MA549: Topology

(Assignment 2: Continuous map and homeomorphism)

July - November, 2023

1. Let X, Y be topological spaces and let $x_0 \in X$. If $f : X \to Y$ is continuous at x_0 and G is an open set in Y containing $f(x_0)$, then is it necessary that $f^{-1}(G)$ is an open set in X? Justify.

2. Let A be a nonempty subset of a topological space X and let $f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}$
Prove that $f : X \to (\mathbb{R}, \tau_u)$ is continuous iff $\partial A = \emptyset$, where τ_u is the usual topology on $\mathbb{R}$.

3. Let τ_f and τ_c denote respectively the cofinite topology and the cocountable topology on $\mathbb{R}$. If $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$
then determine all the points of $\mathbb{R}$ at which $g : (\mathbb{R}, \tau_c) \to (\mathbb{R}, \tau_f)$ is continuous.

4. Let τ_l denote the lower limit topology on $\mathbb{R}$ and let $f(x) = -x$ for all $x \in \mathbb{R}$. Show that $f : (\mathbb{R}, \tau_l) \to (\mathbb{R}, \tau_l)$ is discontinuous at each point of $\mathbb{R}$.

5. Let $f : X \to \mathbb{R}$ be continuous at a point $x_0 \in X$, where X is a topological space and $\mathbb{R}$ is equipped with the usual topology. If $f(x_0) > 0$, then show that there exists an open set G in X containing x_0 such that $f(x) > 0$ for all $x \in G$.

6. Let (X, τ) and (Y, τ') be topological spaces such that every map from (X, τ) to (Y, τ') is continuous. Prove that τ is the discrete topology on X or τ' is the indiscrete topology on Y.

7. Let X, Y be nonempty sets and let $x_0 \in X$, $y_0 \in Y$. If $\tau = \{G \subset X : x_0 \in G\} \cup \{\emptyset\}$ and $\tau' = \{H \subset Y : y_0 \in H\} \cup \{\emptyset\}$, then show that a map $f : (X, \tau) \to (Y, \tau')$ is continuous iff $f(x_0) = y_0$ or f is constant.

8. Let τ_f denote the cofinite topology on $\mathbb{R}$. Examine whether $\varphi : (\mathbb{R}, \tau_f) \to (\mathbb{R}, \tau_f)$ is continuous, if φ is defined as
(a) $\varphi(x) = x^3 - 4x - 1$ for all $x \in \mathbb{R}$.
(b) $\varphi(x) = \sin x$ for all $x \in \mathbb{R}$.
(c) $\varphi(x) = x + \sin x$ for all $x \in \mathbb{R}$.
(d) $\varphi(x) = e^x$ for all $x \in \mathbb{R}$.

9. Show that $\tau = \{G \subset \mathbb{N} : n \in G \text{ and } m|n \Rightarrow m \in G\}$ is a topology on $\mathbb{N}$ which is different from the discrete topology on $\mathbb{N}$.
Show also that a map $f : (\mathbb{N}, \tau) \to (\mathbb{N}, \tau)$ is continuous iff for all $m, n \in \mathbb{N}$, $m|n \Rightarrow f(m)|f(n)$.

10. Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be a family of topologies on a nonempty set X, let (Y, τ) be a topological space and let $f : X \to Y$. Prove that $f : \left(X, \bigcap_{\alpha \in \Lambda} \tau_\alpha\right) \to (Y, \tau)$ is continuous iff for each $\alpha \in \Lambda$, $f : (X, \tau_\alpha) \to (Y, \tau)$ is continuous.

11. Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be a family of topologies on a nonempty set Y, let (X, τ) be a topological space and let $f : X \to Y$. If τ' is the unique smallest topology on Y containing $\bigcup_{\alpha \in \Lambda} \tau_\alpha$, then show that $f : (X, \tau) \to (Y, \tau')$ is continuous iff for each $\alpha \in \Lambda$, $f : (X, \tau) \to (Y, \tau_\alpha)$ is continuous.

12. Let τ_f and τ_u denote respectively the cofinite topology and the usual topology on $\mathbb{R}$. If $\varphi : (\mathbb{R}, \tau_f) \to (\mathbb{R}, \tau_u)$ is continuous, then prove that φ is a constant map.

13. Let X, Y be topological spaces and let $f : X \to Y$. Prove that f is continuous iff $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for all $B \subset Y$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $B \subset Y$.

14. Let X, Y be topological spaces and let $f : X \to Y$. Prove that f is continuous iff $\partial(f^{-1}(B)) \subset f^{-1}(\partial B)$ for all $B \subset Y$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $B \subset Y$.

15. Let X, Y be topological spaces and let $f : X \to Y$. Prove that f is continuous iff $f^{-1}(B^0) \subset (f^{-1}(B))^0$ for all $B \subset Y$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $B \subset Y$.

16. Let X, Y be topological spaces and let $f : X \to Y$. Prove that f is continuous iff $f(A') \subset \overline{f(A)}$ for all $A \subset X$.
Also, show that the continuity of f need not give the equality in the above inclusion for some $A \subset X$.

17. Let $f : X \to Y$ be continuous, where X, Y are topological spaces. If $x \in X$ is a limit point of a subset A of X, then is it necessary that $f(x)$ is a limit point of $f(A)$? Justify.

18. Let X, Y be topological spaces and let $f : X \to Y$ be continuous and onto. If A is a dense set in X, then prove that $f(A)$ is dense in Y and that the onto condition on f is, in general, necessary for this.
Show also that $f(A)$ can be dense in Y even when $A \subset X$ is not dense in X and $f : X \to Y$ is neither continuous nor onto.

19. Consider the topology $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is finite or } 0 \in \mathbb{R} \setminus G\}$ on $\mathbb{R}$. Let τ_u be the usual topology on $\mathbb{R}$ and let $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_u)$ be continuous. Show that there exists a countable subset A of $\mathbb{R}$ such that $f(x) = f(0)$ for all $x \in \mathbb{R} \setminus A$.

20. Consider the topology τ on $\mathbb{R}$ having $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{(a, b) \cap \mathbb{Q} : a, b \in \mathbb{R}, a < b\}$ as a basis. Also, let τ_u be the usual topology on $\mathbb{R}$.
(a) Examine whether $\mathbb{R} \setminus \mathbb{Q}$ is closed in the topological space $(\mathbb{R}, \tau)$.
(b) If $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_u)$ is a continuous map such that $f(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$, then show that $f(x) = 0$ for all $x \in \mathbb{R}$.

21. Let X, Y be topological spaces. Prove that a map $f : X \to Y$ is an open map iff $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$ for all $B \subset Y$.

22. Let X, Y be topological spaces. Show that $f : X \to Y$ is a closed map iff for each $y \in Y$ and for each open set G in X with $f^{-1}(\{y\}) \subset G$, there exists an open set H in Y containing y such that $f^{-1}(H) \subset G$.

23. Let X, Y, Z be topological spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous maps such that $g \circ f : X \to Z$ is a homeomorphism. If g is one-one, then show that both f and g are homeomorphisms.

24. Examine whether the following pairs of topological spaces are homeomorphic.
    (a) $(\mathbb{R}, \tau_u)$ and $(\mathbb{R}, \tau_c)$, where τ_u and τ_c are respectively the usual topology and the cocountable topology on $\mathbb{R}$.
    (b) $(\mathbb{R}, \tau_f)$ and $(\mathbb{R}, \tau_c)$, where τ_f and τ_c are respectively the cofinite topology and the cocountable topology on $\mathbb{R}$.
    (c) $(\mathbb{R}, \tau)$ and $(\mathbb{R}, \tau')$, where $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$ and $\tau' = \{G \subset \mathbb{R} : 0 \notin G\} \cup \{\mathbb{R}\}$.

25. Consider $\mathbb{R}^2$ with the usual topology. Show that the subspaces $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $\{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ and $\{(x, y) \in \mathbb{R}^2 : 2x^2 + 3y^2 < 1\}$ of $\mathbb{R}^2$ are homeomorphic.

26. State TRUE or FALSE with justification for each of the following statements.
    (a) If X is an indiscrete topological space, then there cannot exist any non-constant continuous map $f : X \to (\mathbb{R}, \tau_u)$.
    (b) If τ_u is the usual topology on $\mathbb{R}$, then every one-one, onto and continuous map $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$ is a homeomorphism.
    (c) Every continuous map $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$ is open or closed.
    (d) If τ is the cofinite topology on $\mathbb{R}$, then every bijection $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ is a homeomorphism.
    (e) If X, Y are topological spaces such that X is homeomorphic to a subspace of Y and Y is homeomorphic to a subspace of X, then X and Y must be homeomorphic.

27. Let X, Y be topological spaces and let $f : X \to Y$. Show that the map φ of X to the subspace $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $X \times Y$, defined by $\varphi(x) = (x, f(x))$ for all $x \in X$, is a homeomorphism iff f is continuous.
    Deduce that every topological space X is homeomorphic to the diagonal $\triangle = \{(x, x) : x \in X\}$ of $X \times X$.

28. Let τ_l be the lower limit topology on $\mathbb{R}$. Examine whether the set $\{(x, y) \in \mathbb{R}^2 : x + y < 0\}$ is closed in the product space $(\mathbb{R}, \tau_l) \times (\mathbb{R}, \tau_l)$.

29. If $X = (\mathbb{R}, \tau_c)$, where τ_c is the cocountable topology on $\mathbb{R}$, then determine the interior of the set $(\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q}) \times \cdots$ in the countably infinite product $X^{\mathbb{N}}$ with the product topology.

30. If τ_c is the cocountable topology on $\mathbb{R}$, then show that the product topology of $(\mathbb{R}, \tau_c)$ and $(\mathbb{R}, \tau_c)$ is strictly finer than the cocountable topology on $\mathbb{R} \times \mathbb{R}$.

31. Let X, Y be topological spaces and let $A \subset X$, $B \subset Y$. Prove that $\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$.

32. Let A, B, C, D be topological spaces and let $f : A \to B$ and $g : C \to D$ be continuous. If $\varphi(a, c) = (f(a), g(c))$ for all $(a, c) \in A \times C$, then show that $\varphi : A \times C \to B \times D$ is continuous.

33. Let Λ be an infinite set and let for each $\alpha \in \Lambda$, X_α be a discrete space. Prove that $\prod_{\alpha \in \Lambda} X_\alpha$ is a discrete space in the box topology, but if for each $\alpha \in \Lambda$, X_α has more than one point, then

$\prod_{\alpha \in \Lambda} X_\alpha$ is not a discrete space in the product topology.

34. Consider the usual topology on $\mathbb{R}$ and let $A = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \{n \in \mathbb{N} : x_n \neq 0\}$ is finite$\}$. Show that A is dense in $\mathbb{R}^{\mathbb{N}}$ with the product topology.

35. Determine the closure of $\{(x_n) \in \mathbb{R}^{\mathbb{N}} : \{n \in \mathbb{N} : x_n \neq 0\}$ is finite$\}$ in $\mathbb{R}^{\mathbb{N}}$ with the box topology.