

MA550: Measure Theory

(Assignment 4: Integration)

January - April, 2024

- State TRUE or FALSE giving proper justification for each of the following statements.
 - Whether $L^1(X, S, \mu)$ has an almost non-zero function for every measure space (X, S, μ) ?
 - Let $f : (X, S, \mu) \rightarrow [0, \infty]$ be such that $\|f\|_1 > 0$. Does there exist some $n \in \mathbb{N}$ such that $\mu\{x \in X : |f(x)| < n\} > 0$?
 - There exists a Lebesgue measurable function f on (\mathbb{R}, M, m) such that $\int_E f dm$ is finite for every $E \in M$ but $f \notin L^1(\mathbb{R}, M, m)$.
 - For $n \in \mathbb{N}$, define $f_n = \chi_{(n, n+1)}$. Then there exists a measurable set $E \in M(\mathbb{R})$ with $m(E) = \infty$ such that f_n converges to 0 uniformly on E .
 - Suppose $f_n \in L^+(\mathbb{R}, M, m)$ converges to f point-wise. If $\int_{\mathbb{R}} f_n dm \leq M < \infty, \forall n \leq 1$. Then $\int_{\mathbb{R}} f dm = \lim \int_{\mathbb{R}} f_n dm$.
 - For $x \in \mathbb{R}$, define $f(x) = \min\{1, \frac{1}{x^2}\}$. Whether $f \in L^1(\mathbb{R})$?
 - Suppose (X, S, μ) be a finite measure space on the finite set X . Then $L^1(X, S, \mu)$ is a finite dimensional linear space.
 - Does there exist a Lebesgue measurable function f on (\mathbb{R}, M, m) such that $\int_E f$ is finite for every proper Lebesgue measurable set E but $f \notin L^1(\mathbb{R}, M, m)$?
- Let μ be the counting measure on the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and let $f : \mathbb{N} \rightarrow [0, +\infty]$. Show that $\int_E f d\mu = \sum_{n \in E} f(n)$ for every $E \subset \mathbb{N}$ and hence, in particular, $\int_{\mathbb{N}} f d\mu = \sum_{n=1}^{\infty} f(n)$.
- Let δ_x be the Dirac measure at $x \in X$ on the measurable space $(X, \mathcal{P}(X))$. If $f : X \rightarrow [0, +\infty]$ and $E \subset X$, then show that $\int_E f d\delta_x = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$
(Hence, in particular, $\int_X f d\delta_x = f(x)$.)
- Let μ_n be a sequence of measures on (X, S) . For $E \in S$, define $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E)$. If $f \in L^+(X, S, \mu)$, then prove that $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f d\mu_n$.
- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f = \frac{1}{\sqrt{x}}\chi_{(0,1)}$. Let $g(x) = \sum_{r_n \in \mathbb{Q}} 2^{-n} f(x - r_n)$, then show that the function g belongs to $L^1(\mathbb{R}, M, m)$.
- Let $f_n = \chi_{[\frac{1}{n+1}, \frac{1}{n}]}$. Construct an increasing sequence $\{g_n\}$ of measurable functions on (\mathbb{R}, M, m) in terms of f_n such that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} g_n dm < \infty$.
- For each $x \in [0, 1]$, let $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{k}{n} \text{ for some } k, n \in \mathbb{N} \text{ with g.c.d.}(k, n) = 1, \\ 0 & \text{otherwise.} \end{cases}$
Evaluate the Lebesgue integral $\int_{[0,1]} f dm$.
- Let $f, g : (X, S, \mu) \rightarrow [0, +\infty]$ be measurable. If $\lambda(E) = \int_E f d\mu$ for all $E \in S$, then show that λ is a measure on (X, S) and that $\int_X g d\lambda = \int_X gf d\mu$. Does $\lambda(E) = 0$ imply $\mu(E) = 0$?

9. For each $x \in [0, 1]$, let $f(x) = \begin{cases} x^2 & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ x^3 & \text{if } x = \frac{1}{3^n} \text{ for some } n \in \mathbb{N}, \\ x^4 & \text{otherwise.} \end{cases}$

Evaluate the Lebesgue integral $\int_{[0,1]} f \, dm$.

10. Let $f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in [0, \frac{1}{2}] \setminus C, \\ \cos(\pi x) & \text{if } x \in (\frac{1}{2}, 1] \setminus C, \\ x^2 & \text{if } x \in C. \end{cases}$

Evaluate the Lebesgue integral $\int_{[0,1]} f \, dm$, where C denotes the Cantor ternary set in $[0, 1]$.

11. Evaluate the Lebesgue integrals: (a) $\int_{[0,+\infty)} e^{-|x|} \, dm(x)$ (b) $\int_{(0,1]} \frac{1}{\sqrt[3]{x}} \, dm(x)$

12. Let $f(x) = \begin{cases} e^{|x|} & \text{if } x \in \mathbb{Q}, \\ e^{-|x|} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Evaluate the Lebesgue integral $\int_{\mathbb{R}} f \, dm$.

13. Let $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ \frac{1}{x} & \text{if } x > 1. \end{cases}$

Evaluate the Lebesgue integral $\int_{(0,+\infty)} f \, dm$.

14. Evaluate the following: (a) $\lim_{n \rightarrow \infty} \int_{-2}^2 \frac{x^{2n}}{1+x^{2n}} \, dx$ (b) $\lim_{n \rightarrow \infty} \int_{[0,1]} \frac{1+nx}{(1+x)^n} \, dx$ (c) $\int_0^1 (\sum_{n=1}^{\infty} \frac{x^n}{n}) \, dx$
 (d) $\lim_{n \rightarrow \infty} \int_1^{\infty} \frac{1}{1+x^{2n}} \, dx$ (e) $\sum_{n=0}^{\infty} \int_0^1 \frac{x^2}{(1+x^2)^n} \, dx$ (f) $\lim_{n \rightarrow \infty} \int_{[0,\infty)} \frac{n^2 x e^{-x^2}}{n^2 + x^2} \, dx$

15. Let $f : (X, S, \mu) \rightarrow \mathbb{R}$ be measurable. Define a set function $\nu : S \rightarrow \overline{\mathbb{R}}$ by $\nu(E) = \int_E f \, d\mu$, whenever $E \in S$. Show that $\nu(X)$ is finite if $f \in L^1(X, S, \mu)$. Does the converse true?

16. For $f \in L^+ \cap L^1(\mathbb{R}, M, m)$, define $g(x) = \sum_{n=1}^{\infty} f(2^n x + \frac{1}{n})$. Show that $g \in L^1(\mathbb{R}, M, m)$ and $\int_{\mathbb{R}} g \, dm = \int_{\mathbb{R}} f \, dm$.

17. Construct a function $f \in L^1(\mathbb{R}, M, m)$ such that $\lim_{n \rightarrow \infty} n^2 m\{x \in \mathbb{R} : |f(x)| \geq n\} = \infty$.

18. Let $f \in L(X, S, \mu)$. Suppose there exists an increasing sequence $E_n \in S$ such that $\cup_{n=1}^{\infty} E_n = X$ and $\lim_{n \rightarrow \infty} \int_{E_n} |f| \, d\mu < \infty$. Show that $f \in L^1(X, S, \mu)$.

19. Suppose $f_n, f : (X, S, \mu) \rightarrow [0, \infty]$ are measurable functions such that f_n converges to f point-wise and $f_n \leq f$. Show that $\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu$.

20. Let $f_n : (X, S, \mu) \rightarrow \overline{\mathbb{R}}$ be sequence of measurable functions that f_n increases to f point-wise. If $f, f_n \in L^1(X, S, \mu)$, then show that $\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu$.

21. Let $f_n : X \rightarrow [0, \infty]$ be a sequence of measurable functions and $f_n \rightarrow f$ point wise. Suppose there exists $M > 0$ such that $\sup_{n \geq 1} \int_X f_n \leq M$. Show that $f \in L^1(X, S, \mu)$.

22. Let $f \in L^1(X, S, \mu)$. Then show that for each $\epsilon > 0$ there exists $\delta > 0$ and set $E \in S$ such that $\int_E |f| \, d\mu < \epsilon$, whenever $\mu(E) < \delta$.

23. Let $f \in L^1(X, S, \mu)$ be arbitrary and let $E_n = \{x \in X : |f(x)| \geq n\}$. If $0 < p \leq 1$, then show that $\lim_{n \rightarrow \infty} n^p \mu(E_n) = 0$.

24. Let $f \in L^1(\mathbb{R}, M, m)$ be such that $\int_I f = 0$, for any open interval $I \subset \mathbb{R}$, then show that $f = 0$.

25. Let $\mu(\mathbb{R}) < \infty$ and $f_n \in L^1(X, S, \mu)$ be such that $f_n \rightarrow f$ uniformly. Show that $f \in L^1(X, S, \mu)$ and $\int_X f = \lim \int_X f_n$.
26. Let $f_n : X \rightarrow [0, \infty]$ be a decreasing sequence of measurable functions and $f_n \rightarrow f$ point wise. If $f_1 \in L^1(X, S, \mu)$. Then show that $\int_X f = \lim \int_X f_n$.
27. Let $f_n, g : X \rightarrow \overline{\mathbb{R}}$ be measurable functions such that $f_n \leq g, \forall n \in \mathbb{N}$ and $g \in L^1(X, S, \mu)$. Show that $\limsup \int_X f_n \leq \int_X \limsup f_n$.
28. Let $f_n : X \rightarrow [0, \infty]$ be a sequence of measurable functions and $f_n \rightarrow f$ point wise such that $\int_X f = \lim \int_X f_n < \infty$. Show that $\int_E f = \lim \int_E f_n$, for any $E \in S$.
29. Let $f, g, f_n, g_n \in L^1(X, S, \mu)$ be such that $|f_n| \leq g_n, f_n \rightarrow f$ and $g_n \rightarrow g$ point wise. Show that $\int_X g = \lim \int_X g_n$ implies $\int_X f = \lim \int_X f_n$.
30. Let $f_n, f \in L^1(X, S, \mu)$ be such that $f_n \rightarrow f$ point wise. Prove that $\lim \int_X |f_n - f| = 0$ if and only if $\int_X |f| = \lim \int_X |f_n|$.
31. Let $|f_n| \leq g \in L^1(\mathbb{R})$. Let f_{n_k} be subsequence of f_n such that $f_{n_k} \rightarrow f$ point wise a.e. on \mathbb{R} . If $\lim_{k \rightarrow \infty} \|f_{n_k} - f\| = \overline{\lim}_n \|f_n - f\|_1 < \infty$. Show that $f_n \rightarrow f$ in $L^1(\mathbb{R})$.
32. $f : X \rightarrow [0, \infty]$ be a measurable function. Show that f is integrable on (X, S, μ) if and only if $\sum_{n=-\infty}^{\infty} 2^n \mu\{x \in X : 2^n \leq f(x) \leq 2^{n+1}\} < \infty$.
33. Let $\mu(X) < \infty$ and $f : X \rightarrow [0, \infty]$ be a measurable function. Show that $f \in L^1(X, S, \mu)$ if and only if $\sum_{n=0}^{\infty} \mu\{x \in X : f(x) \geq n\} < \infty$.