

MA550: Measure Theory

(Assignment 1: Outer measures)

January - April, 2024

1. State TRUE or FALSE giving proper justification for each of the following statements.
 - (a) There exists an unbounded subset A of \mathbb{R} such that $m^*(A) = 5$.
 - (b) There exists an open subset A of \mathbb{R} such that $[\frac{1}{2}, \frac{3}{4}] \subset A$ and $m^*(A) = \frac{1}{4}$.
 - (c) There exists an open subset A of \mathbb{R} such that $m^*(A) < \frac{1}{5}$ but $A \cap (a, b) \neq \emptyset$ for all $a, b \in \mathbb{R}$ with $a < b$.
 - (d) If A and B are open subsets of \mathbb{R} such that $A \subsetneq B$, then it is necessary that $m^*(A) < m^*(B)$.
 - (e) There is a Lebesgue measurable set $A \subset \mathbb{R}$ such that $m(A) = 0$ but $m(\text{boundary}(A)) = \infty$.
 - (f) If A and $A \cup B$ are Lebesgue measurable subsets of \mathbb{R} , then B is necessarily Lebesgue measurable subset of \mathbb{R} .
 - (g) There exists a non-zero finite measure μ on $M(\mathbb{R})$, which is constant on every bounded open interval (a, b) with $a < b$.
 - (h) Let E be a subset of \mathbb{R} such that $m^*(E) = 0$. Then it imply that E is contained in a Borel set B with $m^*(B) = 0$.
 - (i) For $A \subseteq \mathbb{R}$, define $\mu_o(A) = \begin{cases} 0 & \text{if } A \text{ is a compact subset of } \mathbb{R}, \\ 1 & \text{if } A \text{ is a non-compact subset of } \mathbb{R}. \end{cases}$
Then μ_o is necessarily a pre-measure on $\mathcal{B}(\mathbb{R})$.
2. Examine whether \mathcal{A} is a σ -algebra of subsets of \mathbb{R} , where
 - (a) $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) = 0 \text{ or } m^*(\mathbb{R} \setminus A) = 0\}$.
 - (b) $\mathcal{A} = \{A \subset \mathbb{R} : m^*(A) < +\infty \text{ or } m^*(\mathbb{R} \setminus A) < +\infty\}$.
 - (c) $\mathcal{A} = \{A \subset \mathbb{R} : A \text{ or } \mathbb{R} \setminus A \text{ is an open subset of } \mathbb{R}\}$.
3. Let X be an uncountable set. Show that the class $\{\{x\} : x \in X\}$ generates the σ -algebra $\{A \subset X : A \text{ is countable or } X \setminus A \text{ is countable}\}$.
4. Let \mathcal{S} be a class of subsets of a nonempty set X and let $A \subset X$. Show that $\sigma(\mathcal{S} \cap A) = \sigma(\mathcal{S}) \cap A$, where for each class \mathcal{C} of subsets of X , $\mathcal{C} \cap A = \{C \cap A : C \in \mathcal{C}\}$.
5. Let X, Y be nonempty sets and let $f : X \rightarrow Y$. If \mathcal{S} is a class of subsets of Y , then show that $\sigma(f^{-1}(\mathcal{S})) = f^{-1}(\sigma(\mathcal{S}))$, where for each class \mathcal{C} of subsets of Y , $f^{-1}(\mathcal{C}) = \{f^{-1}(C) : C \in \mathcal{C}\}$.
6. If \mathcal{S} is a class of subsets of a nonempty set X and if $A \in \sigma(\mathcal{S})$, then show that there exists a countable subclass \mathcal{S}_0 of \mathcal{S} such that $A \in \sigma(\mathcal{S}_0)$.
7. Prove that every infinite σ -algebra on an infinite set is uncountable.
8. Show that $\mathcal{B}(\mathbb{R})$ is generated by each of the following classes.
 - (a) $\{(a, +\infty) : a \in \mathbb{R}\}$
 - (b) $\{(-\infty, a] : a \in \mathbb{Q}\}$
 - (c) $\{[a, b) : a, b \in \mathbb{Q}, a < b\}$
 - (d) $\{A \subset \mathbb{R} : A \text{ is compact}\}$
9. Let E be a Borel subset of \mathbb{R} and let $x \in \mathbb{R}$. Show that $x + E$ is a Borel subset of \mathbb{R} .
10. Examine whether μ^* is an outer measure on \mathbb{R} , where for each $A \subset \mathbb{R}$,
 - (a) $\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is bounded,} \\ 1 & \text{if } A \text{ is unbounded.} \end{cases}$
 - (b) $\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \text{ is nonempty and bounded,} \\ +\infty & \text{if } A \text{ is unbounded.} \end{cases}$

11. Consider the outer measure μ^* on \mathbb{R} , where for each $A \subset \mathbb{R}$, $\mu^*(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ 1 & \text{if } A \text{ is uncountable.} \end{cases}$
Determine all the μ^* -measurable subsets of \mathbb{R} .
12. If $\mathcal{S} = \{\emptyset, [1, 2]\}$ and if $\mu(\emptyset) = 0$, $\mu([1, 2]) = 1$, then determine the outer measure μ^* on \mathbb{R} induced by the set function $\mu : \mathcal{S} \rightarrow [0, +\infty)$.
Also, determine all the μ^* -measurable subsets of \mathbb{R} .
13. Let (X, \mathcal{A}) be a measurable space and let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ be finitely additive with $\mu(\emptyset) = 0$. Show that μ is a measure on \mathcal{A} if either of the following conditions is satisfied.
(a) For every increasing sequence $\{A_n\}_{n=1}^\infty$ of sets in \mathcal{A} , $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(\bigcup_{n=1}^\infty A_n)$.
(b) For every decreasing sequence $\{A_n\}_{n=1}^\infty$ of sets in \mathcal{A} with $\bigcap_{n=1}^\infty A_n = \emptyset$, $\lim_{n \rightarrow \infty} \mu(A_n) = 0$.
14. Let μ^* be an outer measure generated by a finite premeasure μ_o on an algebra \mathcal{A} on a nonempty set X . Show that $E \subseteq X$ is μ^* -measurable iff for each $\varepsilon > 0$, there exists a μ^* -measurable set G containing E such that $\mu^*(G \setminus E) < \varepsilon$.
15. Let μ^* be an outer measure on a non-empty set X . Let $F = \{A \subseteq X : \mu^*(A) = 0\}$. Find the σ -algebra generated by F .
16. Let \mathcal{A} be a σ -algebra of on \mathbb{R} . Write $\bar{\mathcal{A}} = \{E \cup N : E \in \mathcal{A} \text{ and } N \subseteq F \in \mathcal{A} \text{ with } m(F) = 0\}$. Show that $\bar{\mathcal{A}}$ is a σ -algebra. Further, deduce that $\overline{B(\mathbb{R})} = M(\mathbb{R})$.
17. Let μ^* be an outer measure generated by a pre-measure μ_o on an algebra \mathcal{A} on \mathbb{R} . Let E be μ^* measurable (in Caratheodory's sense). Show that there exists a set $G \in \sigma(\mathcal{A})$ such that $\mu^*(E) = \mu^*(G)$.
18. Let (X, τ) be a topological space. Let $\mathcal{B}(X)$ be the σ -algebra generated by τ . Let μ^* be the outer measure generated by a σ -finite pre-measure μ_o on $\mathcal{B}(X)$. Show that $E \in M_{\mu^*}$ if and only if there exists $G \in \mathcal{B}(X)$ such that $\mu^*(G \setminus E) = 0$.
19. Let $f : [0, 2) \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1, \\ 3 - x & \text{if } 1 < x < 2. \end{cases}$
Find $m^*(A)$, where $A = f^{-1}((\frac{9}{16}, \frac{5}{4})) = \{x \in [0, 2) : f(x) \in (\frac{9}{16}, \frac{5}{4})\}$.
20. Let $B \subset A \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A \setminus B) = m^*(A)$.
21. Let $A \subset \mathbb{R}$ such that $m^*(A) > 0$. Show that there exists $B \subset A$ such that B is bounded and $m^*(B) > 0$.
22. If G is a nonempty open subset of \mathbb{R} , then show that $m^*(G) > 0$.
23. Let A be a countable subset of \mathbb{R} and let $B \subset \mathbb{R}$ such that $m^*(B) = 0$. Show that $m^*(A+B) = 0$.
24. Prove or disprove: A subset E of \mathbb{R} is Lebesgue measurable iff $m^*(A \cup B) = m^*(A) + m^*(B)$ for each $A \subset E$ and for each $B \subset \mathbb{R} \setminus E$.
25. Let $E = \{x \in [0, 1] : \text{The decimal representation of } x \text{ does not contain the digit } 5\}$. Show that $m(E) = 0$.

26. Let $A_n \subset \mathbb{R}$ for $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} m^*(A_n) < \infty$. If $E = \{x \in \mathbb{R} : x \in A_n \text{ for infinitely many } n\}$, then show that $m(E) = 0$.
27. Show that a subset E of \mathbb{R} is Lebesgue measurable iff $m^*(I) = m^*(I \cap E) + m^*(I \setminus E)$ for every bounded open interval I of \mathbb{R} .
28. Let $A \subset E \subset B \subset \mathbb{R}$ such that A, B are Lebesgue measurable and $m(A) = m(B) < \infty$. Show that E is Lebesgue measurable.
More generally, let $A \subset B \subset \mathbb{R}$ such that A is Lebesgue measurable and $m^*(B) = m(A) < \infty$. Show that B is Lebesgue measurable.
29. Let $A, B \subset \mathbb{R}$ such that $m^*(A) = 0$ and $A \cup B$ is Lebesgue measurable. Show that B is Lebesgue measurable.
30. Let $A, B \subset \mathbb{R}$ such that A is Lebesgue measurable and $m^*(A \Delta B) = 0$. Show that B is Lebesgue measurable.
31. Let $A, B \subset \mathbb{R}$ be such that $A \cap B$ is Lebesgue measurable and $m^*(A \Delta B) = 0$. Show that A and B are Lebesgue measurable and $m(A) = m(B)$.
32. Let $A \subset \mathbb{R}$ such that $A \cap B$ is Lebesgue measurable for every bounded subset B of \mathbb{R} . Show that A is Lebesgue measurable.
33. Let A and B be subsets of $[0, 1]$ which satisfy $m^*(A \cup B) = m^*(A) + m^*(B)$. If $A \Delta B$ is Lebesgue measurable then prove that A and B are Lebesgue measurable.
34. Let E be a Lebesgue measurable subset of \mathbb{R} and let $A \subset \mathbb{R}$. Show that $m^*(E \cap A) + m^*(E \cup A) = m^*(E) + m^*(A)$.
35. Let A be a subset of \mathbb{R} such that $m^*(A \cup B) = m^*(A) + m^*(B)$ for every subset B of \mathbb{R} . Show that A is Lebesgue measurable. Further, if $m(A) < \infty$, then show that $m(A) = 0$.
36. Let I and J be disjoint open intervals in \mathbb{R} and let $A \subset I, B \subset J$. Show that $m^*(A \cup B) = m^*(A) + m^*(B)$.
37. Let A be a subset of \mathbb{R} with $0 < m^*(A) < \infty$. Show that for each $\epsilon > 0$ there exist an open set O containing A and a compact set $K \subset \mathbb{R}$ such that $m(O \setminus K) < \epsilon$.