**DEPARTMENT OF MATHEMATICS**
**Indian Institute of Technology Guwahati**

MA549: Topology — MidSem
Instructor: Rajesh Srivastava — September 20, 2023
Time duration: 02 hours — Maximum Marks: 30

**N.B.** Answer without proper justification will attract zero mark.

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1. (a) If $(X, \tau_X)$ and $(Y, \tau_Y)$ are second countable topological spaces. Does it imply that the product topology on $X \times Y$ is also second countable? **[1]**
   (b) Let $\tau_o$ be the discrete topology on a nonempty set $X$. Does there exists subbasis $\mathcal{S}$ of $\tau_o$ such that $\sharp(\mathcal{S}) < \sharp(X)$? **[1]**
   (c) Let $A$ be subset of a topological space $X$. Suppose that the boundary of $A$ is a finite set. What conclusion can be drawn about the cardinality of the set $X$? **[1]**
   (d) Is it possible to find a non-trivial subbasis (that is not equal to a basis) for every topological space? **[1]**
   (e) Let $\tau_l$ be the lower limit topology on $\mathbb{R}$. Let $f : (\mathbb{R}, \tau_l) \to (\mathbb{R}, \tau_u)$ be the map defined by $f(x) = -x$. What is the set of points of continuity of $f$? **[1]**

2. Let $(X, d)$ be a separable metric space. Show that for given $\epsilon > 0$ there is a countable set $A$ in $X$ such that $X = \bigcup_{a_i \in A} B_\epsilon(a_i)$. What is the least possible cardinality of a basis of the metric topology on $X$? **[2]**

3. Show that the collection $\mathcal{S} = \{[a, \infty), (-\infty, b) : a, b \in \mathbb{R}\}$ is a subbasis for some topology $\tau$ on $\mathbb{R}$, which is finer that the usual topology $\tau_u$ on $\mathbb{R}$. **[2]**

4. Let $X = \{a, b, c, d\}$ and $\mathcal{S} = \{\{a, b, c\}, \{b, c, d\}\}$. Find the topology generated by $\mathcal{S}$ as a subbasis. **[2]**

5. Let $f : (\mathbb{R}, \tau_l) \to (\mathbb{R}, \tau_u)$ be map such that there exists a closed set $A$ containing $f(0)$. Show that $f^{-1}(A)$ need not be a closed set. **[3]**

6. Let $f : X \to f(X)$ be a continuous map. If $D$ is dense set for $X$, then show that $f(D)$ is a dense set for $f(X)$. **[3]**

7. Let $(X, \tau_X)$ be a topological space, and Y be an arbitrary set with $\sharp(X) = \sharp(Y)$. Find a topology $\tau_Y$ on $Y$ such that $(X, \tau_X)$ is homeomorphic to $(Y, \tau_Y)$. Does the conclusion hold, in general, if $\sharp(X) < \sharp(Y)$? **[5]**

8. Let $X$ be a topological space, and the map $f_A : X \to \mathbb{R}$ be the characteristic function of the set $A$ in $X$. Show that $f_A$ is lower semi-continuous if and only if $A$ is open. What is the final conclusion if the boundary of the set $A$ is nowhere dense? **[5]**

9. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subbases of two non-homeomorphic topologies on $X$. Prove/disprove that $\mathcal{S}_1 \cup \mathcal{S}_2$ is a subbasis for some topology $\tau$ which is finer than the smallest topology containing $\tau(\mathcal{B}(\mathcal{S}_1))$ and $\tau(\mathcal{B}(\mathcal{S}_2))$. **[3]**

**END**