \mu\left(f^{-1}(B)\right)$. Show that μ_f is a measure on $\mathcal{B}(\mathbb{R})$.

29. If $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that the function g defined by $g(x) = \inf\{|f(t)| : \ x < t < x+1\}$ is Lebesgue measurable. Does the conclusion hold if f is bounded Lebesgue measurable function?

30. Let $E \subset \mathbb{R}$ with $m(E) < \infty$. Let $f_n : E \to \overline{\mathbb{R}}$ be sequence of Lebesgue measurable functions such that for each $x \in X$, there exists $M_x > 0$ with $|f_n(x)| \leq M_x < \infty, \ \forall\, n \in \mathbb{N}$. Then for each $\epsilon > 0$, there exists a compact set $K \subset E$ such that f_n is uniformly bounded on K, where $m(E \setminus K) < \epsilon$.

31. Let (X, S, μ) be a finite measure space and $f : X \to \bar{\mathbb{R}}$ be an almost finite S-measurable function. show that for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu\{x \in X : |f(x)| > n_0\} < \epsilon$.

32. Let $f : (\mathbb{R}, \mathcal{M}, m) \to [0, \infty]$ be such that for each $\epsilon > 0$ there exists a Lebesgue measurable set $E \subset \mathbb{R}$ with $m(E) < \epsilon$ and f is continuous on $\mathbb{R} \setminus E$. Show that f is a Lebesgue measurable function.

33. Let $E \subset \mathbb{R}$ be Lebesgue measurable and $m(E) = \infty$. Define a function $f : \mathbb{R} \to \bar{\mathbb{R}}$ by $f(x) = m(E \cap (-\infty, x))$. Show that f is a Borel measurable function.

34. Let $g : [0, 1] \to [0, 2]$ be a bijection with $m(g(C)) = 1$, where C is the Cantor set. Construct a Lebesgue measurable function f on $[0, 1]$ such that $f \circ g^{-1}$ is not Lebesgue measurable.

Problem Set 4: Lebesgue integration and convergence theorems

1. State TRUE or FALSE giving proper justification for each of the following statements.

   (a) Whether $L^1(X, S, \mu)$ has an almost non-zero function for every measure space (X, S, μ)?

   (b) Let $f : (X, S, \mu) \to [0, \infty]$ be such that $\|f\|_1 > 0$. Does there exist some $n \in \mathbb{N}$ such that $\mu\{x \in X : |f(x)| < n\} > 0$?

   (c) There exists a Lebesgue measurable function f on $(\mathbb{R}, \mathcal{M}, m)$ such that $\int_E f dm$ is finite for every $E \in \mathcal{M}$ but $f \notin L^1(\mathbb{R}, \mathcal{M}, m)$.

(d) For $n \in \mathbb{N}$, define $f_n = \chi_{(n,n+1)}$. Then there exists a measurable set $E \in \mathcal{M}(\mathbb{R})$ with $m(E) = \infty$ such that f_n converges to 0 uniformly on E.

(e) Suppose $f_n \in L^+(\mathbb{R}, \mathcal{M}, m)$ converges to f point-wise. If $\int_{\mathbb{R}} f_n dm \leq M < \infty$, $\forall\, n \in \mathbb{N}$. Then $\int_{\mathbb{R}} f dm = \lim \int_{\mathbb{R}} f_n dm$.

(f) For $x \in \mathbb{R}$, define $f(x) = \min\left\{1, \frac{1}{x^2}\right\}$. Whether $f \in L^1(\mathbb{R})$?

(g) Suppose (X, S, μ) be a finite measure space on the finite set X. Then $L^1(X, S, \mu)$ is a finite dimensional linear space.

(h) Does there exist a Lebesgue measurable function f on $(\mathbb{R}, \mathcal{M}, m)$ such that $\int\limits_E f$ is finite for every proper Lebesgue measurable set E but $f \notin L^1(\mathbb{R}, \mathcal{M}, m)$?

2. Let μ be the counting measure on the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and let $f : \mathbb{N} \to [0, +\infty]$. Show that $\int\limits_E f\, d\mu = \sum\limits_{n \in E} f(n)$ for every $E \subset \mathbb{N}$ and hence, in particular, $\int\limits_{\mathbb{N}} f\, d\mu = \sum\limits_{n=1}^{\infty} f(n)$.

3. Let δ_x be the Dirac measure at $x \in X$ on the measurable space $(X, \mathcal{P}(X))$. If $f : X \to [0, +\infty]$ and $E \subset X$, then show that $\int\limits_E f\, d\delta_x = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$
(Hence, in particular, $\int\limits_X f\, d\delta_x = f(x)$.)

4. Let μ_n be a sequence of measures on (X, S). For $E \in S$, define $\mu(E) = \sum\limits_{n=1}^{\infty} \mu_n(E)$. If $f \in L^+(X, S, \mu)$, then prove that $\int\limits_X f d\mu = \sum\limits_{n=1}^{\infty} \int\limits_X f d\mu_n$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f = \frac{1}{\sqrt{x}} \chi_{(0,1)}$. Let $g(x) = \sum\limits_{r_n \in \mathbb{Q}} 2^{-n} f(x - r_n)$, then show that the function g belongs to $L^1(\mathbb{R}, \mathcal{M}, m)$.

6. Let $f_n = \chi_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}$. Construct an increasing sequence $\{g_n\}$ of measurable functions on $(\mathbb{R}, M, m,)$ in terms of f_n such that $\lim\limits_{n \to \infty} \int\limits_{\mathbb{R}} g_n dm < \infty$.

7. For each $x \in [0, 1]$, let $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{k}{n} \text{ for some } k, n \in \mathbb{N} \text{ with g.c.d.}(k, n) = 1, \\ 0 & \text{otherwise.} \end{cases}$

Evaluate the Lebesgue integral $\int\limits_{[0,1]} f\, dm$.

8. Let $f, g : (X, S, \mu) \to [0, +\infty]$ be measurable. If $\lambda(E) = \int\limits_E f\, d\mu$ for all $E \in \mathcal{S}$, then show that λ is a measure on $(X, \mathcal{S})$ and that $\int\limits_X g\, d\lambda = \int\limits_X gf\, d\mu$. Does $\lambda(E) = 0$ imply $\mu(E) = 0$?

9. For each $x \in [0,1]$, let $f(x) = \begin{cases} x^2 & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ x^3 & \text{if } x = \frac{1}{3^n} \text{ for some } n \in \mathbb{N}, \\ x^4 & \text{otherwise.} \end{cases}$ Evaluate the Lebesgue integral $\int\limits_{[0,1]} f\,dm$.

10. Let $f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in [0, \frac{1}{2}] \setminus C, \\ \cos(\pi x) & \text{if } x \in (\frac{1}{2}, 1] \setminus C, \\ x^2 & \text{if } x \in C. \end{cases}$ Evaluate the Lebesgue integral $\int\limits_{[0,1]} f\,dm$, where C denotes the Cantor ternary set in $[0,1]$.

11. Evaluate the Lebesgue integrals: (a) $\int\limits_{[0,+\infty)} e^{-[x]}\,dm(x)$ (b) $\int\limits_{(0,1]} \frac{1}{\sqrt[3]{x}}\,dm(x)$

12. Let $f(x) = \begin{cases} e^{|x|} & \text{if } x \in \mathbb{Q}, \\ e^{-|x|} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Evaluate the Lebesgue integral $\int\limits_{\mathbb{R}} f\,dm$.

13. Let $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ \frac{1}{x} & \text{if } x > 1. \end{cases}$ Evaluate the Lebesgue integral $\int\limits_{(0,+\infty)} f\,dm$.

14. Evaluate the following: (a) $\lim\limits_{n\to\infty} \int\limits_{-2}^{2} \frac{x^{2n}}{1+x^{2n}}\,dx$ (b) $\lim\limits_{n\to\infty} \int\limits_{[0,1]} \frac{1+nx}{(1+x)^n}\,dx$ (c) $\int\limits_{0}^{1} (\sum\limits_{n=1}^{\infty} \frac{x^n}{n})\,dx$ (d) $\lim\limits_{n\to\infty} \int\limits_{1}^{\infty} \frac{1}{1+x^{2n}}\,dx$ (e) $\sum\limits_{n=0}^{\infty} \int\limits_{0}^{1} \frac{x^2}{(1+x^2)^n}\,dx$

    (f) $\lim\limits_{n\to\infty} \int\limits_{[0,\infty)} \dfrac{n^2 x e^{-x^2}}{n^2 + x^2}\,dx$

15. Let $f : (X, S, \mu) \to \mathbb{R}$ be measurable. Define a set function $\nu : S \to \overline{\mathbb{R}}$ by $\nu(E) = \int_E f\,d\mu$, whenever $E \in S$. Show that $\nu(X)$ is finite if $f \in L^1(X, S, \mu)$. Does the converse true?

16. For $f \in L^+ \cap L^1(\mathbb{R}, \mathcal{M}, m)$, define $g(x) = \sum\limits_{n=1}^{\infty} f(2^n x + \frac{1}{n})$. Show that $g \in L^1(\mathbb{R}, \mathcal{M}, m)$ and $\int\limits_{\mathbb{R}} g\,dm = \int\limits_{\mathbb{R}} f\,dm$.

17. Construct a function $f \in L^1(\mathbb{R}, \mathcal{M}, m)$ such that $\lim n^2 m\{x \in \mathbb{R} : |f(x)| \geq n\} = \infty$.

18. Let $f \in L(X, S, \mu)$. Suppose there exists an increasing sequence $E_n \in S$ such that $\cup_{n=1}^{\infty} E_n = X$ and $\lim \int_{E_n} |f|\,d\mu < \infty$. Show that $f \in L^1(X, S, \mu)$.

19. Suppose $f_n, f : (X, S, \mu) \to [0, \infty]$ are measurable functions such that f_n converges to f point-wise and $f_n \leq f$. Show that $\int_X f d\mu = \lim \int_X f_n d\mu$.

20. Let $f_n : (X, S, \mu) \to \bar{\mathbb{R}}$ be sequence of measurable functions that f_n increases to f point-wise. If $f, f_n \in L^1(X, S, \mu)$, then show that $\overline{\lim} \int_X f_n d\mu \leq \int_X f d\mu$.

21. Let $f_n : X \to [0, \infty]$ be a sequence of measurable functions and $f_n \to f$ point wise. Suppose there exists $M > 0$ such that $\sup\limits_{n \geq 1} \int_X f_n \leq M$. Show that $f \in L^1(X, S, \mu)$.

22. Let $f \in L^1(X, S, \mu)$. Then show that for each $\epsilon > 0$ there exists $\delta > 0$ and set $E \in S$ such that $\int\limits_E |f| \, d\mu < \epsilon$, whenever $\mu(E) < \delta$.

23. Let $f \in L^1(X, S, \mu)$ be arbitrary and let $E_n = \{x \in X : |f(x)| \geq n\}$. If $0 < p \leq 1$, then show that $\lim\limits_{n \to \infty} n^p \, \mu(E_n) = 0$.

24. Let $f \in L^1(\mathbb{R}, \mathcal{M}, m)$ be such that $\int\limits_I f = 0$, for any open interval $I \subset \mathbb{R}$, then show that $f = 0$.

25. Let $\mu(\mathbb{R}) < \infty$ and $f_n \in L^1(X, S, \mu)$ be such that $f_n \to f$ uniformly. Show that $f \in L^1(X, S, \mu)$ and $\int\limits_X f = \lim \int\limits_X f_n$.

26. Let $f_n : X \to [0, \infty]$ be a decreasing sequence of measurable functions and $f_n \to f$ point wise. If $f_1 \in L^1(X, S, \mu)$. Then show that $\int\limits_X f = \lim \int\limits_X f_n$.

27. Let $f_n, g : X \to \overline{\mathbb{R}}$ be measurable functions such that $f_n \leq g, \ \forall \, n \in \mathbb{N}$ and $g \in L^1(X, S, \mu)$. Show that $\limsup \int\limits_X f_n \leq \int\limits_X \limsup f_n$.

28. Let $f_n : X \to [0, \infty]$ be a sequence of measurable functions and $f_n \to f$ point wise such that $\int\limits_X f = \lim \int\limits_X f_n < \infty$. Show that $\int\limits_E f = \lim \int\limits_E f_n$, for any $E \in S$.

29. Let $f, g, f_n, g_n \in L^1(X, S, \mu)$ be such that $|f_n| \leq g_n, \ f_n \to f$ and $g_n \to g$ point wise. Show that $\int\limits_X g = \lim \int\limits_X g_n$ implies $\int\limits_X f = \lim \int\limits_X f_n$.

30. Let $f_n, f \in L^1(X, S, \mu)$ be such that $f_n \to f$ point wise. Prove that $\lim \int\limits_X |f_n - f| = 0$ if and only if $\int\limits_X |f| = \lim \int\limits_X |f_n|$.

31. Let $|f_n| \le g \in L^1(\mathbb{R})$. Let f_{n_k} be subsequence of f_n such that $f_{n_k} \to f$ point wise a.e. on $\mathbb{R}$. If $\lim\limits_{k\to\infty} \|f_{n_k} - f\| = \overline{\lim\limits_{n}} \|f_n - f\|_1 < \infty$. Show that $f_n \to f$ in $L^1(\mathbb{R})$.

32. $f : X \to [0, \infty]$ be a measurable function. Show that f is integrable on (X, S, μ) if and only if $\sum\limits_{n=-\infty}^{\infty} 2^n \mu\{x \in X : 2^n \le f(x) \le 2^{n+1}\} < \infty$.

33. Let $\mu(X) < \infty$ and $f : X \to [0, \infty]$ be a measurable function. Show that $f \in L^1(X, S, \mu)$ if and only if $\sum\limits_{n=0}^{\infty} \mu\{x \in X : f(x) \ge n\} < \infty$.

Problem Set 5: Lp-spaces and product measures

1. State TRUE or FALSE giving proper justification for each of the following statements.

   (a) $L^\infty(X, S, \mu)$ contains an almost non-zero function for every measure space (X, S, μ).

   (b) If $f : (X, S, \mu) \to \mathbb{R}$ is bounded almost everywhere, then f is measurable.

   (c) If for $1 \le p < \infty$, $L^\infty(X, S, \mu) \subset L^p(X, S, \mu)$, then μ is a finite measure.

   (d) For $f \in L^\infty(X, S, \mu)$, it is necessary that $\mu\{x \in X : |f(x)| = \|f\|_\infty\} = 0$.

   (e) Let $\mathcal{S}(\mathbb{R})$ be the space of all continuous functions f on $\mathbb{R}$ such that $|x|^\alpha f(x)$ is bounded, for any $\alpha \in \mathbb{N} \cup \{0\}$. Then $\mathcal{S}(\mathbb{R})$ is dense $L^2(\mathbb{R})$.

   (f) Let (X, S, μ) be a σ- finite measure space with $\mu(\{x\}) = 0$ for all $x \in X$. Is it possible that $(\mu \times \mu)(\{(x, y) \in X \times X : x = y\}) > 0$?

   (g) Let $F(x, y) = f(x)g(y)$, where $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. Does it imply that F is finite a.e. $m \times m$?

   (h) The set $\{(x, y) \in \mathbb{R}^2 : \ y = \sin \frac{1}{x}\}$ belongs to $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

2. For $1 \le p < \infty$ and $f \in L^p(X, S, \mu)$ and $\alpha > 0$ show that $\mu\{x \in X : |f(x)| \ge \alpha\} \le \left(\frac{\|f\|_p}{\alpha}\right)^p$. Moreover, for $1 < p < \infty$, show that $\sum\limits_{n=1}^{\infty} \mu\{x \in X : |f(x)| \ge n\}$ is convergent.

3. Let $1 \le p < \infty$ and $f \in L^+(X, S, \mu) \cap L^p(X, S, \mu)$. Define $f_n(x) = \min\{n, f(x)\}$. Then show that f_n increases to f point wise a.e. and $\lim\limits_{n\to\infty} \int\limits_X |f_n - f|^p d\mu = 0$.

4. Suppose $f_n \to f$ in $L^p(\mathbb{R})$ for $1 \le p < \infty$. Let $g_n \in L^\infty(\mathbb{R})$ and $\|g_n\| \le 1$. If g_n converges to g uniformly a.e., then $f_n g_n \to fg$ in $L^p(\mathbb{R})$.

5. Suppose $f_n \in L^p(X, S, \mu)$, for $1 \le p < \infty$, with $\|f_n\|_p \le 1$ and $f_n \to f$ point-wise a.e. Show that $f \in L^p(X, S, \mu)$ and $\|f\|_p \le 1$.

6. Let (X, S, μ) be a σ- finite measure space. Suppose for each $\epsilon > 0$ there exists some $p > 1$ such that $\|f\|_p < \epsilon$ for every $f \in L^p(X, S, \mu)$. Show that $\mu = 0$.

7. Let (X, S, μ) be a measure space and $0 < p < 1$. Then for $f, g \in L^+ \cap L^p(X, S, \mu)$ show that $\|f + g\|_p \ge \|f\|_p + \|g\|_p$.

8. Let $\{E_n\}$ be sequence of disjoint measurable sets. Show that $\sum\limits_{n=1}^{\infty} \alpha_i \chi_{E_i} \in L^p(X, S, \mu)$ if and only if $\sum\limits_{n=1}^{\infty} |\alpha_i|^p \mu(E_i) < \infty$.

9. Let f and g be disjointly supported functions in $L^p(X, S, \mu)$. Prove that $\|f + g\|_p^p = \|f\|_p^p + \|g\|_p^p$.

10. Let $1 \le p < \infty$ $f \in L^p(\mathbb{R}, \mathcal{M}, m)$. Then show that $\|f(x+h) - f(x)\|_p \to 0$ as $|h| \to 0$.

11. For $1 < p < \infty$, prove that $L^1(\mathbb{R}, \mathcal{M}, m) \cap L^p(\mathbb{R}, \mathcal{M}, m)$ is a proper dense subspace of $L^p(\mathbb{R}, \mathcal{M}, m)$.

12. Let $1 \le p,\ q \le \infty$ and $p^{-1} + q^{-1} = r^{-1}$. If $f \in L^p(X, S, \mu)$ and $g \in L^q(X, S, \mu)$, then prove that $fg \in L^1(X, S, \mu)$ and $\|fg\|_r \le \|f\|_p \|g\|_q$. (A generalized Holder's inequality.)

13. Let $1 \le p < q < r \le \infty$. Then prove that $L^q(X, S, \mu) \subset L^p(X, S, \mu) + L^r(X, S, \mu)$.

14. Let $1 \le p < q < r \le \infty$. Show that $L^p(X, S, \mu) \cap L^r(X, S, \mu) \subset L^q(X, S, \mu)$ and $\|f\|_q \le \|f\|_p^\lambda \|f\|_r^{1-\lambda}$, where $\lambda \in (0, 1)$ is given by $q^{-1} = \lambda p^{-1} + (1 - \lambda) r^{-1}$.

15. Let $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$. For $f \in L^p(X, S, \mu)$, prove that
$$\|f\|_p = \sup \left\{ \left| \int_X fg d\mu \right| : g \in L^q(X, S, \mu) \text{ and } \|g\|_q = 1 \right\}.$$

16. Let (X, S, μ) be a σ-finite measure space. Then show that $\|f\|_\infty = \sup\limits_{\|g\|_1 = 1} \left| \int\limits_X fg d\mu \right|$.

17. Let $\mathcal{A}$ be the monotone class generated by all closed sets in $\mathbb{R}$. If E and F are closed subsets $\mathbb{R}$, then show that $E+F$ belongs to $\mathcal{A}$.

18. Let P be a polynomial on $\mathbb{R}^2$. Show that $S=\left\{(x,y)\in\mathbb{R}^2 : P(x,y)=1\right\}\in\mathcal{M}(\mathbb{R})\otimes\mathcal{M}(\mathbb{R})$. Compute $m\times m\left(S\right)$.

19. Let $f:\left(\mathbb{R}^2,\mathcal{M}\otimes\mathcal{M},m\times m\right)\to\overline{\mathbb{R}}$ be a measurable function. If either of f^+ or f^- belongs to $L^1\left(\mathbb{R}^2,\mathcal{M}\otimes\mathcal{M},m\times m\right)$, then show that $\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f\,dm\,dm=\int\limits_{\mathbb{R}^2} f\,d(m\times m)$.

20. Let $f:(X,S,\mu)\to\mathbb{R}$ be measurable. Show that $G_f=\{(x,y)\in X\times\mathbb{R},\ y=f(x)\}\in S\otimes\mathcal{B}(\mathbb{R})$. If $(X,S,\mu)=(\mathbb{R},\mathcal{M},m)$, then show that $m\times m(G_f)=0$.

21. Let(X,S,μ) be a σ-finite measure space. Let $f:(X,S,\mu)\to[0,\infty]$ be measurable. Show that $A_f=\{(x,y)\in X\times[0,\infty],\ y\le f(x)\}\in S\otimes\mathcal{B}(\mathbb{R})$ and $\mu\times m(A_f)=\int_X f(x)d\mu(x)$.

22. Let (X,S,μ) be a finite measure space and $f:X\to[1,\infty]$ be a measurable function. Compute $\mu\times m\left\{(x,y)\in X\times\mathbb{R}:\ y<f(x)\right\}$.

23. Show that $\mathbb{D}=\left\{(x,y)\in\mathbb{R}^2:\ y\ge x^2 \text{ and } y\le 1\right\}\in\mathcal{M}(\mathbb{R})\otimes\mathcal{M}(\mathbb{R})$. Find $m\times m(\mathbb{D})$.

24. Let $f\in L^1(X,S,\mu)$ and $g\in L^1(Y,T,\nu)$. Define $\varphi(x,y)=f(x)g(y)$. Show that φ is measurable and $\varphi\in L^1(X\times Y,S\otimes T,\mu\times\nu)$.

25. Let $E,F\in\mathcal{M}(\mathbb{R})$ and $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=\chi_E(x)\chi_F(x-y)$. Then show that f is $\mathcal{M}(\mathbb{R})\otimes\mathcal{M}(\mathbb{R})$-measurable and $\int\limits_{\mathbb{R}^2} f\,d(m\times m)=m(E)m(F)$.

26. For $E,F\in\mathcal{M}(\mathbb{R})$, define $h(y)=\int\limits_{\mathbb{R}}\chi_E(x)\chi_F(x-y)dx$. Show that h is a Borel measurable function on $\mathbb{R}$.

27. Let $X=Y=[0,1],\ S=T=\mathcal{B}[0,1]$ and $\mu=\nu=m$. Define $f:[0,1]\times[0,1]\to\mathbb{R}$ by

$$f(x,y)=\begin{cases}1 & \text{if } x\in\mathbb{Q},\\ 2y & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}.\end{cases}$$

Compute $\int\limits_0^1\int\limits_0^1 f(x,y)dydx$ and $\int\limits_0^1\int\limits_0^1 f(x,y)dxdy$. Whether $f\in L^1(m\times m)$?

28. Let $f(x,y) = e^{-xy}\sin x$ and $D = [0,\infty)\times[1,\infty)$. Show that $f\chi_D \in L^1(\mathbb{R}^2, \mathcal{M}\otimes\mathcal{M}, m\times m)$ and $\int\limits_0^\infty\int\limits_1^\infty f(x,y)dydx = \int\limits_1^\infty\int\limits_0^\infty f(x,y)dxdy$.

29. Let $f(x,y) = e^{-xy} - 2e^{-2xy}$ and $D = [0,1]\times[1,\infty)$. Show that $f\chi_D \notin L^1(\mathbb{R}^2, \mathcal{M}\otimes\mathcal{M}, m\times m)$.

30. Let $f \in L^1(0,a)$ and define $g(x) = \int_x^a \frac{f(t)}{t}dt$. Then show that $g \in L^1(0,a)$ and compute $\int_0^a g(x)dx$.

31. For $f \in L^1(\mathbb{R}, \mathcal{M}, m)$, define $F(x) = \int\limits_0^x f(t)dt$. Show that $F \in L^1([0,1], M, m)$ and deduce that $\|F\|_1 \le \|f\|_1$.

32. Let $f \in L^1(\mathbb{R}, \mathcal{M}, m)$. If $\varphi(x,y) = \frac{f(x+y)}{1+y^2}$, then show that φ is $\mathcal{M}\otimes\mathcal{M}$-measurable, and $\varphi \in L^1\left(\mathbb{R}^2, \mathcal{M}\otimes\mathcal{M}, m\times m\right)$.

33. Let $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ be defined by $T(f)(x) = \int\limits_{\mathbb{R}} \frac{f(x+y)}{1+y^2}dy$. Show that T is bounded and satisfies $\|T\| = \pi$.

34. Define a linear functional on $L^1(\mathbb{R}, \mathcal{M}, m)$ by $T(f) = \int\limits_{\mathbb{R}} \frac{f(x)}{1+|x|}$. Show that T is bounded and verifies $\|T\| \le 1$.

Problem Set 6: Signed measures and the Radon–Nikodym theorem

1. State TRUE or FALSE, with justification.

   (a) If ν is a signed measure on (X,Σ), then its total variation $|\nu|$ is always finite.

   (b) If $\nu \ll \mu$ and $\mu(E) = 0$, then $\nu(E) = 0$.

   (c) If $\nu \ll \mu$ and $\mu \ll \nu$, then the Radon–Nikodym derivative $\frac{d\nu}{d\mu}$ is bounded away from 0 and ∞.

   (d) If $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

   (e) If $\nu_1, \nu_2 \ll \mu$, then $\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$ μ-a.e.

2. (Hahn decomposition) Let ν be a signed measure on (X,Σ). Show that there exist disjoint measurable sets P, N with $P \cup N = X$ such that $\nu(E) \ge 0$ for all $E \subset P$ measurable and $\nu(E) \le 0$ for all $E \subset N$ measurable. Show that P, N are unique up to ν-null sets.

3. (Jordan decomposition) With ν as above, define

$$\nu^+(E) = \nu(E \cap P), \qquad \nu^-(E) = -\nu(E \cap N).$$

Show that $\nu^\pm$ are mutually singular finite measures on every set of finite $|\nu|$-measure, $\nu = \nu^+ - \nu^-$, and that this decomposition is unique.

4. (Total variation) Define $|\nu| = \nu^+ + \nu^-$. Prove that for every $E \in \Sigma$,

$$|\nu|(E) = \sup\Big\{\sum_{k=1}^{\infty} |\nu(E_k)| : \ E = \dot{\bigcup_{k=1}^{\infty}} E_k, \ E_k \in \Sigma\Big\}.$$

Deduce that $|\nu|$ is a measure and $|\nu(E)| \le |\nu|(E)$ for all E.

5. (Radon–Nikodym) Let (X, Σ, μ) be σ-finite and let ν be a signed measure with $\nu \ll \mu$. Prove that there exists $f \in L^1_{\text{loc}}(\mu)$ such that $\nu(E) = \int_E f \, d\mu$ for all $E \in \Sigma$, and that f is unique μ-a.e.

6. (Concrete RN derivatives)

   (a) On $(\mathbb{R}, \mathcal{M}, m)$ let $\nu(E) = \int_E e^{-|x|} \, dm(x)$. Compute $\frac{d\nu}{dm}$.

   (b) Let μ be counting measure on $\mathbb{N}$ and $\nu(\{n\}) = 2^{-n}$. Compute $\frac{d\nu}{d\mu}$.

7. (Lebesgue decomposition) Let μ, ν be σ-finite measures on (X, Σ). Show that there exist unique measures $\nu_{\text{ac}}, \nu_{\text{s}}$ such that $\nu = \nu_{\text{ac}} + \nu_{\text{s}}$, with $\nu_{\text{ac}} \ll \mu$ and $\nu_{\text{s}} \perp \mu$.

8. (Absolute continuity via total variation) Let ν be a signed measure and μ a measure. Show that $\nu \ll \mu$ if and only if $|\nu| \ll \mu$.

9. (Duality, challenge) Let (X, Σ, μ) be σ-finite and $1 < p < \infty$ with $1/p + 1/q = 1$. Show that for $g \in L^q(X, \mu)$ the functional $T_g(f) = \int_X fg \, d\mu$ is bounded on L^p and $\|T_g\| = \|g\|_q$. Conversely, show that every bounded linear functional on L^p arises this way.

Problem Set 7: Absolute continuity and differentiation of integrals

1. State TRUE or FALSE, with justification.

   (a) Every function of bounded variation on $[a, b]$ is absolutely continuous.

   (b) If $F(x) = \int_a^x f(t) \, dt$ with $f \in L^1([a, b])$, then F is absolutely continuous.

   (c) If F is absolutely continuous on $[a, b]$, then F' exists everywhere on $[a, b]$.

   (d) If $f \in L^1_{\text{loc}}(\mathbb{R})$, then $\displaystyle\lim_{r \downarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} f(t) \, dt = f(x)$ for every x.

2. (Absolute continuity $\Longrightarrow$ integral representation) Let $F : [a, b] \to \mathbb{R}$ be absolutely continuous. Show that $F' \in L^1([a, b])$ and

$$F(x) = F(a) + \int_a^x F'(t) \, dt \qquad \text{for all } x \in [a, b].$$

3. (Integral representation $\Longrightarrow$ absolute continuity) Let $f \in L^1([a,b])$ and define $F(x) = \int_a^x f(t)\,dt$. Prove that F is absolutely continuous and $F'(x) = f(x)$ for a.e. $x \in [a,b]$.

4. (Fundamental theorem of calculus, Lebesgue version) Assume F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b])$, and F is absolutely continuous. Show that $\int_a^b F'(t)\,dt = F(b) - F(a)$.

5. (Lebesgue differentiation theorem, core case) Let $f \in L^1_{\text{loc}}(\mathbb{R})$. Prove that for a.e. $x \in \mathbb{R}$,

$$\lim_{r \downarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} f(t)\,dt = f(x).$$

(Hint: reduce to characteristic functions of measurable sets and use density points.)

6. (Density points) Let $E \subset \mathbb{R}$ be Lebesgue measurable. Show that for a.e. $x \in E$,

$$\lim_{r \downarrow 0} \frac{m(E \cap (x-r, x+r))}{2r} = 1,$$

and for a.e. $x \notin E$ the same limit is 0.

7. (A.e. differentiability of indefinite integrals) Let $f \in L^1([a,b])$ and $F(x) = \int_a^x f$. Show that F is differentiable a.e. and $F' = f$ a.e.

8. (Vitali-type control, challenge) Let F be absolutely continuous on $[a,b]$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite disjoint family of intervals $\{(a_k, b_k)\}$ in $[a,b]$ with $\sum_k (b_k - a_k) < \delta$ we have $\sum_k |F(b_k) - F(a_k)| < \varepsilon$.

Problem Set 8: Regular Borel measures and the Riesz representation theorem

1. State TRUE or FALSE, with justification.

   (a) Every finite Borel measure on a compact metric space is automatically regular.

   (b) If μ is a regular Borel measure on K and $E \subset K$ is Borel, then for every $\varepsilon > 0$ there exist compact $F \subset E$ and open $O \supset E$ such that $\mu(O \setminus F) < \varepsilon$.

   (c) If $L : C(K) \to \mathbb{R}$ is a positive linear functional, then $\|L\| = L(1)$.

2. (Regularity approximation) Let K be a compact metric space and μ a finite regular Borel measure on K. Prove that for every Borel set $E \subset K$ and $\varepsilon > 0$ there exist compact $F \subset E$ and open $O \supset E$ such that $\mu(O \setminus F) < \varepsilon$.

3. (Urysohn functions) Let K be a compact metric space and let $F \subset O \subset K$ with F closed and O open. Construct a continuous function $\varphi : K \to [0,1]$ such that $\varphi = 1$ on F and $\varphi = 0$ on $K \setminus O$.

4. (Dirac measures) For $x_0 \in K$ define $\delta_{x_0}(E) = \mathbf{1}_E(x_0)$. Show that δ_{x_0} is a regular Borel probability measure. Compute the corresponding functional $L_{x_0}(f) = \int_K f \, d\delta_{x_0}$ on $C(K)$.

5. (Riesz representation, reconstruction step) Let $L : C(K) \to \mathbb{R}$ be a positive linear functional and define for open $U \subset K$

$$\mu_0(U) = \sup\{L(\varphi) : \ \varphi \in C(K), \ 0 \le \varphi \le 1, \ \operatorname{supp}(\varphi) \subset U\}.$$

   Show that μ_0 is finitely additive on disjoint open sets and extends to a finite regular Borel measure μ on K.

6. (Support of a measure) Let μ be a finite regular Borel measure on K. Define the support $\operatorname{supp}(\mu)$ as the complement of the largest open set of μ-measure zero. Show that $\operatorname{supp}(\mu)$ is closed and $\mu(K \setminus \operatorname{supp}(\mu)) = 0$.

7. (Uniqueness) Assume μ, ν are finite regular Borel measures on K and $\int f \, d\mu = \int f \, d\nu$ for all $f \in C(K)$. Show that $\mu = \nu$ on $\mathcal{B}(K)$.

Bibliography

[1] D. L. Cohn, *Measure Theory*, Birkhäuser, Boston, 1994.

[2] G. de Barra, *Measure Theory and Integration*, New Age International, New Delhi, 1981.

[3] M. Capiński and E. Kopp, *Measure, Integral and Probability*, 2nd ed., Springer, London, 2007.

[4] H. L. Royden, *Real Analysis*, 3rd ed., Prentice Hall, Englewood Cliffs, NJ, 1988.

[5] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw–Hill, New York, 1987. (Reprint: McGraw–Hill Education (India), 2017.)

[6] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., John Wiley & Sons, New York, 1999.

[7] N. L. Carothers, *Real Analysis*, Cambridge University Press, Cambridge, 2000.

[8] P. R. Halmos, *Measure Theory*, Springer, New York, 1974.

[9] V. I. Bogachev, *Measure Theory*, Vols. I–II, Springer, Berlin, 2007.

[10] P. Billingsley, *Probability and Measure*, 3rd ed., John Wiley & Sons, New York, 1995.