DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA549: Topology | MidSem
Instructor: Rajesh Srivastava | September 20, 2023
Time duration: 02 hours | Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) If (X, τ_X) and (Y, τ_Y) are second countable topological spaces. Does it imply that the product topology on $X \times Y$ is also second countable? **[1]**

   (b) Let τ_o be the discrete topology on a nonempty set X. Does there exists a subbasis $\mathcal{S}$ of τ_o such that $\sharp(\mathcal{S}) < \sharp(X)$? **[1]**

   (c) Let A be subset of a topological space X. Suppose that the boundary of A is a finite set. What conclusion can be drawn about the cardinality of the set X? **[1]**

   (d) Is it possible to find a non-trivial subbasis (that is not equal to a basis) for every topological space? **[1]**

   (e) Let τ_l be the lower limit topology on $\mathbb{R}$. Let $f : (\mathbb{R}, \tau_l) \to (\mathbb{R}, \tau_u)$ be the map defined by $f(x) = -x$. What is the set of points of continuity of f? **[1]**

2. Let (X, d) be a separable metric space. Show that for given $\epsilon > 0$ there is a countable set A in X such that $X = \bigcup_{a_i \in A} B_\epsilon(a_i)$. What is the least possible cardinality of a basis of the metric topology on X? **[2]**

3. Show that the collection $\mathcal{S} = \{[a, \infty), (-\infty, b) : a, b \in \mathbb{R}\}$ is a subbasis for some topology τ on $\mathbb{R}$, which is finer that the usual topology τ_u on $\mathbb{R}$. **[2]**

4. Let $X = \{a, b, c, d\}$ and $\mathcal{S} = \{\{a, b, c\}, \{b, c, d\}\}$. Find the topology generated by $\mathcal{S}$ as a subbasis. **[2]**

5. Let $f : (\mathbb{R}, \tau_l) \to (\mathbb{R}, \tau_u)$ be map such that there exists a closed set A containing $f(0)$. Show that $f^{-1}(A)$ need not be a closed set. **[3]**

6. Let $f : X \to f(X)$ be a continuous map. If D is dense set for X, then show that $f(D)$ is a dense set for $f(X)$. **[3]**

7. Let (X, τ_X) be a topological space, and Y be an arbitrary set with $\sharp(X) = \sharp(Y)$. Find a topology τ_Y on Y such that (X, τ_X) is homeomorphic to (Y, τ_Y). Does the conclusion hold, in general, if $\sharp(X) < \sharp(Y)$? **[5]**

8. Let X be a topological space, and the map $f_A : X \to \mathbb{R}$ be the characteristic function of the set A in X. Show that f_A is lower semi-continuous if and only if A is open. What is the final conclusion if the boundary of the set A is nowhere dense? **[5]**

9. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subbases of two non-homeomorphic topologies on X. Prove/disprove that $\mathcal{S}_1 \cup \mathcal{S}_2$ is a subbasis for some topology τ which is finer than the smallest topology containing $\tau(\mathcal{B}(\mathcal{S}_1))$ and $\tau(\mathcal{B}(\mathcal{S}_2))$. **[3]**

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