MA549: Topology

Assignment 4: Connected spaces

July - November, 2023

1. State TRUE or FALSE with justification for each of the following statements.
   (a) It is possible that $\mathbb{R}^2$ can be written as countable union of connected paths.
   (b) The cardinality of set of all the polynomials on $\mathbb{R}$ such that complement of their zero set are connected is countable.
   (c) There exists a non-empty open and connected set $A \subset \mathbb{R}^n$ such that every real valued function on A is continuous.
   (d) If a metric space X is path connected, then there exists a continuous function $f : [0,1] \to X$ which is onto.
   (e) There exists a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ such that the graph G_f is connected in $\mathbb{R}^2$ but $\text{int}(\overline{G}_f)$ is non-empty in $\mathbb{R}^2$.
   (f) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^3 \in \mathbb{R} \setminus \mathbb{Q}\}$ is a disconnected subset of $\mathbb{R}^2$ (with the usual topology).
   (g) If a topological space X has only finitely many components, then each component of X must be open in X.
   (h) If τ_l denotes the lower limit topology on $\mathbb{R}$, then the topological space $(\mathbb{R}, \tau_l)$ is totally disconnected.
   (i) $\mathbb{R}$ with the cofinite topology is a locally connected space.

2. Show that a topological space X is connected iff for every $x, y \in X$ with $x \neq y$, there exists a connected subspace of X containing both x and y.

3. Prove that a topological space X is connected iff every nonempty proper subset of X has a nonempty boundary.

4. Let C be a connected subspace of a topological space X and let $A \subset X$ such that $C \cap A \neq \emptyset$ and $C \cap (X \setminus A) \neq \emptyset$. Prove that $C \cap \partial A \neq \emptyset$.

5. Consider the topology τ on $\mathbb{R}$ having $\{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \cap \mathbb{Q} : a, b \in \mathbb{R}, a < b\}$ as a basis. Show that the topological space $(\mathbb{R}, \tau)$ is connected.

6. Show that $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is a compact set in } (\mathbb{R}, \tau_u)\} \cup \{\emptyset\}$ is a topology on $\mathbb{R}$, where τ_u is the usual topology on $\mathbb{R}$. Examine whether the topological space $(\mathbb{R}, \tau)$ is (a) compact (b) connected.

7. Let A and B be nonempty closed (or, open) subsets of a topological space such that both $A \cup B$ and $A \cap B$ are connected. Show that A and B are also connected.

8. If A and B are connected subsets of a topological space such that $A \cap \overline{B} \neq \emptyset$, then prove that $A \cup B$ is connected.

9. If X, Y are connected spaces and if $A \subsetneq X$, $B \subsetneq Y$, then show that $(X \times Y) \setminus (A \times B)$ is connected.

10. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subspaces of a topological space X such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Show that $\bigcup_{n=1}^{\infty} A_n$ is connected.

11. If $\{A\} \cup \{A_\alpha : \alpha \in \Lambda\}$ is a class of connected subsets of a topological space such that $A \cap A_\alpha \neq \emptyset$ for all $\alpha \in \Lambda$, then show that $A \cup \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right)$ is connected.

12. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a class of connected subsets of a topological space such that $A_\alpha \cap A_\beta \neq \emptyset$ for all $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$. Prove that $\bigcup_{\alpha \in \Lambda} A_\alpha$ is connected.

13. Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty compact connected sets in a Hausdorff space. Prove that $\bigcap_{n=1}^{\infty} A_n$ is nonempty, compact and connected.

14. Let X be a connected space and let there exist a non-constant continuous map from X to $\mathbb{R}$ (with the usual topology). Show that X is uncountable.

15. Prove that every connected T_3-space containing at least two points must be uncountable.

16. Let X be a connected space and let τ_l be the lower limit topology on $\mathbb{R}$. Prove that every continuous map from X to $(\mathbb{R}, \tau_l)$ is a constant map.
(Hence, in particular, every continuous map from $(\mathbb{R}, \tau_u)$ to $(\mathbb{R}, \tau_l)$ is a constant map, where τ_u denotes the usual topology on $\mathbb{R}$.)

17. For each $m, n \in \mathbb{N}$, let $B_{m,n} = \{mk + n : k \in \mathbb{Z}\} \cap \mathbb{N}$. Show that
   (a) $\{B_{m,n} : m, n \in \mathbb{N}, \text{g.c.d.}(m, n) = 1\}$ is a basis for some topology τ on $\mathbb{N}$.
   (b) for each prime $p \in \mathbb{N}$, $\{pk : k \in \mathbb{N}\}$ is a closed set in $(\mathbb{N}, \tau)$ and hence deduce that there exist infinitely many prime numbers.
   (c) $(\mathbb{N}, \tau)$ is connected, Hausdorff but not compact.
   (d) if P is the set of all primes, then $P^0 = \emptyset$.

18. Consider $\mathbb{R}^2$ with the usual topology. If A is a countable subset of $\mathbb{R}^2$, then show that $\mathbb{R}^2 \setminus A$ is path connected.

19. Prove that every open connected subspace of $\mathbb{R}^2$ (with the usual topology) is path connected.