Compact Sets:

Compactness is property of a set can be compared with finite set, which are readily satisfied many properties, e.g. every function on finite set is uniformly cont. And on real line, we have a generalization to closed & bounded sets. By a fundamental result due to Heine-Borel-Lebesgue every closed & bounded set K can be understood by the fact that every open cover of K reduces to finite sub-cover. This result has extraordinary profound consequences, and it turns out to be a definition in general topology.

Def^n: A topological space X is said to be compact if every open cover $\mathcal{G} = \{O_i \in \mathfrak{T}_X : i \in I\}$ has finite sub-cover.

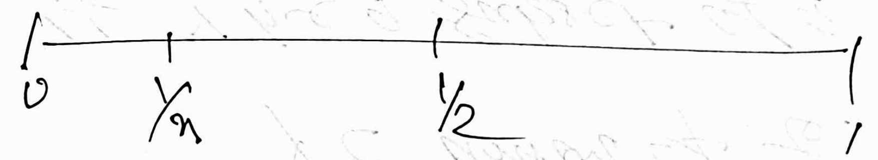
i.e. $X = \bigcup_{i\in I} O_i \Rightarrow X = \bigcup_{i=1}^{n} O_i$.

Ex. $(\mathbb{R}, \mathcal{U})$ is not compact because

$$\mathcal{G} = \{(n, n+2) : n \in \mathbb{Z}\}$$

has no finite sub-cover.

Ex. $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is cpt

$K =$ 

is compact as any cover K has an open interval containing 0, which contains most of the pts of K except finitely many.

Ex. $A = (0,1]$ is not cpt as the cover $\mathcal{G} = \{(\frac{1}{n}, 1] : n \in \mathbb{N}\}$ has no finite subcover of A.

Notice that A itself is not too large, but it can be stretched through homeo. to $(-\infty, 1]$, however this is not possible for $[0,1]$. Hence, we can guess that closeness of set is necessary for compactness. (104)

In general, it takes a little effort to decide the compactness of a set.

We say $Y \subset X$ is covered by an open cover $\mathcal{C}$ in X if

$$Y \subset \text{union of } \mathcal{C}.$$

Lemma: Let Y be a subset of a top. space X. Then Y is compact iff every cover of Y in X has finite sub-cover of Y.

pf: Suppose Y is cpt & $\mathcal{C} = \{O_i : i \in I\}$ be a cover of Y in X. Then

$$Y = \bigcup_{i \in I} (O_i \cap Y)$$

$$\Rightarrow Y = \bigcup_{i=1}^{n} (O_i \cap Y) \Rightarrow Y \subset \bigcup_{i=1}^{n} O_i.$$

(since X is cpt in its own right.

Conversely, suppose every open cover of Y in X has a finite sub-cover. (105)

claim. Y is cpt in its own right.

Let $Y = \bigcup_{i \in I} O_i'$; $O_i' \subset Y$.

Then $O_i' = O_i \cap Y$ (by defn of subspace)

$\Rightarrow Y \subset \bigcup_{i \in I} O_i$

$\Rightarrow Y \subset \bigcup_{i=1}^{n} O_i \Rightarrow Y = \bigcup_{i=1}^{n} O_i \cap Y$

$\therefore Y = \bigcup_{i=1}^{n} O_i'$.

<u>Theorem</u>: Every closed subspace of a cpt space is compact.

Pf: let Y be a closed subspace of a cpt space X.

Let $Y \subseteq \bigcup_{i \in I} O_i$; $O_i \subset X$.

Then $X = (X \setminus Y) \cup \bigcup_{i \in I} O_i$

Then $X = (X \setminus Y) \bigcup_{i=1}^{n} O_i$ ($\because X$ is cpt)

$\Rightarrow Y = X \cap Y = \bigcup_{i=1}^{n} (O_i \cap Y)$ (106)

$\Rightarrow Y \subset \bigcup_{i=1}^{n} O_i$.

Ex. In indiscrete top. space $(X, \{\phi, X\})$ every set is compact. Does it imply every set is closed?

Theorem: Every compact subspace of a Hausdorff space is closed.

Pf: Let Y be a compact subspace of a Hausdorff space X. Claim $X \setminus Y$ is open in X.

For $x_0 \in X \setminus Y$, it is enough to prove that $\exists$ a nbd of x_0 which does not intersect Y.

For each $y \in Y$, choose two disjoint open nbd sets U_y & V_y of x_0 & y resp.

then $Y \subset \bigcup_{y \in Y} V_y \Rightarrow Y \subset \bigcup_{i=1}^{n} V_{y_i}$.

let $V = \bigcup_{i=1}^{n} V_{y_i}$ &

$U = \bigcap_{i=1}^{n} U_{y_i}$.

Then U will be desire open nhd.

If $z \in V$, then $z \in V_{y_i}$ for some i and hence $z \notin U_{y_i} \Rightarrow z \notin U$.

i.e. $Y \subset V$ & $Y \cap U \subset V \cap U = \emptyset$.

The above statement we note down for later use.

<u>Lemma</u>: If Y is a cpt subspace of a Hausdorff space X, then for $x_0 \notin Y$, $\exists$ disjoint open sets U & V of X containing x_0 & Y resp.

Ex. $[a,b]$ is compact in $\mathbb{R}$, because any closed sub-space of $[a,b]$ is cpt.

Ex. $[a,b]$ & (a,b) are not cpt in $(\mathbb{R}, \mathcal{U})$ because they are not closed in Hausdorff space $(\mathbb{R}, \mathcal{U})$. (108)

Remark: 1. We need Hausdorff criteria in the previous theorem, because in $(\mathbb{R}, \mathcal{T}_{\text{co-finite}})$ top. space, the only proper set which are closed are finite sets, but every subset in co-finite top. is compact.

Let $A \subset \mathbb{R}$, & $A \subset \bigcup A_i$. Pick $A_i \neq \Phi$

Then $A_i = \mathbb{R} \setminus \{a_1, \dots, a_k\}$. Choose $a_j \in A_j$

Then $A \subset A_i \cup A_1 \cup \dots \cup A_k$.

Remark 2 $(\mathbb{R}, \mathcal{T}_{\text{co-countable}})$ is not compact.

$\mathbb{R} = \bigcup_{i}^{\infty} ((\mathbb{R} \setminus \mathbb{N}) \cup \{i\})$ has no finite sub cover.

Theorem: Continuous image of compact set is compact.

Pf: Let $f: X \longrightarrow Y$ be cont & X be a compact ~~subset~~ space.

let $f(X) = \bigcup_{i \in I} V_i$, $V_i \in \tau_Y$

then $X = \bigcup_{i \in I} f^{-1}(V_i)$, $f^{-1}(V_i)$ - open X,

since f is continuous.

$\Rightarrow X = \bigcup_{i=1}^{n} f^{-1}(V_i)$

$\Rightarrow f(X) = \bigcup_{i=1}^{n} V_i$.

<u>Theorem</u>: let $f: X \longrightarrow Y$ (Hausdorff) be cont.

(a) If X is cpt, then f is a closed map.

(b) If X is cpt, and f is a bijection then f is a homeo.

Pf: (a) let $F \subset X$ be closed, then F is compact $\Rightarrow f(F)$ is cpt in Y

$\Rightarrow f(F)$ is closed as Y is T^2-space. (Hausdorff).

(b) If f is a bijection, then

$$(f^{-1})^{-1}(F) = f(F) \text{ is closed}$$

$\Rightarrow f^{-1}$ is cont. & f is a homeo.

<u>Theorem:</u> The product of finitely many compact top. spaces is compact.

Pf is based on the following assertion:

Let X be a top. space & Y be compact. Let $x_0 \in X$ & N be an open set in $X \times Y$ s.t. $x_0 \times Y \subset N \subset X \times Y$.

We claim that $\exists$ a nbd W of x_0 in X s.t. $W \times Y \subset N$.

(Note the set $W \times Y$ is called tube about $x_0 \times Y$).

We can cover $x_0 \times Y$ by basic open sets lying in N

i.e. $x_0 \times Y \subset \bigcup_{i \in I} U_i \times V_i \subset N$.

Since $x_0 \times Y \cong Y \Rightarrow x_0 \times Y$ is compact.

(111)

$\Rightarrow x_0 \times Y \subset \bigcup_{i=1}^{n} U_i \times V_i$.

(we assume that each $U_i \times V_i$ intersects $x_0 \times Y$, otherwise basis open sets are superflous, and hence can be discarded)

Let $W = U_1 \cap \cdots \cap U_n$. Then

$x_0 \in W$, (because $U_i \times V_i \cap (x_0 \times Y) \neq \Phi$)

we prove that

$$W \times Y \subset \bigcup_{i=1}^{n} U_i \times V_i.$$

Let $(x,y) \in W \times Y$. Then $(x_0, y) \in x_0 \times Y$

$\Rightarrow (x_0, y) \in U_i \times V_i$ for some i

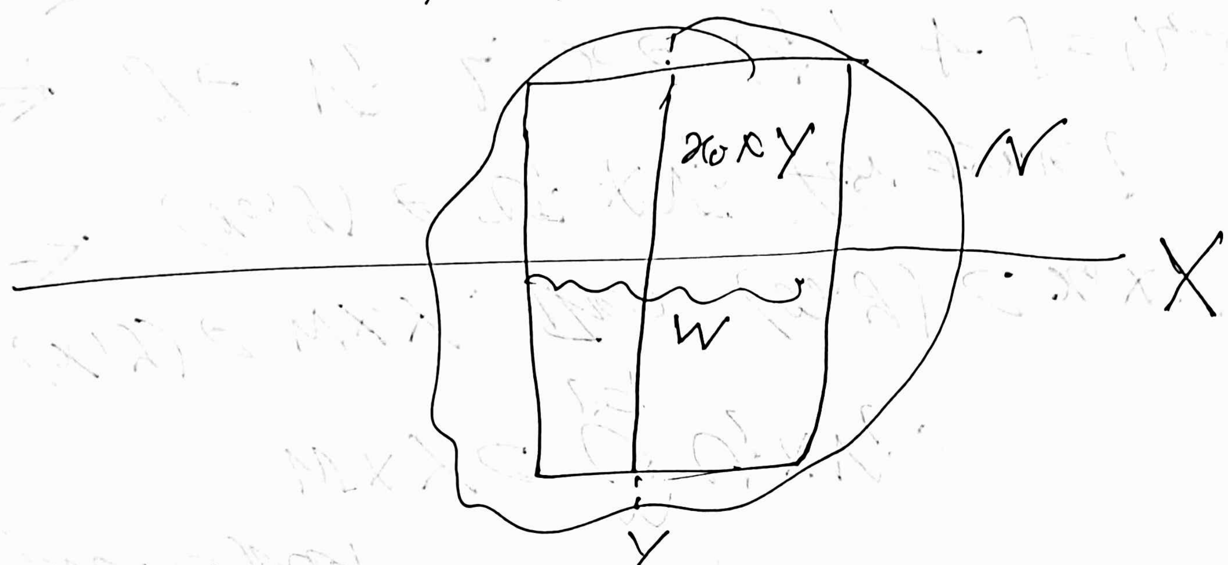
$\Rightarrow y \in V_i$ & $x \in U_j$, $\forall j = 1, 2, \ldots n$.

$\Rightarrow (x,y) \in U_j \times V_j$

i.e. $W \times Y \subset \bigcup_{i=1}^{n} U_i \times V_i \subset N$.

2.c. If $x_0 \times Y \subset N$ (open nbd), then we can fit a rectangle $W \times Y \subset N$ st (112)

$x_0 \times Y \subset W \times Y \subset N$.



proof of the main theorem:

let X & Y be compact &

$X \times Y = \bigcup_{i \in I} O_i$; O_i - open in $X \times Y$.

Then for $x \in X$, $x \times Y \subset \bigcup_{i=1}^{m} O_i = N_x$ (say)

Then $\exists$ tube $W_x \times Y \subset N_x$.

Hence each $W_x \times Y$ is covered by

finitely many O_i's. Also,

$$X = \bigcup_{x \in X} W_x \quad \text{— open cover}$$

$$\Rightarrow \quad X = \bigcup_{j=1}^{k} W_j$$

$$\Rightarrow \quad \bigcup_{j=1}^{k} (W_j \times Y) = X \times Y$$

But $W_j \times Y \subset N_j = \bigcup_{i=1}^{m} O_i$

$$\Rightarrow \quad X \times Y = \bigcup_{j=1}^{k} \left(\bigcup_{i=1}^{m} O_i \right).$$

Hence $X \times Y$ is compact.

Lemma (Tube lemma):

Let X be a top. space, and Y be a cpt space. If N is an open set in $X \times Y$ containing $x_0 \times Y$ ($\subset N$), then N contains a tube $W \times Y$ about $x_0 \times Y$, where W is an open nbd of x_0 in X.

Ex. The lemma fails if Y is not compact.

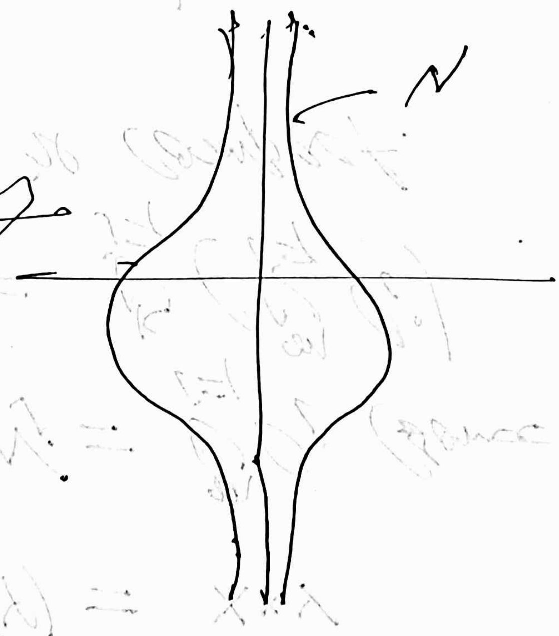
(114)

$Y = y$-axis in $\mathbb{R}^2$.

$N = \{(x,y) : |x| < \frac{1}{1+y^2}\}$

Then N is an open set containing $0 \times \mathbb{R}$ but it contains no tube around $0 \times \mathbb{R}$.

We cannot cut a rectangle containing $0 \times \mathbb{R}$.



Next, we describe compactness through finite intersection property (FIP). This leads to a complete characterization of compact metric spaces too.

Def^n: $\mathcal{C} \subseteq P(X)$ is said to satisfy FIP if $\forall$ finite sub-collection of $\mathcal{C}$ has non-empty intersection.

Ex. $A_1 \supset A_2 \supset \cdots A_n \supset A_{n+1} \supset \cdots$, $A_i \subseteq X$ has finite intersection property.

<u>Theorem:</u>

A top. space X is compact iff every collection $\mathcal{C}$ of closed sets in X having F.I.P has non-empty intersection.

Pf: A top. space X is compact iff every open cover of X reduces to finite sub-cover.

We prove this via contrapositive.

That is if a collection $\mathcal{A}$ of open sets in X has no finite sub-cover, then.

Then $\mathcal{A}$ does not cover X.

If every finite sub-collection $\mathcal{B}$ of $\mathcal{A}$ does not cover X, then $\mathcal{B}^c = \{X \setminus A : A \in \mathcal{B}\}$ has non-empty intersection, then

$$\bigcap A^c \neq \Phi$$

$\Rightarrow$ $\mathcal{A}$ is not a cover of X.

Ex. Let $G_1 \supset G_2 \supset \cdots G_n \supset G_{n+1} \supset \cdots$ be a seqn of closed sets in a compact space, with $G_n \neq \Phi$. Then $\bigcap G_n \neq \Phi$.

Ex. Show that every closed & bounded subset of $\mathbb{R}^2$ (and $\mathbb{R}^n$) is cpt.

Ex. $A = \{(x, \frac{1}{x}) : 0 < x \leq 1\}$ is closed but not bdd.

Ex. On $\mathbb{R}^2$, Show that projection of closed set need not be closed & same is true for bounded set

Ex $B = \{(x, \sin\frac{1}{x}) : 0 < x \leq 1\}$

is bdd. but not closed.

Theorem: (Extreme value theorem)

Let X be a compact top. space.
If $f: X \to \mathbb{R}$ be a continuous function, then f is bounded and attains its inf. & sup.

i.e $\exists$ $c, d \in X$ st

$$f(c) \leq f(x) \leq f(d), \forall x \in X.$$

proof: Since f is cont & X is cpt, it implies that $f(X) = A$ ~~has~~ is compact.
We show 1st that A has sup & inf.
If A has no largest element, then

Then $\{(-\infty, a) : a \in A\}$ is an open cover of A. Since A is cpt,

$$A \subset \bigcup_{i=1}^{n} (-\infty, a_i).$$

Let $a' = \max_{1 \le i \le n} a_i$. Then $a' \notin A$, which is a contradiction. By similar argument A has smallest element.

i.e. $m = \inf A = f(c)$ &

$M = \sup A = f(d)$,

where $c, d \in X$.

Now, try to characterize compact sets in a metric space.

For $A \subset X$,

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

is uniformly cont on X.

$$d(A) = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$$

is called diameter of A.

Defⁿ: A number $\delta > 0$ is called Lebesgue number for an open cover $\mathcal{A}$ of X if every subset $A \subset X$ with $\delta(A) < \delta$ $\Rightarrow$ $A \subset A_i$ for some $A_i \in \mathcal{A}$.

(119)

<u>Theorem</u>: (<u>Lebesgue Covering lemma</u>)

If X is a compact metric space, then every cover of X has a Lebesgue no.

Pf: Let $\mathcal{A}$ be an open cover of X.

Then $X = \bigcup_{i \in I} A_i$; $\mathcal{A} = \{A_i : i \in I\}$.

If $X = A_i$ for some i, then every positive no. is Lebesgue no for $\mathcal{A}$.

So assume $X \neq A_i$ $\forall i$.

Since X is Cpt,

$$X = \bigcup_{i=1}^{n} A_i$$

Let $C_i = X \setminus A_i$; $i = 1, 2, \dots n$.

Define a function $f: X \to \mathbb{R}$ by average of $d(x, C_i)$.

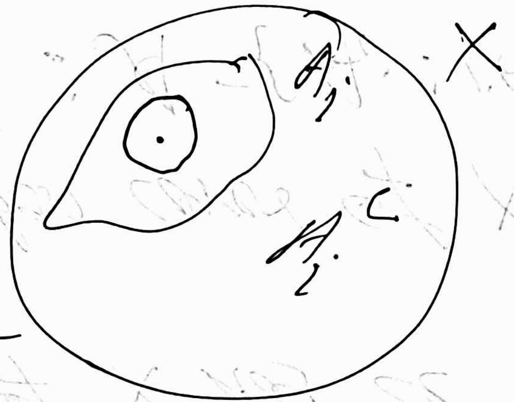
i.e. $f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i)$.

Then $f(x) > 0$. For this, let $x \in X$, then $x \in A_i$ — open $\Rightarrow \exists\ \epsilon > 0$ s.t.

$B(x, \epsilon) \subseteq A_i$

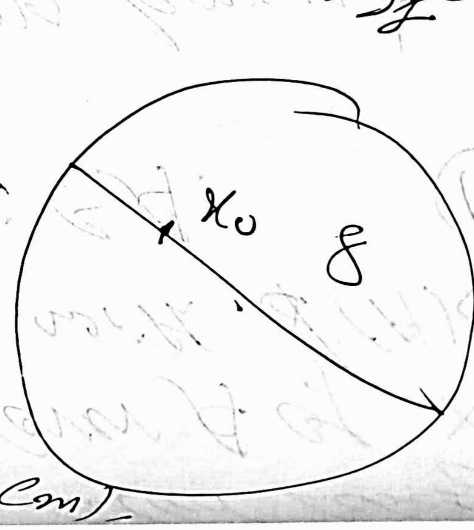
$\Rightarrow d(x, C_i) \geq \epsilon$

$\Rightarrow f(x) \geq \epsilon/n > 0$.



Since f is cont on cpt space X, f has its infimum, δ, $\delta > 0$.
We claim δ is the required Lebesgue no. for A.

Let $B \subset X$ & $d(B) < \delta$.
Choose a pt $x_0 \in B$.
Then $B \subset B(x_0, \delta)$



Now, $\delta \leq f(x_0) \leq d(x_0, C_m)$

where $d(x_0, x_{cm}) =$ maxi of $d(x_0, c_i)$.
Then $B(x_0, \delta) \subset A_m = X \setminus C_m$ (12)
ie $B \subset A_m$ for some m.

Defn: $f : (X, d_X) \longrightarrow (Y, d_Y)$
is said to be unif. cont if $\forall \epsilon > 0$,
$\exists$ a $\delta > 0$ s.t. $\forall$ pair of point
$x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$
$\Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$.

Theorem: Every Continuous function on compact metric space is uniformly continuous.

proof: let X be a cpt space &
$f : (X, d_X) \longrightarrow (Y, d_Y)$
be a continuous map.

For $\epsilon > 0$, let $Y = \bigcup_{y \in Y} B(y, \epsilon/2)$.

Let $\mathcal{A} = \{ f^{-1}\{B(y, \epsilon/2)\} : y \in Y \}$.

Then $\mathcal{A}$ is a cover of X. Hence, $\mathcal{A}$ has a Lebesgue no say δ.

Let $\{x_1, x_2\} \subset X$ & $d_X(x_1, x_2) < \delta$.

Then $\{x_1, x_2\}$ has diameter $< \delta$.

So $\{f(x_1), f(x_2)\} \subset B(y, \epsilon/2)$ for some $y \in Y$

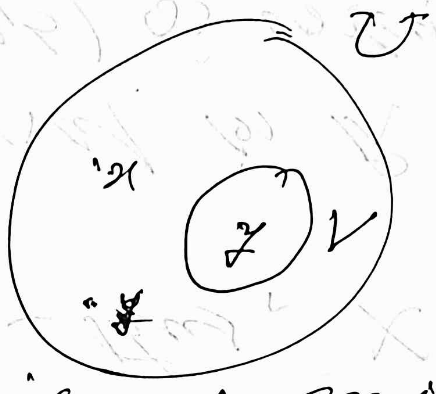
$\Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$.

Hence, f is uniformly continuous.

Defn: A pt $x \in X$ (top space) is said to be isolated if $\{x\}$ is open in X.

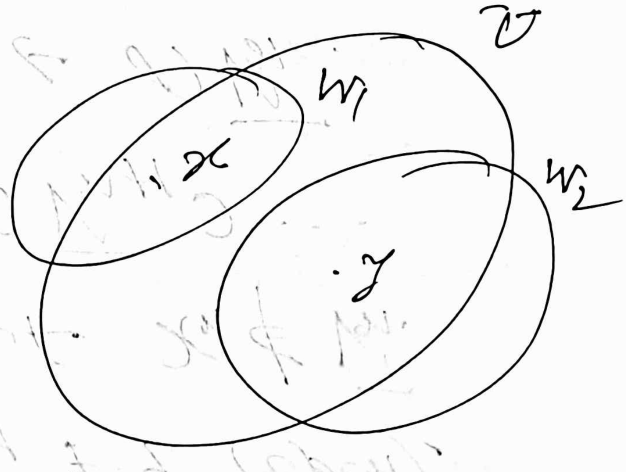
<u>Theorem</u>: Let X be a non-empty compact Hausdorff space. If X has no isolated pt then X is uncountable.

Pf: First we show that for given
non-empty open set U & $x \in X$,
$\exists$ an open set $V \subset U$ s.t
$$x \notin \overline{V}.$$



Choose $y \in U$ s.t.
$y \neq x$. This is possible
even if $x \in U$ since x is not an isolated pt

Since the space is T_2-space, $\exists$ two disjoint open set W_1 & W_2 s.t
$$x \in W_1 \ \& \ y \in W_2.$$



Let $V = W_2 \cap U \subset U$.

Then $x \notin \overline{V}$. If
$x \in \overline{V}$, then
$$x \in V \cap W_1 = \emptyset.$$

We complete the proof by showing that any function $f : \mathbb{N} \to X$ is not

Surjective. Let $x_n = f(n)$. Then for $x_1 \in X = U$, $\exists$ open set V s.t. (124)

$x_1 \notin \overline{V_1}$.

In general, for $V_{n-1} \neq \emptyset$ (open),

choose $V_n \subset V_{n-1}$ s.t. $x_n \notin \overline{V_n}$.

$\Rightarrow \overline{V_1} \supset \overline{V_2} \supset \cdots \supset \overline{V_n} \supset \overline{V_{n+1}} \supset \cdots$

Since X is cpt, $\exists\ x \in \bigcap \overline{V_n}$.

Then $x \neq x_n$ for any n, since $x \in \overline{V_n}$, but $x_n \notin \overline{V_n}$.

i.e. $x \neq f(n)$, $\forall\ n \geq 1$.

Hence f is not onto. Thus, X is uncountable.

Cor. Every closed interval in $\mathbb{R}$ is uncountable & hence (a,b) is uncountable.

Ex. Let $\mathbb{R}_K$ be endowed with K-top. where K has basis

$\mathcal{B} = \{(a,b) : a,b \in \mathbb{R}\} \cup \{K \setminus (c,d) : c,d \in \mathbb{R}\}$,
when $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. (125)

Show that $[0,1]$ is not compact in $\mathbb{R}_K$.

Limit point Compactness:

Def^n: A top. space X is said to be limit point compact (LPC) if every infinite subset of X has limit point.

Theorem: Compactness $\Rightarrow$ limit pt cpt, but converse need not be true.

proof: To claim every infinite set has limit pt, it is enough to show that if a set has no ~~limit~~ point, then it is finite.

Suppose $A \subset X$ has no limit pt.

~~Suppose~~ Then A is closed. Further, (12b)
for each $a \in A$, $\exists$ U_a st

$$U_a \cap A = \{a\}$$

Then $X = (X \setminus A) \cup \bigcup_{a \in A} U_a$ is an
open cover of X.

$$\Rightarrow X = (X \setminus A) \bigcup_{i=1}^{n} U_{a_i}$$

$$\Rightarrow A \subset \bigcup_{i=1}^{n} U_{a_i} \Rightarrow A \text{ is finite.}$$

Ex. Let $Y = \{y_1, y_2\}$, $\mathcal{T}_Y = \{\phi, Y\}$
and $X = \mathbb{N} \times Y$, when $\mathbb{N}$ has
usual topology. Then X is LPC
but not compact as covering

$$\{V_n = \{n\} \times Y\}$$

has no finite subcover.
(Since $\mathcal{T}_Y = \{\phi, Y\}$, every non-empty
~~poi~~ set has limit point.

Theorem: Let X be a metrizable space. Then TFAE: (127)

(i) X is compact.

(ii) X is limit point compact

(iii) X is sequentially compact

proof: (i) $\Rightarrow$ (ii) is already done.

(ii) $\Rightarrow$ (iii):

Let (x_n) be a seq^n in X. If $A = \{x_n : n \in \mathbb{N}\}$ is finite, then it is fixed.

If A is infinite, then A has a limit pt. say x.

We find a subsequence of (x_n) which converges to x.

Choose x_{n_1} so that $x_{n_1} \in B(x, 1)$.

Since $B(x, \frac{1}{1})$ intersects A in ~~~~ infinitely many points, we can

~~we~~ Choose $n_i > n_{i-1}$ s.t.

$x_{n_i} \in B(x, r_i)$

Thus $x_{n_i} \to x$.

<u>(iii) ⇒ (i):</u>

(128)

First we prove that if X is seq^n compact then every cover has Lebesgue no. We prove this via Contradiction

Suppose there is a Cover $\mathcal{A}$ of X having no Lebesgue number. Then for each $\delta = 1/n > 0$, ∃ a set C_n of diameter $d(C_n) < \frac{1}{n}$ but

$$C_n \not\subset A \text{ for any } A \in \mathcal{A}.$$

Choose $x_n \in C_n$. then by hypothesis ∃ Subseq^n $x_{n_i} \to a$, and $a \in A$ for some $A \in \mathcal{A}$. But A is open.

So $B(a,\epsilon) \subset A$ for some $\epsilon > 0$.
Note that $\frac{1}{n_i} < \epsilon/2$ for large i,
and also for large n_i,

$$d(x_{n_i}, a) < \epsilon/2$$

$$\Rightarrow C_{n_i} \subset B(a,\epsilon) \subset A, \left(\because \frac{1}{n_i} < \epsilon/2\right)$$

which is a contradiction.

Secondly, we prove that if X is seq^n compact, then for given $\epsilon > 0$, $\exists\, x_i$; $i = 1, 2, \ldots n$ st

$$X = \bigcup_{i=1}^{n} B(x_i, \epsilon) \text{ ——— } (*)$$

We prove this too via contradiction.
Assume that $\exists\, \epsilon > 0$ such that $(*)$ is false.
Let $x_1 \in X$, then $B(x_1, \epsilon) \subsetneq X$.
Let $x_2 \in X \setminus B(x_1, \epsilon)$.
In general, given $x_1, x_2, \ldots x_n$, we

Can choose $x_{n+1} \in X$ s.t.

$$x_{n+1} \notin B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon).$$

$\Rightarrow d(x_{n+1}, x_i) \geq \epsilon \quad \forall i = 1, 2, \ldots, n.$

$\Rightarrow (x_n)$ has no convergent subseq^n.

Finally, we show X is compact.

Let $\mathcal{A}$ be an open cover of X.

Since ~~$\mathcal{A}$~~ X is seq^n cpt, $\mathcal{A}$ has Lebesgue no say $\delta > 0$. Let $\epsilon = \delta/3$.

Also, $X = \bigcup_{i=1}^{n} B(x_i, \epsilon)$

Each of these ball has diameter $2\delta/3 < \delta$.

So it lies in some of A_i

$$\Rightarrow X = \bigcup B(x_i, \epsilon) = \bigcup_{i=1}^{n} A_i.$$

Ex. Let $X = [0,1]^{\omega} = [0,1] \times [0,1] \times \cdots$

for $x = (x_n)$ & $y = (y_n)$ in X, let

$d(x,y) = \sup_n |x_n - y_n|$.

Find an infinite subset of X having no limit point.

Ex. Show that $[0,1]$ is not compact in R_l.

Ex. A space X is said to be countably compact if countable cover has finite subcover.

If X is a T_1-space, show that X is countably compact iff X is limit point compact.

Ex. Let $f: X \longrightarrow X$ s.t. $d(f(x), f(y)) = d(x,y)$.

If X is compact, then f is a homeo.

Now, we discuss a most important result relating to compactness of sets in topology, known as Tychonoff theorem.

Remark: (i) A top. space X is compact if every basis open cover has finite sub-cover.

(ii) A class $\mathcal{C}$ is called closed basis for top. space X if

$$\mathcal{B} = \{ X \setminus C : C \in \mathcal{C} \}$$

is a basis for X.

Similarly, closed subbasis can be defined.

Theorem:

A top. space X is compact if every sub-basis cover has a finite sub-cover or every class of sub-basis closed sets with F.I.P has non-empty intersection.

proof: It is each to see the equivalence of two statements in the theorem.

Let $\mathcal{C}$ be a sub-basis closed set class with F.I.P. ~~with F has~~ has non-empty intersection.

If $\mathcal{B} = \{B_i\}$ be the basis generated by $\mathcal{G}$ of closed sets. Then it is enough to show that every such $\mathcal{B}$ with F.I.P has non-empty intersection. (133)

i.e. if $\mathcal{B} = \{B_i\}$ ~~with~~ F.I.P
$\Rightarrow \bigcap B_i \neq \Phi$.

We use Zorn's Lemma to prove this.

Let $\mathcal{F} = \{ \{B_j\} \subset \{B_i\}$ with $\{B_j\}$ has F.I.P $\}$.

Then $\mathcal{F}$ is a partially ordered set with inclusion.

If we consider a chain in $\mathcal{F}$, then its union has also F.I.P. Hence, each chain has an upper bound.

By Zorn's lemma, $\mathcal{F}$ has a maximal class say $\{B_K\}$ ~~w.r.t.~~ having F.I.P

Since $\bigcap B_K \subseteq \bigcap B_i$, it is enough to prove that $\bigcap B_K \neq \Phi$.

Since each $B_k \in \mathcal{B}(\mathcal{C})$ is finite union of member from $\mathcal{C}$. For instance
$B_1 = C_1 \cup C_2 \cup \cdots \cup C_{n_1}$.

It is enough to show that at least one of the set $C_i \in \{B_k\}$. If so, then $\exists$ C_i for each set B_k s.t.

$C_i = B_k$ for some k. (134)

Then $\bigcap C_i \in \bigcap B_k$

But C_i's has F.I.P $\Rightarrow$ non-empty intersection.

We finish the proof by showing that each set $C_i = B_{k_i}$ for some i.

Suppose $C_i \notin \{B_k\}$ for each i

Then $\{B_k, C_i\} \supset \{B_k\}$, but $\{B_k\}$ is the maximal class with F.I.P. Hence $\{B_k, C_i\}$ lacks that

So $C_i \cap (B_{k_i}^1 \cap \cdots \cap B_{k_i}^j) = \Phi$
for some choice of B_j's from $\{B_k\}$.

If this is true for each C_i, then

$$B_1 \cap (B_1' \cap \ldots \cap B_{k_c}') = \Phi.$$

which contradicts that $\{B_K\}$ has F.I.P.

We use this result to derive a proof classical Heine-Borel theorem.

Theorem (Heine-Borel):

Every closed & bounded sub-space of the real line is compact.

Proof: Since a closed & bounded set can be put into a closed and bounded interval $[a,b]$, and the fact that every closed sub-space of compact space is compact, it is enough to show that $[a,b]$ is compact.

We know that $\mathcal{J} = \{[a,d), (c,b], \ a < c, d < b\}$ forms a sub-basis for $[a,b]$.

Let $\mathcal{J}' = \{[a, c_i], [d_j, b]\}$ be a subclass of $\mathcal{J}$ with F.I.P.

We claim ~~$\mathcal{J}\cap$~~ $\cap \mathcal{J}' \neq \Phi$.

Notice that if $\mathcal{J}'$ contains sets of type $[a, c_i]$ only or $[d_j, b]$ only then their intersection is a or b.

So let $\mathcal{J}'$ contains sets of both types.

Let $d = \sup\{d_j\}$.

We complete the proof by showing that $d \leq c_i \ \forall i$. If not, let $c_{i_0} < d$ for some i_0. By def^n of d, $\exists$ d_{j_0} st $c_{i_0} < d_{j_0}$.

Thus $[a, c_{i_0}] \cap [d_{j_0}, b] = \Phi$.

This is a contradiction that $\mathcal{J}'$ has F.I.P.

Theorem (Tychonoff):

The product of any nonempty class of compact spaces is compact.

Proof: Let $X = \prod_i X_i$; X_i — be compact.

Let $\{F_j\}$ be a non-empty sub-basis of the defining closed subbase for the product topology X. (137)

This means, each F_j can be expressed as $F_j = \prod_i F_{ij}$,

where F_{ij} is a closed subset of X_i with $F_{ij} = X_i$ for all i except one.

By previous theorem, it is enough to show that if $\{F_j\}$ has F.I.P, then $\bigcap_j F_j \neq \Phi$.

Notice that for each fixed i, $\{F_{ij}\}$ is a class of closed sub-basis of X_i with F.I.P, and by compactness of X_i, $\exists\ x_i \in X_i$ s.t. $x_i \in \bigcap_j F_{ij}$.

This is true for each i. s.t. we

write $x = (x_i)$, then $x \in \prod_i X_i = X$
& $x \in \bigcap_j F_j$. Hence, X is cpt.

(138)

Theorem (Generalized Heine-Borel theorem)

Every closed & bounded subspace of $\mathbb{R}^n$ is compact.

Proof: By open rectangle in $\mathbb{R}^n$ we mean $\prod_{i=1}^{n} (a_i, b_i)$ and similarly closed rectangle by $\prod_{i=1}^{n} [a_i, b_i]$.

Since every closed & bdd subspace of $\mathbb{R}^n$ can be put in a closed rectangle, it is enough to prove that $X = \prod_{i=1}^{n} [a_i, b_i]$ is compact. Since each $[a_i, b_i]$ is compact due to classical Heine-Borel theorem, by Tychonoff's theorem, it is enough to show that product top. on X is same as its relative topology in $\mathbb{R}^n$. This follows by the fact that open rectangles forms a basis for the product top. on $\mathbb{R}^n$.

Defn: A topological space X is said to be locally compact if every point $x \in X$ has an open neighbourhood V s.t. $\overline{V}$ is compact.

Ex $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact.

Ex. Any infinite dim Hilbert space is not locally compact.
(Hint: $0 \in H$ (Hilbert space), $B_{1/n}(0)$ need to be cpt which is false in infinite dim space)

<u>Proposition</u> Let X be a Hausdorff space. Then X is locally compact iff $\forall$ nhd W of every $x \in X$, $\exists$ open set V s.t. $\overline{V}$ is compact & $x \in V \subset \overline{V} \subset W$.

<u>Proof</u>: If condition is satisfied, then X is necessarily locally compact.

On the other hand let $x \in U$ (nhd) & $\overline{U}$ is compact. Since X is Hausdorff, $\overline{U}$ is closed in X. (140)

Let $Y = \overline{U} \setminus W \cap U$, where W is a nhd of x. Then Y is ~~open~~ closed & hence compact.

Note that $x \notin Y$. $\exists$ open set $C \supset Y$ and $D \ni x$ s.t. $C \cap D = \Phi$.

By intersecting D with U, we may assume that $D \subset U$. Since $\overline{D} \subset \overline{U}$

$\Rightarrow \overline{D} \cap Y = \Phi$.

$\Rightarrow \overline{D} \subset \overline{U} \setminus Y = W \cap U$.

$\Rightarrow x \in D \subset \overline{D} \subset W \cap U \subset W$.

$\overline{D}$ is cpt since $\overline{D} \subset \overline{U}$.

Compactification:

In study of non-compact topological spaces X, it is often useful to construct a compact top. space X^* which contains

X. E.g, ~~B~~y adjoining two points $-\infty$ & $+\infty$ to $\mathbb{R}$ to get the set $\mathbb{R}^* = [-\infty, \infty]$. 141

It turns out that every non-empty subset of $\mathbb{R}^*$ (extended real nos) has infimum & supremum. Hence, $\mathbb{R}^*$ is a compactification of $\mathbb{R}$.

The simplest way of compactification of a top. space is adjoining a ~~point~~ single point. E.g. Sphere is constructed by adjoining a single point ∞ to the Euclidean plane $\mathbb{R}^2$, and we specify nbd of ∞ as the complement of bounded sets in $\mathbb{R}^n$.

This construction can be generalized to an arbitrary top. space

The one point compactification of a top. space X is the set $X^* = X \cup \{\infty\}$ such that $U \in \mathcal{T}^*$ if either U is open

X or $U \subset X^*$ s.t. $X^* \setminus U$ is closed & compact in X. Then (X^*, τ^*) is a compactification space for (X, τ). This is known as one point compactification. (142)

Ex. Let $X = (-1,1)$. Then $X \subset X^{**} = [-1,1]$, which is compact. But also $(-1,1) \subset S^1$, and S^1 is compact, which is one point compactification of $(-1,1)$. But also $[-1,1]$ is not homeo. to S^1. Hence, we get different compactification.

However, the following theorem insure that every top. space one point compactification.

Theorem (Alexandroff):

The one-point compactification X^* of top. space X is compact, and X is a sub-space of X^*.

The space X^* is Hausdorff iff X is locally compact and Hausdorff. (143)

Proof: $(X^*, \mathcal{T}^*)$ is a top. space.

(i) If $\infty \in U_1 \cap U_2$, $U_1, U_2 \in \mathcal{T}^*$;

then $X^* \setminus U_1 \cap U_2 = (X^* \setminus U_1) \cup (X^* \setminus U_2)$

= closed & compact.

$\Rightarrow U_1 \cap U_2 \in \mathcal{T}^*$.

(ii) if $\infty \in \bigcup_{i \in I} U_i \Rightarrow \infty \in U_{i_0}$. Then

$X^* \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X^* \setminus U_i) \subset X^* \setminus U_{i_0}$

closed   closed & compact

$\Rightarrow X^* - \bigcup_{i \in I} U_i$ is closed & compact

Hence $\bigcup_{i \in I} U_i$ is open in X^*.

(iii) let $X^* = \bigcup_{i \in I} U_i$ (open cover)

Then $\infty \in U_{i_0} \Rightarrow X^* \setminus U_{i_0}$ is cpt

$\Rightarrow X^* \setminus U_{i_0} \subset \bigcup_{i=1}^{n} U_i \Rightarrow X^* = U_{i_0} \bigcup_{i=1}^{n} U_i$.

$\Rightarrow$ X^* is compact in X^*
and X is a subspace of X^*.

Suppose X^* is Hausdorff space. Then X is locally compact & Hausdorff
$(x \in X \subset X^* \Rightarrow x \subset X \cap U^* \subset X \cap V^*)$

If X is locally compact & Hausdorff, we need to show X^* is Hausdorff. For this, it is enough to find disjoint open nbd of $x \in X$ & $\infty \in X^*$ in X^*. Since X is l.c. for $x \in X$, $\exists$ open set U st $x \in U \subset \bar{U}$, where $\bar{U}$ is cpt. Let $V = X \setminus \bar{U}$, then $\infty \in V$ (open).

<u>Remark</u>: If X is compact, then ∞ is an isolated point of X^*.
(i.e $\{\infty\}$ is both open & closed in X^*).
Conversely, if ∞ is an isolated point

of X^*, then X is closed in X^*, and hence compact.

Stone-Čech Compactification:

Notice that if K is compact set in $\mathbb{R}$ and $f: (0,1) \longrightarrow K$ be unif. cont., then f can be extended to compactification $[0,1]$ uniformly.

Let $x \in [0,1]$, then $\exists\, x_n \in (0,1)$ s.t. $x_n \to x$. Now, $f(x_n)$ is a seq^n in compact space K, hence by passing to sub-sequence, we can assume that $f(x_n) \longrightarrow \tilde{f}(x)$.

Let $\tilde{f}(x) := \lim f(x_n)$, when $x_n \to x$.

Suppose $x, y \in [0,1]$ and $|x-y| < \delta$ (where δ to be chosen later). Then for large n, $|x_n - y_n| < \delta$,

Hence, by uniform continuity of f

$\Rightarrow$ X^* is compact in X^*.

On $(0,1)$, it follows that for $\epsilon > 0$, (146)
$\exists\ \delta > 0$ s.t

$|x_n - y_n| < \delta$ & $|f(x_n) - f(y_n)| < \epsilon$

i.e $|\tilde{f}(a) - \tilde{f}(y)| < \epsilon$.

Hence $\tilde{f}$ is unif cont. on compactification $[0,1]$.

The above example can be generalized to topological set up, known as Stone - Cech compactification.

note that in the above example we assume uniform continuity instead of continuity to avoid complication.