DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA550: Measure Theory — EndSem
Instructor: Rajesh Srivastava — May 5, 2024
Time duration: Three hours — Maximum Marks: 50

N.B. Answer without proper justification will attract zero mark.

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1. (a) Whether $\{(x,y) \in \mathbb{R}^2 : x < y\}$ belongs to $B(\mathbb{R}) \otimes B(\mathbb{R})$? **[1]**

   (b) Let $E \in M(\mathbb{R}) \otimes M(\mathbb{R})$ and $m \times m(E) = 0$. Does it imply that $m(E_x) = 0$ for a.e. x in $\mathbb{R}$? **[1]**

   (c) Let (X, S, μ) be a measure space. If $\int_E f\, d\mu > 0$ for every $E \in S$, does it imply $f(x) > 0$ for a.e. x in X? **[1]**

   (d) Let $f \in L^+ \cap L^1(X, S, \mu)$ and $E_n = \{x \in X : \frac{1}{n} \leq f(x) \leq n\}$. Does it imply $\int_{E_n} f d\mu \to \int_X f d\mu$? **[1]**

   (e) Whether $L^\infty([0,1], M, m) = \bigcap_{p \geq 1} L^p([0,1], M, m)$? **[1]**

2. Suppose $f \in L^2(X, S, \mu)$. Show that $\lim_{n\to\infty} n^2 \mu\{x \in X : |f(x)| \geq n\} = 0$. **[4]**

3. Let $f, f_n \in L^1(X, S, \mu)$ and $\int_X |f_n - f| d\mu \to 0$. Show that for each $\epsilon > 0$, there exist $\delta > 0$, $n_o \in \mathbb{N}$ and $E \in S$ such that $\left|\int_E f_n d\mu\right| < \epsilon$, whenever $\mu(E) < \delta$ and $n \geq n_o$ **[4]**

4. Let $f(x,y) = \sin x\, \chi_{\{(x,y):\, y<x<y+2\pi\}}$. Show that $\int_{\mathbb{R}}\int_{\mathbb{R}} (f(x,y) dx dy \neq \int_{\mathbb{R}}\int_{\mathbb{R}} (f(x,y) dy dx$. **[4]**

5. Let $f : (X, S, \mu) \to ([0,\infty), B(\mathbb{R}), m)$ and $A = \{(x,y) : 0 \leq y \leq f(x), x \in X\}$. Show that f is S-measurable if and only if $A \in S \otimes \mathbb{B}(\mathbb{R})$. **[5]**

6. Let $f, f_n : (X, S, \mu) \to [0, \infty]$ be such that $f_n \to f$ pointwise and $\sup_{n\geq 1} \int_X f_n d\mu < \infty$. Show that $\int_X |f_n - f| d\mu \to 0$ if and only if $\int_X f_n d\mu \to \int_X f d\mu$. (Fatou's lemma may be helpful.) **[5]**

7. Let $f : (X, S, \mu) \to [1, \infty)$ be a measurable function. If $f \in L^1(X, S, \mu)$, then show that the series $\sum_{n=1}^{\infty} \mu\{x \in X : f(x) > n\}$ is convergent. **[5]**

8. Let $T : L^2(\mathbb{R}, M, m) \to L^1(\mathbb{R}, M, m)$ be defined by $(Tf)(x) = \int_{\mathbb{R}} \frac{f(x+y)}{(1+y^2)\sqrt{1+x^2}} dy$. Show that T is a bounded linear map and $\|T\| \leq (\pi)^{3/2}$. **[4]**

9. Let $f \in L^1(\mathbb{R}, M, m)$. Show that $\frac{f(nx)}{n} \to 0$ a.e. x in $\mathbb{R}$. (Beppo-Levi theorem may be helpful.) **[5]**

10. If $f \notin L^\infty(\mathbb{R}, M, m)$, then show that $\lim_{p\to\infty} \|f\|_p = \infty$. **[5]**

11. Let $f \in L^+ \cap L^1(X, S, \mu)$. Show that $\int_X f d\mu = \int_0^\infty \mu\{x \in X : f(x) > t\} dt$. (Fubini's theorem may be helpful) **[4]**

END