MA549: Topology

Assignment 4: Countability and separation axioms

July - November, 2023

1. State TRUE or FALSE with justification for each of the following statements.
   (a) If A and B are nonempty subsets of topological spaces X and Y respectively such that $A \times B$ is closed in the product space $X \times Y$, then A must be closed in X and B must be closed in Y.
   (b) If τ_l denotes the lower limit topology on $\mathbb{R}$, then $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$ is an open set in the product space $(\mathbb{R}, \tau_l) \times (\mathbb{R}, \tau_l)$.
   (c) There cannot exist topologies τ, τ' on an infinite set X such that the product topology for (X, τ) and (X, τ') coincides with the cofinite topology on $X \times X$.
   (d) If τ_u and τ_l denote respectively the usual topology and the lower limit topology on $\mathbb{R}$, then the product space $(\mathbb{R}, \tau_u) \times (\mathbb{R}, \tau_l)$ is not metrizable.

2. Prove that every closed (respectively, open) subset of a metrizable space is a G_δ (respectively, an F_σ) set.
   Also, show that the metrizability condition is, in general, necessary.

3. Let (X, τ) be a metrizable topological space and let τ_u be the usual topology on $\mathbb{R}$. Show that τ is the weakest topology on X with respect to which every continuous map from (X, τ) to $(\mathbb{R}, \tau_u)$ remains continuous.

4. Let X be a first countable space and let $A \subset X$. Prove that
   (a) A is closed in X iff for every sequence $(a_n) \subset A$ and for every $x \in X$, $a_n \to x \Rightarrow x \in A$.
   (b) A is open in X iff for every sequence $(x_n) \subset X$ and for every $a \in A$, $x_n \to a \Rightarrow x_n \in A$ eventually.

5. If $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$, then show that the topological space $(\mathbb{R}, \tau)$ is first countable but not second countable.

6. Let X be a first countable space and let $G \subset X$. If for every nonempty countable set A in X, $G \cap A$ is open in the subspace A, then show that G is open in X.

7. Let $X = \mathbb{Z}_+ \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, 2, ...\}$, and let
   $\tau = \mathcal{P}(X \backslash \{(0, 0)\}) \cup \{G \subset X : (0, 0) \in G, \{m \in \mathbb{Z}_+ : \{n \in \mathbb{Z}_+ : (m, n) \notin G\} \text{ is infinite}\} \text{ is finite}\}$.
   Prove that τ is a topology on X which is different from the discrete topology on X.
   Determine all the convergent sequences in the topological space (X, τ) and hence conclude that (X, τ) is not first countable.
   (The topological space (X, τ) is called the Arens-Fort space. It shows that a topological space $(Z, \mathfrak{J})$ need not be first countable even if Z is a countable set.)

8. Prove that a topological space is second countable iff it has a countable subbasis.

9. Let A be an uncountable subset of a second countable space. Show that uncountably many points of A are limit points of A.

10. Let X be a separable space. Prove that every class of pairwise disjoint open sets in X is countable.

Hence deduce that the set of all isolated points of X is countable.

11. Prove that every topological space can be considered as a subspace of a separable space. (From this it follows immediately that a subspace of a separable space need not be separable.)

12. If Y is a nonempty open subset of a separable space (X, τ), then show that the subspace $(Y, \tau|_Y)$ is separable.

13. Let Y be a dense subspace of a first countable separable space X. Show that Y is separable.

14. Let τ_l denote the lower limit topology on $\mathbb{R}$. Prove that every subspace of the topological space $(\mathbb{R}, \tau_l)$ is separable.

15. Prove or disprove: $\mathbb{R}$ with the cocountable topology is a Lindelöf space.

16. Prove that every metrizable Lindelöf space is second countable.

17. Let X be any set with at least two elements. Show that there exists a topology τ on X such that (X, τ) is a T_0-space but not a T_1-space.

18. Prove that a topological space X is a T_0-space iff for all $x, y \in X$ with $x \neq y$, $\overline{\{x\}} \neq \overline{\{y\}}$.

19. For a topological space (X, τ), prove that the following statements are equivalent.
    (a) (X, τ) is a T_1-space.
    (b) For each $x \in X$, $\{x\} = \bigcap\{G \in \tau : x \in G\}$.
    (c) τ is finer than the cofinite topology on X.

20. Let (X, τ) be a T_1-space and $A \subset X$. Show that $A = \bigcap\{G \in \tau : A \subset G\}$.

21. Show that for every convergent sequence in a topological space X to have a unique limit in X, it is necessary but not sufficient that X is a T_1-space.

22. Let X be a T_1-space and let $A \subset X$, $x \in X$. Show that
    (a) $x \in A'$ iff every open set in X containing x contains infinitely many points of A.
    (b) A' is closed in X.

23. Let X be a first countable T_1-space and let $A \subset X$, $x \in X$. Prove that $x \in A'$ iff there exists a sequence of distinct points in A converging to x in X.
    Also, show that both T_1 and first countability conditions are, in general, necessary.

24. Show that in a first countable T_1-space, every singleton set is a G_δ set.
    Show that both first countability and T_1 conditions are, in general, necessary.
    Also, give an example of a topological space which is not first countable but in which every singleton set is a G_δ set.

25. Prove that a topological space (X, τ) is a Hausdorff space iff for each $x \in X$, $\{x\} = \bigcap\{\overline{G} : G \in \tau, x \in G\}$.

26. If X is a Hausdorff space, then show that
    (a) for each $x \in X$, $\bigcap\{F \subset X : F \text{ is closed in } X, x \in F\} = \{x\}$.

(b) for each $x \in X$, $\bigcap\{G \subset X : G \text{ is open in } X, x \in G\} = \{x\}$.
Also, show that a topological space X satisfying (a) and (b) need not be Hausdorff.

27. Show that a topological space X is a Hausdorff space iff $\{(x, x) : x \in X\}$ is a closed subset of the product space $X \times X$.

28. Show that a topological space X is a discrete space iff $\{(x, x) : x \in X\}$ is an open subset of the product space $X \times X$.

29. Let X be a Hausdorff space and let $f : X \to X$ be a continuous map such that $f \circ f = f$. Prove that $f(X)$ is a closed subset of X.

30. Let X, Y be topological spaces and let $f : X \to Y$ and $g : Y \to X$ be continuous such that $g(f(x)) = x$ for all $x \in X$. If Y is a Hausdorff space, then show that X is a Hausdorff space and $f(X)$ is closed in Y.

31. Let D be a dense subset of a topological space X and let Y be a Hausdorff space. If $f : X \to Y$ and $g : X \to Y$ are continuous such that $f(x) = g(x)$ for all $x \in D$, then show that $f(x) = g(x)$ for all $x \in X$.

32. Let X_0 be a dense subspace of a topological space X and let Y be a Hausdorff space. Prove that every continuous map $f_0 : X_0 \to Y$ can have at most one continuous extension $f : X \to Y$. Show also that a continuous map $f_0 : X_0 \to Y$ need not have any continuous extension $f : X \to Y$.

33. Let X and Y be Hausdorff spaces and let $f : X \to Y$ be onto. Show that $f : X \to Y$ is a homeomorphism iff $\overline{A} = f^{-1}(\overline{f(A)})$ for all $A \subset X$.

34. Let $x_1, ..., x_n$ be distinct points of a Hausdorff space X. Prove that there exist pairwise disjoint open sets $G_1, ..., G_n$ in X such that $x_i \in G_i$ for $i = 1, ..., n$.

35. Prove that every infinite Hausdorff space contains an infinite set A such that each point of A is an isolated point of A.
(Hence every infinite Hausdorff space contains an infinite discrete subspace).

36. Prove that a first countable space X is a Hausdorff space iff every convergent sequence in X has a unique limit in X.

37. Let X, Y be topological spaces with Y Hausdorff. If $f : X \to Y$ is continuous, then prove that $G_f = \{(x, f(x)) : x \in X\}$ (the graph of f) is closed in the product space $X \times Y$.
Show that the Hausdorff condition on Y is, in general, necessary.

38. Prove that a topological space Y is Hausdorff iff for every topological space X and for any continuous maps $f : X \to Y$ and $g : X \to Y$, the set $\{x \in X : f(x) = g(x)\}$ is closed in X. Hence show that the set of all fixed points of a continuous map from a Hausdorff space Y to itself is closed in Y.

39. State TRUE or FALSE with justification for each of the following statements.
(a) If τ and τ' are topologies on a nonempty set X such that both (X, τ) and (X, τ') are T_1-spaces, then $(X, \tau \cap \tau')$ must be a T_1-space.

(b) There exists a Hausdorff space with exactly 100 (distinct) open sets.

(c) If $x_1, x_2, ...$ are distinct points in a Hausdorff space X, then there must exist pairwise disjoint open sets $G_1, G_2, ...$ in X such that $x_n \in G_n$ for all $n \in \mathbb{N}$.

(d) Every second countable Hausdorff space is metrizable.

(e) If X and Y are topological spaces such that the product space $X \times Y$ is Hausdorff, then both X and Y must be Hausdorff.

(f) If $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then the topological space $(\mathbb{R}, \tau)$ is normal but not regular.

40. Prove that every regular T_0-space is a T_3-space but that a normal T_0-space need not be a T_4-space.

41. Let X be a T_3-space and let $x, y \in X$ with $x \neq y$. Prove that there exist open sets G and H in X containing x and y respectively such that $\overline{G} \cap \overline{H} = \emptyset$.

42. Let A and B be disjoint closed subsets of a normal space X. Prove that there exist open subsets G and H of X containing A and B respectively such that $\overline{G} \cap \overline{H} = \emptyset$.

43. Let X be a T_4-space and let Y be a topological space. If $f : X \to Y$ is continuous, closed and onto, then show that Y is a T_4-space.

44. Let X be a normal space. Show that X is regular iff X is completely regular.

45. Let (X, τ) be a completely regular space. Show that the weak topology on X induced by the family of all continuous maps $f : (X, \tau) \to (\mathbb{R}, \tau_u)$ is the same as τ.

46. Let K be a compact set and C be a closed set in a regular space X such that $K \cap C = \emptyset$. Prove that there exist disjoint open sets G and H in X such that $K \subset G$ and $C \subset H$.

47. Show that a compact Hausdorff space is metrizable iff it is second countable.