MA550: Measure Theory

(Assignment 5: L^p-spaces and product measures)
January - April, 2024

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) $L^\infty(X, S, \mu)$ contains an almost non-zero function for every measure space (X, S, μ).
   (b) If $f : (X, S, \mu) \to \mathbb{R}$ is bounded almost everywhere, then f is measurable.
   (c) If for $1 \leq p < \infty$, $L^\infty(X, S, \mu) \subset L^p(X, S, \mu)$, then μ is a finite measure.
   (d) For $f \in L^\infty(X, S, \mu)$, it is necessary that $\mu\{x \in X : |f(x)| = \|f\|_\infty\} = 0$.
   (e) Let $\mathcal{S}(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$ such that $|x|^\alpha f(x)$ is bounded, for any $\alpha \in \mathbb{N}$. Then $\mathcal{S}(\mathbb{R})$ is dense $L^2(\mathbb{R})$.
   (f) Let (X, S, μ) be a σ- finite measure space with $\mu(\{x\}) = 0$ for all $x \in X$. Is it possible that $(\mu \times \mu)(\{(x, y) \in X \times X : x = y\}) > 0$?
   (g) Let $F(x, y) = f(x)f(y)$, where $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. Does it imply that F is finite a.e. $m \times m$?
   (h) The set $\{(x, y) \in \mathbb{R}^2 : \; y = \sin \frac{1}{x}\}$ belongs to $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.
2. For $1 \leq p < \infty$ and $f \in L^p(X, S, \mu)$ and $\alpha > 0$ show that $\mu\left\{x \in X : |f(x)| \geq \alpha\right\} \leq \left(\frac{\|f\|_p}{\alpha}\right)^p$. Further, for $1 < p < \infty$, show that $\sum\limits_{n=1}^{\infty} \mu\{x \in X : |f(x)| \geq n\}$ is convergent.
3. Let $1 \leq p < \infty$ and $f \in L^+(X, S, \mu) \cap L^p(X, S, \mu)$. Define $f_n(x) = \min\left\{n, f(x)\right\}$. Then show that f_n increases to f point wise a.e. and $\lim\limits_{n\to\infty} \int\limits_X |f_n - f|^p d\mu = 0$.
4. Suppose $f_n \to f$ in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. Let $g_n \in L^\infty(\mathbb{R})$ and $\|g_n\| \leq 1$. If g_n converges to g uniformly a.e., then $f_n g_n \to fg$ in $L^p(\mathbb{R})$.
5. Suppose $f_n \in L^p(X, S, \mu)$, for $1 \leq p < \infty$, with $\|f_n\|_p \leq 1$ and $f_n \to f$ point-wise a.e. Show that $f \in L^p(X, S, \mu)$ and $\|f\|_p \leq 1$.
6. Let (X, S, μ) be a σ- finite measure space. Suppose for each $\epsilon > 0$ there exists some $p > 1$ such that $\|f\|_p < \epsilon$ for every $f \in L^p(X, S, \mu)$. Show that $\mu = 0$.
7. Let (X, S, μ) be a measure space and $0 < p < 1$. Then for $f, g \in L^+ \cap L^p(X, S, \mu)$ show that $\|f + g\|_p \geq \|f\|_p + \|g\|_p$.
8. Let $\{E_n\}$ be sequence of disjoint measurable sets. Show that $\sum\limits_{n=1}^{\infty} \alpha_i \chi_{E_i} \in L^p(X, S, \mu)$ if and only if $\sum\limits_{n=1}^{\infty} |\alpha_i|^p \mu(E_i) < \infty$.
9. Let f and g be disjointly supported functions in $L^p(X, S, \mu)$. Prove that $\|f+g\|_p^p = \|f\|_p^p + \|g\|_p^p$.
10. Let $1 \leq p < \infty$ $f \in L^p(\mathbb{R}, M, m)$. Then show that $\|f(x + h) - f(x)\|_p \to 0$ as $|h| \to 0$.
11. For $1 < p < \infty$, prove that $L^1(\mathbb{R}, M, m) \cap L^p(\mathbb{R}, M, m)$ is a proper dense subspace of $L^p(\mathbb{R}, M, m)$.
12. Let $1 \leq p, \, q \leq \infty$ and $p^{-1} + q^{-1} = r^{-1}$. If $f \in L^p(X, S, \mu)$ and $g \in L^q(X, S, \mu)$, then prove that $fg \in L^1(X, S, \mu)$ and $\|fg\|_r \leq \|f\|_p \|g\|_q$. (A generalized Holder's inequality.)
13. Let $1 \leq p < q < r \leq \infty$. Then prove that $L^q(X, S, \mu) \subset L^p(X, S, \mu) + L^r(X, S, \mu)$.

14. Let $1 \leq p < q < r \leq \infty$. Show that $L^p(X,S,\mu) \cap L^r(X,S,\mu) \subset L^q(X,S,\mu)$ and $\|f\|_q \leq \|f\|_p^{\lambda}\|f\|_r^{1-\lambda}$, where $\lambda \in (0,1)$ is given by $q^{-1} = \lambda p^{-1} + (1-\lambda)r^{-1}$.

15. Let $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$. For $f \in L^p(X,S,\mu)$, prove that

$$\|f\|_p = \sup\left\{\left|\int_X fg d\mu\right| : g \in L^q(X,S,\mu) \text{ and } \|g\|_q = 1\right\}.$$

16. Let (X,S,μ) be a σ-finite measure space. Then show that $\|f\|_\infty = \sup\limits_{\|g\|_1=1}\left|\int\limits_X fg d\mu\right|$.

17. Let $\mathcal{A}$ be the monotone class generated by all closed sets in $\mathbb{R}$. If E and F are closed subsets $\mathbb{R}$, then show that $E+F$ belongs to $\mathcal{A}$.

18. Let P be a polynomial on $\mathbb{R}^2$. Show that $S = \{(x,y) \in \mathbb{R}^2 : P(x,y) = 1\} \in M(\mathbb{R}) \otimes M(\mathbb{R})$. Compute $m \times m\,(S)$.

19. Let $f : (\mathbb{R}^2, M \otimes M, m \times m) \to \overline{\mathbb{R}}$ be a measurable function. If either of f^+ or f^- belongs to $L^1(\mathbb{R}^2, M \otimes M, m \times m)$, then show that $\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} f\,dm\,dm = \int\limits_{\mathbb{R}^2} f\,d(m \times m)$.

20. Let $f : (X,S,\mu) \to \mathbb{R}$ be measurable. Show that $G_f = \{(x,y) \in X \times \mathbb{R},\ y = f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$. If $(X,S,\mu) = (\mathbb{R}, M, m)$, then show that $m \times m(G_f) = 0$.

21. Let(X,S,μ) be a σ-finite measure space. Let $f : (X,S,\mu) \to [0,\infty]$ be measurable. Show that $A_f = \{(x,y) \in X \times [0,\infty],\ y \leq f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$ and $\mu \times m(A_f) = \int_X f(x)d\mu(x)$.

22. Let (X,S,μ) be a finite measure space and $f : X \to [1,\infty]$ be a measurable function. Compute $\mu \times m\{(x,y) \in X \times \mathbb{R} :\ y < f(x)\}$.

23. Show that $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 :\ y \geq x^2 \text{ and } y \leq 1\} \in M(\mathbb{R}) \otimes M(\mathbb{R})$. Find $m \times m(\mathbb{D})$.

24. Let $f \in L^1(X,S,\mu)$ and $g \in L^1(Y,T,\nu)$. Define $\varphi(x,y) = f(x)g(y)$. Show that φ is measurable and $\varphi \in L^1(X \times Y, S \otimes T, \mu \times \nu)$.

25. Let $E, F \in M(\mathbb{R})$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \chi_E(x)\chi_F(x-y)$. Then show that f is $M(\mathbb{R}) \otimes M(\mathbb{R})$-measurable and $\int\limits_{\mathbb{R}^2} f\,d(m \times m) = m(E)m(F)$.

26. For $E, F \in M(\mathbb{R})$, define $h(y) = \int\limits_{\mathbb{R}} \chi_E(x)\chi_F(x-y)dx$. Show that h is a Borel measurable function on $\mathbb{R}$.

27. Let $X = Y = [0,1]$, $S = T = \mathcal{B}[0,1]$ and $\mu = \nu = m$. Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 2y & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Compute $\int\limits_0^1\int\limits_0^1 f(x,y)dydx$ and $\int\limits_0^1\int\limits_0^1 f(x,y)dxdy$. Whether $f \in L^1(m \times m)$?

28. Let $f(x,y) = e^{-xy}\sin x$ and $D = [0,\infty) \times [1,\infty)$. Show that $f\chi_D \in L^1(\mathbb{R}^2, M \otimes M, m \times m)$ and $\int\limits_0^\infty\int\limits_1^\infty f(x,y)dydx = \int\limits_1^\infty\int\limits_0^\infty f(x,y)dxdy$.

29. Let $f(x,y) = e^{-xy} - 2e^{-2xy}$ and $D = [0,1] \times [1,\infty)$. Show that $f\chi_D \notin L^1(\mathbb{R}^2, M \otimes M, m \times m)$.

30. Let $f \in L^1(0,a)$ and define $g(x) = \int_x^a \frac{f(t)}{t}dt$. Then show that $g \in L^1(0,a)$ and compute $\int_0^a g(x)dx$.

31. For $f \in L^1(\mathbb{R}, M, m)$, define $F(x) = \int\limits_0^x f(t)dt$. Show that $F \in L^1([0,1], M, m)$ and deduce that $\|F\|_1 \leq \|f\|_1$.

32. Let $f \in L^1(\mathbb{R}, M, m)$. If $\varphi(x,y) = \frac{f(x+y)}{1+y^2}$, then show that φ is $M \otimes M$-measurable, and $\varphi \in L^1\left(\mathbb{R}^2, M \otimes M, m \times m\right)$.

33. Let $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ be defined by $T(f)(x) = \int\limits_{\mathbb{R}} \frac{f(x+y)}{1+y^2} dy$. Show that T is bounded and satisfies $\|T\| = \pi$.

34. Define a linear functional on $L^1(\mathbb{R}, M, m)$ by $T(f) = \int\limits_{\mathbb{R}} \frac{f(x)}{1+|x|}$. Show that T is bounded and verifies $\|T\| = 1$.