# MA550: Measure Theory

( Assignment 3: Measurable functions)

January - April, 2024

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous $m$-a.e. on $\mathbb{R}$, then there must exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ $m$-a.e. on $\mathbb{R}$.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and if $g : \mathbb{R} \to \mathbb{R}$ is such that $f = g$ $m$-a.e. on $\mathbb{R}$, then $g$ must be continuous $m$-a.e. on $\mathbb{R}$.
   (c) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous such that $f = g$ $m$-a.e. on $\mathbb{R}$, then it is necessary that $f(x) = g(x)$ for all $x \in \mathbb{R}$.
   (d) An almost everywhere vanishing Lebesgue measurable function need not be continuous.
   (e) There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f = \chi_{[0,1]}$ $m$-a.e. on $\mathbb{R}$.
   (f) Let $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 1$. Then $f$ is Borel measurable on $\mathbb{R}$.
   (g) For $n \in \mathbb{N}$, define $f_n = \chi_{(n,n+1)}$. Does there exist a measurable set $E$ in $\mathbb{R}$ with $m(E) = \infty$ such that $f_n$ converges to 0 uniformly on $E$?
   (h) Let $f, g : \mathbb{R} \to [0, \infty)$ be Lebesgue measurable such that $m\{x \in \mathbb{R} : f\,g \neq 0\} = 0$. Does it imply that $\max\{f, g\} = f + g$?
   (i) Let $\text{supp}h = \{x \in \mathbb{R} : \ h(x) \neq 0\}$. Suppose $f,\ g : \mathbb{R} \to [0, \infty)$ are such that $\text{supp}f \cap \text{supp}g = \emptyset$. Does it imply that $\max\{f, g\} = f + g$?

2. If $(X, \mathcal{A})$ is a measurable space and $A \subset X$, then show that $\chi_A : X \to \mathbb{R}$ is $\mathcal{A}$-measurable iff $A$ is $\mathcal{A}$-measurable.

3. If $(X, \mathcal{A})$ is a measurable space, then show that $f : X \to [-\infty, +\infty]$ is $\mathcal{A}$-measurable iff $\{x \in X : f(x) > r\} \in \mathcal{A}$ for each $r \in \mathbb{Q}$.

4. Let $D$ be a dense subset of $\mathbb{R}$. Show that $f : \mathbb{R} \to \bar{\mathbb{R}}$ is a Lebesgue measurable function if and only if $\{x \in \mathbb{R} : f(x) > r\}$ is a Lebesgue measurable set for each $r \in D$.

5. Let $f : \mathbb{R} \to [0, \infty]$ be such that $m^*(\{x \in \mathbb{R} : f(x) \geq 2^n\}) < \frac{1}{2^n}$, whenever $n \in \mathbb{N}$. Show that $\{x \in \mathbb{R} : f(x) = \infty\}$ is Lebesgue measurable.

6. Let $f_n,\ f$ be real valued measurable functions on $\mathbb{R}$. Let $E = \{x \in \mathbb{R} : \ \lim f_n(x) = f(x)\}$. Show that $E$ is Lebesgue measurable.

7. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq 5, \\ 0 & \text{if } |f(x)| > 5. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

8. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} 0 & \text{if } f(x) \in \mathbb{Q}, \\ 1 & \text{if } f(x) \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

9. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} -2 & \text{if } f(x) < -2, \\ f(x) & \text{if } -2 \leq f(x) \leq 3, \\ 3 & \text{if } f(x) > 3. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

10. Let $f : [0,\ 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$
    Find the Lebesgue measure of the set $\{x \in \mathbb{R} : \ f(x) \geq 0\}$.

11. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then show that $g \circ f$ is $\mathcal{A}$-measurable.

12. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If $G$ is an open subset of $\mathbb{R}^2$, then show that $\{x \in X : (f(x), g(x)) \in G\}$ is $\mathcal{A}$-measurable.

13. If $f : \mathbb{R} \to \mathbb{R}$ is continuous $m$-a.e. on $\mathbb{R}$, then show that $f$ is Lebesgue measurable.

14. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, then show that $f' : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x,\ .)$ and $f(.,\ y)$ are continuous then $f$ is Lebesgue measurable.

16. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x,\ .)$ is measurable and $f(.,\ y)$ is continuous. Show that $f$ is Lebesgue measurable.

17. Let $f, g : (X, \mathcal{A}) \to \mathbb{R}$. Define $\varphi(x) = (f(x), g(x))$. Then show that $f$ and $g$ are $\mathcal{A}$-measurable if and only if $\varphi$ is $\mathcal{A}$-measurable.

18. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$ and let $f : X \to \mathbb{R}$ be measurable. Let $A_n = \{x \in X : |f(x)| > n\}$. Show that $A_n$ is $\mathcal{A}$-measurable and $\lim \mu(A_n) = 0$.

19. Let $f : X \to \overline{\mathbb{R}}$ be an almost finite measurable function on a finite measure space $(X, S, \mu)$. Let $A_n = \{x \in X :\ |f(x)| > n\}$. Show that $\lim \mu(A_n) = 0$.

20. Let $f : [a, b] \to \mathbb{R}$ be Lebesgue measurable. Let $N = \{x \in [a, b] : f(x) = 0\}$. Show that $g = \chi_N + \frac{1}{f}\chi_{N^c}$ is Lebesgue measurable.

21. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose for each $\epsilon > 0$ there exists an open set $O$ such that $m(O) < \epsilon$ and $f$ is constant on $\mathbb{R} \smallsetminus O$. Show that $f$ is Lebesgue measurable.

22. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous one-one and onto map. Then show that $f$ sends Borel sets onto Borel sets.

23. Let $\mathbb{Q}$ denotes set of rationals. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = \begin{cases} 1 & \text{if } x + y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

    and $g(x, y) = \begin{cases} 1 & \text{if } \frac{x}{y} \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$
    Show that $f$ and $g$ are Lebesgue measurable.

24. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Show that $\{x \in \mathbb{R} : f$ is continuous at x $\}$ is Lebesgue measurable.

25. Let $C$ be the Cantor's ternary set. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in C \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$
    Show that $f$ is Lebesgue measurable. By letting $C$ has a non-Borel measurable subset, construct a Lebesgue measurable function which is not Borel measurable.

26. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $E$ be Lebesgue measurable $E \subset [a, b]$. Show that $m(E) = 0$, implies $m(f(E)) = 0$ if and only if for every Lebesgue measurable subset $A \subset [a, b]$ the set $f(A)$ is Lebesgue measurable.

27. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sup\{|x + y| :\ y \in [0, 1]\}$. Show that $f$ is Borel measurable.

28. Let $f : (X, S, \mu) \to \mathbb{R}$ be measurable and $\mathcal{B}(\mathbb{R})$ denotes the Borel sigma algebra on $\mathbb{R}$. Define a set function $\mu_f : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ by $\mu_f(B) = \mu\left(f^{-1}(B)\right)$. Show that $\mu_f$ is a measure on $\mathcal{B}(\mathbb{R})$.

29. If $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that the function $g$ defined by $g(x) = \inf\{|f(t)| : \ x < t < x+1\}$ is Lebesgue measurable. Does the conclusion hold if $f$ is bounded Lebesgure measurable function?

30. Let $E \subset \mathbb{R}$ with $m(E) < \infty$. Let $f_n : E \to \overline{\mathbb{R}}$ be sequence of Lebesgue measurable functions such that for each $x \in X$, there exists $M_x > 0$ with $|f_n(x)| \leq M_x < \infty$, $\forall\, n \in \mathbb{N}$. Then for each $\epsilon > 0$, there exists a compact set $K \subset E$ such that $f_n$ is uniformly bounded on $K$, where $m(E \setminus K) < \epsilon$.

31. Let $(X, S, \mu)$ be a finite measure space and $f : X \to \bar{\mathbb{R}}$ be an almost finite $S$-measurable function. show that for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu\{x \in X : |f(x)| > n_0\} < \epsilon$.

32. Let $f : (\mathbb{R}, M, m) \to [0, \infty]$ be such that for each $\epsilon > 0$ there exists a Lebesgue measurable set $E \subset \mathbb{R}$ with $m(E) < \epsilon$ and $f$ is continuous on $\mathbb{R} \setminus E$. Show that $f$ is a Lebesgue mesurable function.

33. Let $E \subset \mathbb{R}$ be Lebesgue measurable and $m(E) = \infty$. Define a function $f : \mathbb{R} \to \bar{\mathbb{R}}$ by $f(x) = m(E \cap (-\infty, x))$. Show that $f$ is a Borel measurable function.

34. Let $g : [0,1] \to [0,2]$ be a bijection with $m(g(C)) = 1$, where $C$ is the Cantor set. Construct a Lebesgue measurable function $f$ on $[0,1]$ such that $f \circ g^{-1}$ is not Lebesgue measurable.