MA549: Topology

Assignment 3: Quotient spaces and compact spaces

July - November, 2023

1. State TRUE or FALSE with justification for each of the following statements.
   (a) If $A = \{x \in \mathbb{R} : x^8 - x^7 \leq 200\}$ and $B = \{x^2 - 2x : x \in (0, \infty)\}$, then $A \cap B$ is a compact set in $\mathbb{R}$ with the usual topology.
   (b) If every proper closed subset of a topological space X is compact, then X must be compact.
   (c) If every subset of a Hausdorff space X is compact, then X must be a discrete space.
   (d) A topological space in which every compact subset is closed must be Hausdorff.
   (e) If X is a topological space such that every continuous map $f : X \to \mathbb{R}$ (with the usual metric on $\mathbb{R}$) is bounded, then X must be compact.
   (f) Every topological space which is a continuous image of a topological space with the Bolzano-Weierstrass property must have the Bolzano-Weierstrass property.
   (g) If τ_c denotes the cocountable topology on $\mathbb{R}$, then the topological space $(\mathbb{R}, \tau_c)$ is locally compact.
   (h) The one-point compactification of $\mathbb{N}$ (with the usual topology) is homeomorphic to $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ (with the usual topology).

2. Let $P : X \to Y$ be continuous. Let $f : Y \to X$ be a continuous function such that $p \circ f = I$. Show that p is a quotient map.

3. Let A be subset of topological space X, and let $r : X \to A$ be such that $r(a) = a$ for each $a \in A$. Show that (retraction) r is a quotient map.

4. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $\pi_1(x, y) = x$. Let $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} :$ either $x \geq 0$ or $y = 0($ or both$)\}$. Let q be the restriction of p to A. Show that q is a quotient map that is neither open nor closed.

5. Let $p : X \to Y$ be an open map and A is open in X. Show that $q :\to P(A)$, where $q = p|A$, is an open map.

6. Define equivalence relations on $X = \mathbb{R}^2$ by
   (a) $x_o, y_o) \sim (x_1, y_1)$ if $x_o + y_o^2 = x_1 + y_1^2$,
   (b) $x_o, y_o) \sim (x_1, y_1)$ if $x_o^2 + y_o^2 = x_1^2 + y_1^2$.
   Let X^* be the corresponding quotient space. Find the surfaces in each case which is homeomorphic to corresponding X^*.

7. Let Y be the quotient space obtained from $K-$topology $\mathbb{R}_K$ by collapsing the set K to a point. Let $p : \mathbb{R}_K \to Y$ be a quotient map.
   (a) Show that Y satisfies T_1 axiom but not Hausdorff.
   (b) Show that $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map.

8. Show that $[0, 1]$ is not limit point compact in the lower limit topological space $\mathbb{R}_l$.

9. If X is compact Hausdorff space under two topologies τ and τ', then show that either $\tau = \tau'$ or not comparable.

10. Let X be any nonempty set and let $x_0 \in X$. If $\tau = \{G \subset X : x_0 \notin G\} \cup \{X\}$, then find all the compact subsets of the topological space (X, τ).

11. Consider the topology τ on $\mathbb{R}$ having
$\mathcal{B} = \{(a,b) : a, b \in \mathbb{R}, a < b\} \cup \{(a,b) \setminus K : a, b \in \mathbb{R}, a < b\}$ as a basis, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$.
Examine whether $[0,1]$ is a compact set in the topological space $(\mathbb{R}, \tau)$.

12. Let τ_c be the cocountable topology on a nonempty set X. Find all the (a) compact (b) sequentially compact subsets of the topological space (X, τ_c).

13. If $a, b \in \mathbb{R}$ such that $a < b$, then show that $[a,b]$ is not a compact subset of $\mathbb{R}$ with the lower limit topology.

14. Let (x_n) be a sequence in a topological space X and let $x_n \to x \in X$. Show that
$\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a compact subset of X.

15. Prove that the union of finitely many compact sets in a topological space is compact.
Also, show that the union of infinitely many compact sets in a topological space need not be compact.

16. Let A and B be compact subsets of a topological space X. Show that $A \cap B$ need not be compact but if X is Hausdorff, then $A \cap B$ must be compact.

17. Let G be topological group.
    (a) If A and B are compact in G, then show that $A \cdot B$ is compact.
    (b) Let H be subgroup of G and $p :\to G/H$ be the quotient map. If H is compact, show that p is closed.
    (c) Let H be a compact subgroup of G. Show that if G/H is compact, then G is compact.

18. Let A and B be compact subspaces of the topological spaces X and Y respectively. If W is an open set in $X \times Y$ containing $A \times B$, then prove that there exist open sets G and H in X and Y respectively such that $A \times B \subset G \times H \subset W$.

19. For each $n \in \mathbb{N}$, let $X_n = [0,1]$ with the usual topology, and $X = \prod\limits_{n=1}^{\infty} X_n$.
    (a) Show that X is not limit point compact with respect to uniform topology.
    (b) Show that $\prod\limits_{n=1}^{\infty} X_n$ with the box topology is not compact.

20. Let X, Y be topological spaces with Y compact. Prove that the projection map $p_X : X \times Y \to X$ is a closed map.

21. Let X, Y be topological spaces with Y compact. If $f : X \to Y$ is such that $\{(x, f(x)) : x \in X\}$ (*i.e.* the graph of f) is closed in the product space $X \times Y$, then prove that f is continuous.

22. Show that the closure of a compact subset of a topological space need not be compact.
Show, however, that the closure of a compact subset of a regular space is compact.

23. Let X be a topological space. If for each $x \in X$, there exists an open set G in X containing x such that $\overline{G}$ is a compact Hausdorff space, then show that X is Hausdorff.

24. Show that a compact topological space need not be separable but that every compact metrizable space must be separable.

25. If X is a Lindelöf space and Y is a compact space, then show that the product space $X \times Y$ is a Lindelöf space.

26. Let A_n be sequence of closed nowhere dense sets in compact Hausdorff space X. Show that $(\cup A_n)^\circ = \emptyset$.

27. Let X and Y be topological spaces such that Y is compact. Let $p : X \to Y$ be continuous, closed and onto such that $p^{-1}(\{y\})$ is a compact set in X for each $y \in Y$. Show that X is compact.

28. Let X and Y be topological spaces such that X is second countable. Let $f : X \to Y$ be continuous, closed and onto such that $f^{-1}(\{y\})$ is a compact set in X for each $y \in Y$. Show that Y is second countable.

29. Let f be a continuous map from a compact space X onto a Hausdorff space Y and let g be a map from Y to a topological space Z. If $g \circ f$ is continuous, then prove that g is continuous.

30. Let X be a metrizable topological space. Prove that the following statements are equivalent.
    (a) X is compact.
    (b) X is bounded with respect to every metric on X that induces the topology of X.
    (c) Every continuous map $f : X \to \mathbb{R}$ (with the usual metric on $\mathbb{R}$) is bounded.

31. Let (X, d) be a compact metric space and let $f : X \to X$ be such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that f is onto and so f is a homeomorphism.
    Show also that the compactness of (X, d) is, in general, necessary.

32. Let X be a compact Hausdorff space and let $f : X \to X$ be continuous. Show that there exists a nonempty compact set K in X such that $f(K) = K$.

33. Show that a T_1-space X has the Bolzano-Weierstrass property iff every countable open cover of X has a finite subcover.

34. Let X be a topological space with the Bolzano-Weierstrass property and let Y be a first countable space. Show that the projection map $p_Y : X \times Y \to Y$ is closed.

35. Let Y be a nonempty dense subset of a Hausdorff space X. If the subspace Y is locally compact, then show that Y is open in X.

36. Let X be a compact Hausdorff space and let $\mathbb{R}$ be equipped with the usual topology. Consider the real vector space $C(X, \mathbb{R})$ of all continuous maps from X to $\mathbb{R}$, where the vector space operations are defined pointwise. Prove that $C(X, \mathbb{R})$ is finite dimensional iff X is finite.

37. Let K be a nonempty compact subset and let C be a nonempty closed subset of a locally compact Hausdorff space X such that $K \cap C = \emptyset$. Prove that there exists a continuous map $f : X \to [0, 1]$ (with the usual topology) such that $f(K) = \{0\}$ and $f(C) = \{1\}$.

38. Let f_n be real-valued sequence of monotone increasing continuous function on a compact topological space X. If f_n converges point wise to f, than show that f_n converges to f uniformly.

39. Show that a compact Hausdorff space is metrizable iff it is second countable.