MA550: Measure Theory

(Assignment 2: Measurable sets)
January - April, 2024

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) $\{x \in \mathbb{R} : x^6 - 6x^4 \text{ is irrational}\}$ is a Lebesgue measurable subset of $\mathbb{R}$.
   (b) If A is a Lebesgue measurable subset of $\mathbb{R}$ and if B is a Lebesgue non-measurable subset of $\mathbb{R}$ such that $B \subset A$, then it is necessary that $m^*(A \setminus B) > 0$.
   (c) Whether the set $E = \bigcup_{x\in\mathbb{R}}(x + \mathbb{Q})$ is Lebesgue measurable?
   (d) Let C be the Cantor set in $[0, 1]$ and $a, b \in \mathbb{R}$ with $a < b$. Then the set $C + (a, b)$ is Borel measurable.
   (e) If A and B are disjoint subsets of $\mathbb{R}$ such that A is Lebesgue measurable and B is Lebesgue non-measurable, then it is possible that $m^*(A \cup B) < m^*(A) + m^*(B)$.
   (f) If A is subset of $\mathbb{R}$ with $m^*(A) < \infty$, then $m^*(A^2) < \infty$.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $F \subset \mathbb{R}$ be a countable set. Show that $E + F$ is Lebesgue measurable.

3. If $A \subset \mathbb{R}$, then show that there exists a Lebesgue measurable subset E of $\mathbb{R}$ such that $m^*(A) = m(E)$.

4. Let $A \subset [0, 1]$ be Lebesgue measurable with $m(A) = 1$. If $B \subset [0, 1]$, then show that $m^*(A \cap B) = m^*(B)$.

5. For $i = 1, ..., n$, let $E_i \subset (0, 1)$ be Lebesgue measurable such that $\sum\limits_{i=1}^{n} m(E_i) > n - 1$. Show that $m(\bigcap\limits_{i=1}^{n} E_i) > 0$.

6. Let $\{E_i\}$ be a decreasing sequence of Lebesgue measurable sets in $[0, 1]$ which satisfying $\sum\limits_{i=1}^{n} m(E_i) > n - \frac{1}{n}$. Show that $m\left(\bigcap\limits_{i=1}^{\infty} E_i\right) = 1$.

7. Let $A \subset \mathbb{R}$ such that $m^*(A) > 0$. Show that there exist $x, y \in A$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$.

8. Let A and B be Lebesgue measurable subsets of $(0, 1)$ such that $m(A) > \frac{1}{2}$ and $m(B) > \frac{1}{2}$. Prove that there exist $a \in A$ and $b \in B$ such that $a + b = 1$.

9. Suppose F is a closed subset of $[0, 1]$ such that $F \cap (a, b) \neq \emptyset$ for all $a, b \in [0, 1]$ with $a < b$. Show that $m(F) = 1$.

10. Let A be an unbounded Lebesgue measurable subset of $\mathbb{R}$ such that $m(A) < \infty$. Show that for each $\varepsilon > 0$, there exists a bounded Lebesgue measurable set B in $\mathbb{R}$ such that $B \subset A$ and $m(A \setminus B) < \varepsilon$.

11. For $n \in \mathbb{N}$, write $E = \bigcup\limits_{n=1}^{\infty} \left[n, n + \frac{1}{n^{3/2}}\right]$. Show that $m(E) < \infty$ and $m^*(\{x^2 : x \in E\}) = \infty$.

12. If $A \subset \mathbb{R}$ such that $m^*(A) = 0$, then show that $m^*(\{x^2 : x \in A\}) = 0$.

13. Let A be a closed subset of $[0, 1]$ that satisfies $A \cap (\alpha, \beta) \neq \emptyset$ for all $\alpha, \beta \in [0, 1]$ with $\alpha < \beta$. Show that $m(A \setminus A^2) = 0$.

14. Let $A, B \subset \mathbb{R}$ such that $A \cup B$ is Lebesgue measurable and $m(A \cup B) = m^*(A) + m^*(B) < \infty$. Show that both A and B are Lebesgue measurable.

15. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}$ and let $\{E_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}$ such that $A_n \subset E_n$ for each $n \in \mathbb{N}$. Show that $m^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m^*(A_n)$.

16. Let $E \subset \mathbb{R}$ and let $\alpha \in \mathbb{R}$. If $\alpha E = \{\alpha x : x \in E\}$, then show that $m^*(\alpha E) = |\alpha| m^*(E)$. Also, show that if E is Lebesgue measurable, then αE is Lebesgue measurable.

17. If E is a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < +\infty$ and if $f(x) = m(E \cap (-\infty, x])$ for all $x \in \mathbb{R}$, then show that $f : \mathbb{R} \to \mathbb{R}$ is continuous.

18. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < \infty$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = m\{E \cap (-\infty, x^2)\}$. Show that f is differentiable at 0 and $f'(0) = 0$.

19. Let $E \subset \mathbb{R}$ and $m^*(E) > 0$. Then for each $0 < \alpha < 1$, there exists an open interval I such that $m^*(E \cap I) \geq \alpha m(I)$.

20. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $m(E) < \infty$. Then there exist a sequence of compact set (K_n) contained in E and a set N Lebesgue measure zero such that $E = F \cup N$, where $F = \cup_{n=1}^{\infty} K_n$.

21. Let $E \subset \mathbb{R}$ be Lebesgue measurable and $m(E) < \infty$. Show that for each $\epsilon > 0$, there exist compact set K and open set O with $K \subseteq E \subseteq O$ such that $m(O \setminus K) < \epsilon$.

22. Let $m^*(A) > 0$. Then show that there exists at least one closed set $F \subset \mathbb{R}$ with $m(F) < \infty$ such that $A \cap F \neq \emptyset$.

23. Let μ be a finite measure on $M(\mathbb{R})$. Suppose for each closed set $F \subset \mathbb{R}$ with $m(F) < \infty$, implies $\mu(F) = 0$. Then show that $\mu = 0$.

24. Let E be a measurable subset of $\mathbb{R}$ with $m(E) < \infty$ and $m\{E \cap (n, n+1)\} < \frac{1}{2^{|n|+2}} m(E)$, for all $n \in \mathbb{Z}$. Show that $m(E) = 0$.

25. Let $\{E_n\}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$ such that $\sum_{n=1}^{\infty} m(E_n) < \infty$. Show that $m\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$.

26. Let $A \subset \mathbb{R}$ be a closed set with $m(A) = 0$. Show that A is nowhere dense in $\mathbb{R}$. But does this conclusion hold true when A is not closed?

27. Let $[-1, 1] \cap \mathbb{Q} = \{r_1, r_2, \ldots\}$. For a Lebesgue measurable set $E \subset [0, 1]$ with $m(E) > 0$, define $E_n = E + r_n;\ n \in \mathbb{N}$. Show that all of E_n's can not be pairwise disjoint. Further, deduce that there exist $x, y \in E$ such that $x - y \in \mathbb{Q}$.

28. Let $\tilde{M}$ be the class of all Lebesgue measurable subset of $[0, 1]$. If $N \notin \tilde{M}$. Prove/disprove $N \cap (\mathbb{R} \setminus \mathbb{Q}) \in \tilde{M}$.

29. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) = \infty$. Show that there exists a sequence $\{E_n\}$ of pairwise disjoint measurable subsets of E such that $m(E_n) < \infty$, for all n and $E = \bigcup_{n=1}^{\infty} E_n$.

30. Let F be a closed subset of $\mathbb{R}$ with $m(F) = 0$. Then for any $A \subset F$, show that $m^*\{x \in \mathbb{R} : d(x, A) = 0\} = 0$.

31. Let A be a bounded subset of $\mathbb{R}$. Show that $m\left(\overline{A}\right) < \infty$.

32. Let K be a compact subset of $\mathbb{R}$ and $O_n = \left\{x \in \mathbb{R} : d(x, K) < \frac{1}{n}\right\}$. Show that each of O_n is Lebesgue measurable and $\lim_{n \to \infty} m(O_n) = m(K)$.

33. Let (X, S, μ) be a finite measure space. For a sequence of sets $A_n \in S$, if we define $\overline{\lim} A_n = \cap_{k \geq 1}(\cup_{n \geq k} A_n)$, then show that $\mu(\overline{\lim} A_n) \geq \overline{\lim}\, \mu(A_n)$.