MA549: Topology

(Assignment 1: Topological spaces and basis)

July - November, 2023

1. List all possible topologies on the set $\{a, b\}$ and also on the set $\{a, b, c\}$.

2. Let X be a set with exactly 4 elements. Does there exist a topology τ on X such that there are precisely 14 open sets in the topological space (X, τ)? Justify.

3. Examine whether τ is a topology on X, where
   (a) $X = \mathbb{R}$ and $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.
   (b) $X = \mathbb{R}$ and $\tau = \{[a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.
   (c) $X = \mathbb{R}$ and $\tau = \{(a, \infty) : a \in \mathbb{Q}\} \cup \{\emptyset, \mathbb{R}\}$.
   (d) $X = \mathbb{R}$ and $\tau = \{[-a, a) : a \in \mathbb{R},\ a > 0\} \cup \{\emptyset, \mathbb{R}\}$.
   (e) $X = \mathbb{N}$ and $\tau = \{\{n, n+1, n+2, ...\} : n \in \mathbb{N}\} \cup \{\emptyset\}$.
   (f) $X = \mathbb{R}$ and $\tau = \{G \subset \mathbb{R} : \mathbb{Q} \not\subset G\} \cup \{\mathbb{R}\}$.
   (g) $X = \mathbb{R}$ and $\tau = \{G \subset \mathbb{R} : G \subset \mathbb{Q} \text{ or } \mathbb{Q} \subset G\}$.
   (h) $X = [0, 1]$ and $\tau = \{G \subset [0, 1] : GG \subset G\}$.
   (For each $A \subset [0, 1]$, define $AA = \{xy : x, y \in A\}$.)
   (i) X is an infinite set and $\tau = \{G \subset X : X \setminus G \text{ is infinite or } \emptyset\}$.
   (j) X is a nonempty set, $x_0 \in X$ and $\tau = \{G \subset X : X \setminus G \text{ is finite or } x_0 \in X \setminus G\}$.
   (k) X is an uncountable set and $\tau = \{G \subset X : G \text{ is countable or } X \setminus G \text{ is countable}\}$.

4. Let $\tau = \{G_k : k \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}^2\}$, where for each $k \in \mathbb{R}$, $G_k = \{(x, y) \in \mathbb{R}^2 : x > y + k\}$. Prove that τ is a topology on $\mathbb{R}^2$.
   Is τ a topology on $\mathbb{R}^2$ if '$k \in \mathbb{R}$' is replaced by (a) '$k \in \mathbb{N}$'? (b) '$k \in \mathbb{Q}$'? Justify.

5. Let X, Y be nonempty sets and let $f : X \to Y$.
   (a) If $\mathfrak{J}'$ is a topology on Y, then show that $\mathfrak{J} = \{f^{-1}(H) : H \in \mathfrak{J}'\}$ is a topology on X.
   (b) If τ is a topology on X and if f is one-one and onto, then show that $\tau' = \{f(G) : G \in \tau\}$ is a topology on Y.
   Show also that both the conditions one-one and onto on f is, in general, necessary.

6. Let $\mathcal{X}$ be the class of all (proper) prime ideals of a commutative ring R with unity and for each $E \subset R$, let $\mathcal{V}(E) = \{P \in \mathcal{X} : E \subset P\}$. Prove that $\tau = \{\mathcal{X} \setminus \mathcal{V}(E) : E \subset R\}$ is a topology on $\mathcal{X}$.
   (τ is called the Zariski topology on $\mathcal{X}$ and the topological space $(\mathcal{X}, \tau)$ is called the prime spectrum of R, denoted by Spec(R).)

7. Let τ_f and τ_c denote respectively the cofinite and the cocountable topologies on a nonempty set X. Find a necessary and sufficient condition on X such that $\tau_f = \tau_c$.

8. Let (X, τ) be a topological space in which every singleton subset of X is open. Prove that τ is the discrete topology on X.

9. Let τ be a topology on an infinite set X such that every infinite subset of X is open in (X, τ). Show that τ is the discrete topology on X.

10. Let τ be a topology on an infinite set X such that every infinite subset of X is closed in (X, τ). Prove that τ is the discrete topology on X.

11. Let τ and τ' be topologies on $\mathbb{R}$ such that every countable subset of $\mathbb{R}$ is open in $(\mathbb{R}, \tau)$ and every uncountable subset of $\mathbb{R}$ is open in $(\mathbb{R}, \tau')$. Determine (with justication) which of the following is true.
(a) τ is strictly finer than τ'.
(b) τ' is strictly finer than τ.
(c) τ and τ' are not comparable.
(d) $\tau = \tau'$.

12. Prove that on a finite set with exactly n elements there are at most $2^{(2^n-2)}$ distinct topologies. (The problem of determining the exact number of topologies on a finite set is still open. However, in some particular cases the results are known. For example, if $k(n)$ denotes the number of distinct topologies on a set with exactly n elements, then it is already known that $k(4) = 355$, $k(5) = 6942$, $k(6) = 209527$, $k(7) = 9535241$.)

13. Prove that uncountably many distinct topologies can be defined on any infinite set.

14. Let X be a nonempty set and let $\lambda > 0$. Consider the metric d on X defined by
$$d(x, y) = \begin{cases} \lambda & \text{if } x \neq y, \\ 0 & \text{if } x = y \ (x, y \in X). \end{cases}$$
Find the topology on X induced by d.

15. Let d be a metric on a nonempty set X and let $\lambda > 0$. Consider the metrics d_1, d_2 and d_3 on X defined by $d_1(x, y) = \lambda d(x, y)$, $d_2(x, y) = \min\{1, d(x, y)\}$ and $d_3(x, y) = \frac{d(x,y)}{1+d(x,y)}$ for all $x, y \in X$. Prove that d, d_1, d_2 and d_3 induce the same topology on X.

16. Let d be a metric on a nonempty set X and let τ_d be the topology on X induced by d. Prove that every finite subset of X is closed in (X, τ_d).
Without using this fact, show also that every finite subset of $\mathbb{R}$ is closed in $\mathbb{R}$ with the usual topology.

17. Let X be a topological space. If A is an open subset of X and B is a closed subset of X, then prove that $A \setminus B$ is an open subset of X and $B \setminus A$ is a closed subset of X.

18. Let $\mathcal{X}$ be the class of all (proper) prime ideals of a commutative ring R with unity. Consider the Zariski topology $\tau = \{\mathcal{X} \setminus \mathcal{V}(E) : E \subset R\}$ on $\mathcal{X}$, where $\mathcal{V}(E) = \{P \in \mathcal{X} : E \subset P\}$ for each $E \subset R$. If $P \in \mathcal{X}$, then show that $\{P\}$ is a closed set in the topological space $(\mathcal{X}, \tau)$ iff P is a maximal ideal of R.

19. Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a class of closed subsets of a topological space X. Assume that for each $x \in X$, there exists an open subset G_x of X containing x such that $\{\alpha \in \Lambda : G_x \cap A_\alpha \neq \emptyset\}$ is a finite set. Show that $\bigcup_{\alpha\in\Lambda} A_\alpha$ is closed in X.

20. Let G be a nonempty open subset of $\mathbb{R}$ (with the usual topology) such that $x - y \in G$ for all $x, y \in G$. Show that $G = \mathbb{R}$.
(Thus $\mathbb{R}$ is the only open subgroup of the additive group $\mathbb{R}$ with the usual topology.)

21. Let F be a nonempty closed subset of $\mathbb{R}$ (with the usual topology) such that $x - y \in F$ for all $x, y \in F$. Show that either $F = \mathbb{R}$ or $F = \alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$.
(This provides the class of all closed subgroups of the additive group $\mathbb{R}$ with the usual topology.)

22. Prove that every subgroup of the group $(\mathbb{R}, +)$ is either cyclic (and hence closed in $(\mathbb{R}, \tau_u)$) or dense in $(\mathbb{R}, \tau_u)$.

23. Consider the topology $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is finite or } 2 \notin G\}$ on $\mathbb{R}$. If τ_u is the usual topology on $\mathbb{R}$, examine whether $\tau \cup \tau_u$ is a topology on $\mathbb{R}$.

24. State TRUE or FALSE with justification for each of the following statements.
    (a) There exists a topological space with exactly 100 (distinct) open sets.
    (b) If $\{\tau_n\}_{n=1}^{\infty}$ is an ascending chain of topologies on a nonempty set X, then $\bigcup_{n=1}^{\infty} \tau_n$ must be a topology on X.

25. Let τ_u be the usual topology on $\mathbb{R}$. Show that $\tau_u|_{\mathbb{N}}$ is the discrete topology on $\mathbb{N}$ but that $\tau_u|_{\mathbb{Q}}$ is not the discrete topology on $\mathbb{Q}$.

26. Consider $\mathbb{R}$ with the usual topology. Find the relative topology on $\{\frac{1}{n} : n \in \mathbb{N}\}$.

27. Is it possible to define a topology τ on $\mathbb{R}$ such that τ is not the discrete topology on $\mathbb{R}$ and the relative topology on $[0, 1]$ induced by τ is the discrete topology on $[0, 1]$?

28. Let τ_l denote the lower limit topology on $\mathbb{R}$. Examine whether the set $\{x \in \mathbb{Q} : x^2 > 5\}$ is closed in the topological space $(\mathbb{Q}, \tau_l|_{\mathbb{Q}})$.

29. Let (X, τ) be a topological space and let $A(\neq \emptyset) \subset X$. Prove that $\tau' = \{G \cup (H \cap A) : G, H \in \tau\}$ is a topology on X such that A is open in the topological space (X, τ') and $\tau|_A = \tau'|_A$.

30. Let (X, τ) be a topological space such that for each nonempty finite subset A of X (or, for each subset A of X containing exactly two elements), $\tau|_A$ is the indiscrete topology on A. Is it necessary that τ is the indiscrete topology on X?

31. Let F be a closed subspace of a topological space X and let G be an open set in F. If H is an open set in X containing G, then prove that $G \cup (H \setminus F)$ is open in X.

32. Let Y and Z be subspaces of a topological space X and let $A \subset Y \cap Z$. If A is open in both Y and Z, then show that A is open in $Y \cup Z$.

33. Let Y be a nonempty subset of a topological space X and let Z be a nonempty open subset of $X \setminus Y$. If $A \subset Z$ such that A is open in $Y \cup Z$, then show that A is open in X.

34. Let X be a topological space and $A \subset X$. Show that A is closed in X iff for every $x \in \overline{A}$, there exists an open set G in X containing x such that $G \cap A$ is closed in G.

35. Let X be a topological space and $A \subset X$. If for each $a \in A$, there exists an open set G in X containing a such that $G \cap A$ is closed in G, then show that there exist an open set V in X and a closed set F in X such that $A = V \cap F$.

36. Consider the subspace $(X, \tau_u|_X)$ of the topological space $(\mathbb{R}, \tau_u)$, where $X = \{m+n\pi : m, n \in \mathbb{Z}\}$ and τ_u is the usual topology on $\mathbb{R}$. Examine whether $\{0\}$ is an open set in $(X, \tau_u|_X)$.

37. Let (X, d) be a metric space and let $Y(\neq \emptyset) \subset X$. If ρ is the metric on Y induced by d (*i.e.* $\rho = d|_{Y\times Y}$), τ_d is the topology on X induced by d and τ_ρ is the topology on Y induced by ρ, then show that $\tau_\rho = \tau_d|_Y$.

38. Consider $\mathbb{R}$ with the usual topology and let $x \in \mathbb{R}$. Find the closure of the set $\{x + r : r \in \mathbb{Q}\}$.

39. Consider $\mathbb{R}$ with the lower limit topology. Does there exist a subset A of $\mathbb{R}$ such that $\overline{A} = [0, 1) \cup (1, 2)$? Justify.

40. Let X be a topological space and let $A, B \subset X$. Prove that $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$, and also that the inclusion can be strict.

41. Let τ, τ' be topologies on a nonempty set X. Prove that $\tau \subset \tau'$ iff $\overline{A}^{\tau} \supset \overline{A}^{\tau'}$ for all $A \subset X$. (Hence $\tau = \tau'$ iff $\overline{A}^{\tau} = \overline{A}^{\tau'}$ for all $A \subset X$. However, even if $\tau \subsetneq \tau'$, it can happen that $\overline{A}^{\tau} = \overline{A}^{\tau'}$ for some $A \subset X$.)

42. If G is an open set in a topological space X, then show that $\overline{G} = \overline{X \setminus \overline{X \setminus \overline{G}}}$.

43. Let A be a subset of a topological space X. Show that $\overline{A} \setminus A$ is closed in X iff there exist a closed set F in X and an open set G in X such that $A = F \cap G$.

44. Let G be an open set in a topological space X and let $A \subset X$. Prove that $G \cap A = \emptyset$ iff $G \cap \overline{A} = \emptyset$.
Hence, or otherwise, show that if U and V are disjoint open sets in X, then $(\overline{U})^0 \cap (\overline{V})^0 = \emptyset$.

45. Prove that a subset G of a topological space X is open in X iff $\overline{G \cap \overline{A}} = \overline{G \cap A}$ for all $A \subset X$.

46. Let Y be a closed subspace of a topological space X. Let $A \subset X$ and let H be an open set in Y such that $A \cap Y \subset H$. Prove that $A \cap \overline{Y \setminus H}^{X} = \emptyset$.

47. Let A, B be subsets of a topological space. Show that $(A \setminus B)^0 \subset A^0 \setminus B^0$ and also show that the equality need not occur in the above inclusion.

48. Prove that a subset F of a topological space X is closed in X iff $(F \cup A^0)^0 = (F \cup A)^0$ for all $A \subset X$.

49. Let Y be a subspace of a topological space X and let $A \subset Y$. Show that $\mathrm{Int}_Y(A) \supset Y \cap \mathrm{Int}_X(A)$ and $\mathrm{Int}_X(A) = \mathrm{Int}_Y(A) \cap \mathrm{Int}_X(Y)$, although in general, $\mathrm{Int}_Y(A) \neq Y \cap \mathrm{Int}_X(A)$.

50. For every subset A of a topological space X, show that $(X \setminus A)^0 = X \setminus \overline{A}$.
(Other similar 'commutative' relations are true for closure, interior and complement. For example, $X \setminus A^0 = \overline{X \setminus A}$.)

51. Let A, B be subsets of a topological space. Prove that $\overline{A} \cap B^0 = \overline{A \cap B} \cap B^0$.

52. Let G be an open subset of a topological space. Prove that $G \subset (\overline{G})^0$, and that the inclusion can be strict. Prove, however, that $\overline{G} = \overline{(\overline{G})^0}$.

53. If F is a closed set in a topological space, then show that $(\overline{F^0})^0 = F^0$.

54. If A is a subset of a topological space, then show that $\overline{A^0} = \overline{(\overline{A^0})^0}$.

55. Let A be an open set in a topological space X and let $B \subset X$. Show that $(\overline{A \cap B})^0 = (\overline{A})^0 \cap (\overline{B})^0$.

56. If $\tau = \{\{n, n+1, ...\} : n \in \mathbb{N}\} \cup \{\emptyset\}$, then in the topological space $(\mathbb{N}, \tau)$, find $\{4, 13, 28, 37\}'$ and also find all the subsets A of $\mathbb{N}$ for which $A' = \mathbb{N}$.

57. If $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then in the topological space $(\mathbb{R}, \tau)$, determine $[4, 10)'$ and $\mathbb{Z}'$.

58. If $\tau = \{G \subset \mathbb{R} : 2 \in G\} \cup \{\emptyset\}$, then determine (with justification) $(2, 4)'$ and $[2, 4]'$ in the topological space $(\mathbb{R}, \tau)$.

59. Let A be a subset of a topological space X such that each subset of A is closed in X. Show that A has no limit point in X.

60. If $\tau = \{G \subset \mathbb{R} : \mathbb{R} \setminus G \text{ is finite or } 0 \in \mathbb{R} \setminus G\}$, then determine (with justification) $\mathbb{Q}^0$, $\overline{\mathbb{R} \setminus \mathbb{Q}}$, $[0, 1]'$ and $\partial\mathbb{Q}$ in the topological space $(\mathbb{R}, \tau)$.

61. Let A be a subset of a topological space. Prove that $\partial A^0 \subset \partial A$ and $\partial\overline{A} \subset \partial A$ and that both these inclusions can be strict.

62. Let A be a subset of a topological space. Show that $(A \cap \partial A)^0 = \emptyset$. Is it necessary that $(\partial A)^0 = \emptyset$?

63. Consider $\mathbb{R}$ with the usual topology. Does there exist a set $A \subset \mathbb{R}$ such that $\partial A = [0, 1]$? Justify.

64. Let A, B be subsets of a topological space X. Prove that $\partial(A \cup B) \subset \partial A \cup \partial B$, and that this inclusion can be strict. Show, however, that if $\overline{A} \cap \overline{B} = \emptyset$, then $\partial(A \cup B) = \partial A \cup \partial B$.

65. Let A, B be subsets of a topological space such that $\partial A \cap \partial B = \emptyset$. Prove that $(A \cup B)^0 = A^0 \cup B^0$.

66. State TRUE or FALSE with justification for each of the following statements.
   (a) Every uncountable subset of $\mathbb{R}$ (with the usual topology) has a limit point in $\mathbb{R}$.
   (b) If every singleton subset of a topological space X is dense in X, then X must be an indiscrete space.
   (c) If no proper subset of a topological space X is dense in X, then X must be a discrete space.
   (d) A nonempty open set in a topological space X cannot be nowhere dense in X.

67. Let A be a nonempty subset of a topological space X. Show that A is dense in the subspace $\overline{A}^X$.

68. Let Y be a subspace of a topological space X. If $D \subset Y$ is dense in Y, then prove that D is dense in the subspace $\overline{Y}^X$.
(Hence, if Y is a dense subspace of a topological space X and if $D \subset Y$ is dense in Y, then D

is dense in X.)

69. Let D be a dense set in a topological space X and let $Y(\neq \emptyset) \subset X$. Show that $D \cap Y$ need not be dense in Y, but if moreover Y is open in X, then $\overline{D \cap Y}^X = \overline{Y}^X$ and hence deduce that $D \cap Y$ is dense in Y.

70. Let D_1 and D_2 be dense sets in a topological space X. Show that $D_1 \cap D_2$ need not be dense in X, but if moreover D_1 or D_2 is open in X, then $D_1 \cap D_2$ is dense in X.

71. Let F be a closed subset of a topological space X. Prove that F is nowhere dense in X iff $X \setminus F$ is dense in X. Is this result true for an arbitrary subset F of X? Justify.

72. Prove that a subset A of a topological space X is nowhere dense in X iff $A \subset \overline{X \setminus \overline{A}}$.

73. Let A be a subset of a topological space X. Prove that the following statements are equivalent.
(a) A is nowhere dense in X.
(b) $(X \setminus A)^0$ is dense in X.
(c) $A \subset \partial\overline{A}$.

74. Let Y be a nonempty open set in a topological space X and let $A \subset Y$. If A is nowhere dense in the subspace Y, then show that A is nowhere dense in X.

75. Prove that for each $1 \leq m < n$, $A = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_{m+1} = x_{m+2} = \cdots = x_n = 0\}$ is nowhere dense in $\mathbb{R}^n$ with the usual topology.

76. Show that for a set A in a topological space X, ∂A need not be nowhere dense in X, but that ∂A is nowhere dense in X if A is either open or closed in X.

77. Prove that in a topological space X, every closed nowhere dense set is the boundary of some open set in X.

78. Prove that every finite union of nowhere dense sets in a topological space X is nowhere dense in X.
Give examples to show that an infinite union of nowhere dense sets in a topological space X can/need not be nowhere dense in X.

79. Let X be a topological space and let $x \in X$. If x has a finite local basis in X, then show that x has a local basis in X consisting of precisely one member.

80. Let $\mathcal{B}$ be a class of subsets of a nonempty set X. Show that $\mathcal{B}$ is a basis for the discrete topology on X iff $\mathcal{B} \supset \{\{x\} : x \in X\}$.

81. Show that $\{(a, b) : a, b \in \mathbb{Q}, a < b\}$ is a basis for the usual topology on $\mathbb{R}$.

82. Show that $\{[a, b) : a, b \in \mathbb{Q}, a < b\}$ is a basis for a topology τ on $\mathbb{R}$ which is strictly coarser than the lower limit topology on $\mathbb{R}$ but strictly finer than the usual topology on $\mathbb{R}$.
Determine the closures of $(0, \sqrt{2})$ and $(\sqrt{2}, 3)$ in the topological space $(\mathbb{R}, \tau)$.

83. Examine whether each of the following classes is a basis for some topology on $\mathbb{R}$.
(a) $\{[a, b] : a \in \mathbb{Q}, b \in \mathbb{R} \setminus \mathbb{Q}, a < b\}$

(b) $\{[a,b] : a,b \in \mathbb{R}, a < b\}$
(c) $\{[a,b] : a,b \in \mathbb{Q}, a < b\}$

84. Let τ_u and τ_c denote respectively the usual topology and the cocountable topology on $\mathbb{R}$. Examine whether $\tau_u \cup \tau_c$ is a basis for some topology on $\mathbb{R}$.

85. Let X be any nonempty set. Prove that $\mathcal{S} = \{X \setminus \{x\} : x \in X\}$ is a subbasis for the cofinite topology τ_f on X. Is $\mathcal{S}$ a basis for τ_f on X? Is $\mathcal{S}$ a basis for some topology on X? Justify.

86. For each $(a,b,c) \in \mathbb{R}^3$ with $a^2 + b^2 \neq 0$, let $E(a,b,c) = \{(x,y) \in \mathbb{R}^2 : ax + by + c = 0\}$ and let τ be the topology on $\mathbb{R}^2$ generated by the class $\{E(a,b,c) : a,b,c \in \mathbb{R}, a^2 + b^2 \neq 0\}$. Examine whether $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is an open set in the topological space $(\mathbb{R}^2, \tau)$.

87. Let τ be the topology on $\mathbb{R}$ having $\mathcal{S} = \{\mathbb{R} \setminus \{x\} : x \in \mathbb{R}\} \cup \{\{x\} : x (\neq 2) \in \mathbb{R}\}$ as a subbasis. If A is an infinite subset of $\mathbb{R}$, then show that $A' \neq \emptyset$ in the topological space $(\mathbb{R}, \tau)$.

88. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ and let $\mathcal{B} = \{(a,b) : a,b \in \mathbb{R}, a < b\} \cup \{(a,b) \setminus K : a,b \in \mathbb{R}, a < b\}$. Show that $\mathcal{B}$ is a basis for some topology on $\mathbb{R}$. (This topology is called the K-topology on $\mathbb{R}$.)
Prove that the K-topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$ but that it is not comparable with the lower limit topology on $\mathbb{R}$.

89. Let $\mathcal{B}$ be a basis for a topology τ on a nonempty set X. Prove that a set $D \subset X$ is dense in X iff $D \cap B \neq \emptyset$ for every $B (\neq \emptyset) \in \mathcal{B}$.
Does this remain true if $\mathcal{B}$ is merely a subbasis for τ on X? Justify.

90. Let τ be the topology on $\mathbb{R}$ having $\{[a,b) : a,b \in \mathbb{R}, a < b\} \cup \{(a,b] : a,b \in \mathbb{R}, a < b\}$ as a subbasis. Examine whether $\mathbb{Q}$ is a dense set in the topological space $(\mathbb{R}, \tau)$.

91. Let $\{\tau_\alpha\}_{\alpha \in \Lambda}$ be a family of topologies on a nonempty set X. Show that there is a unique smallest topology on X containing all τ_α $(\alpha \in \Lambda)$ and a unique largest topology on X contained in all τ_α $(\alpha \in \Lambda)$.

92. Consider the topologies $\tau_1 = \{\emptyset, \{a\}, \{a,b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b,c\}, X\}$ on the set $X = \{a,b,c\}$. Find the smallest topology on X containing both τ_1 and τ_2 and the largest topology on X contained in both τ_1 and τ_2.

93. State TRUE or FALSE with justification for each of the following statements.
(a) If τ is the topology on $\mathbb{R}$ having $\{(a,b) : a,b \in \mathbb{R}, a < b\} \cup \{(a,b) \setminus K : a,b \in \mathbb{R}, a < b\}$ as a basis, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$, and if τ_u is the usual topology on $\mathbb{R}$, then the relative topology on $A = \{x \in \mathbb{R} : x > 0\}$ induced by τ is strictly finer than the relative topology on A induced by τ_u.
(b) If a topology τ on $\mathbb{R}$ is strictly finer than both the lower limit topology and the cocountable topology on $\mathbb{R}$, then τ must be the discrete topology on $\mathbb{R}$.
(c) If $\mathcal{B}$ and $\mathcal{B}'$ are bases for topologies τ and τ' respectively on a nonempty set X, then $\{B \cap B' : B \in \mathcal{B}, B' \in \mathcal{B}'\}$ must be a basis for some topology on X.
(d) If τ and τ' are topologies on a nonempty set X, then $\tau \cup \tau'$ must be a basis for some topology on X.

94. It is known that if $\mathbb{R}$ is equipped with either the indiscrete topology or the cofinite topology, then the sequence $(\frac{1}{n})_{n=1}^{\infty}$ converges to 1. Mention (with justification) another topology on $\mathbb{R}$

with respect to which the sequence $(\frac{1}{n})_{n=1}^{\infty}$ converges to 1.

95. Consider $X = (0, 1]$ with the relative topology τ induced by the usual topology on $\mathbb{R}$. Show that the sequence $(\frac{1}{n})_{n=1}^{\infty}$ in X does not converge in the topological space (X, τ).

96. If $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then in the topological space $(\mathbb{R}, \tau)$, prove that the sequence
    (a) $(n)_{n=1}^{\infty}$ converges to every point of $\mathbb{R}$.
    (b) $(-n)_{n=1}^{\infty}$ does not converge.
    (c) $((-1)^n)_{n=1}^{\infty}$ converges to x iff $x \in (-\infty, -1]$.

97. If $\tau = \{G \subset \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$, then determine (with justification) all the convergent sequences and all their limits in the topological space $(\mathbb{R}, \tau)$.

98. State TRUE or FALSE with justification for each of the following statements.
    (a) The sequence $\left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty}$ converges in $\mathbb{R}$ with the lower limit topology.
    (b) There exists a topology τ on $\mathbb{R}$ such that the sequence $(1, 2, 3, ...)$ converges to the unique limit 0 in $(\mathbb{R}, \tau)$.
    (c) If (X, τ) is a topological space such that every sequence in X converges to every point of X in (X, τ), then τ must be the indiscrete topology on X.