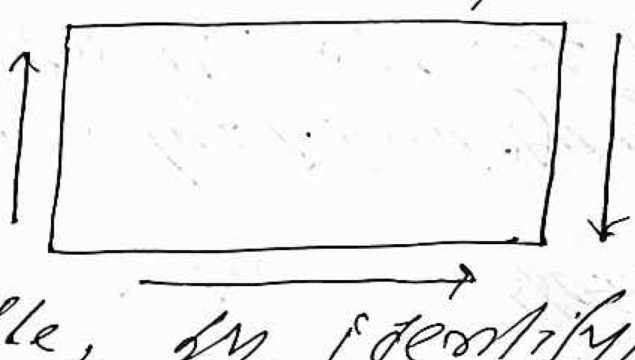
Quotient Spaces:

It is a way to construct complicated objects out of simple ones, via cut-paste technique.

For circle, can be obtained by identifying two points of the real line as one which differs by an integer (i.e. $x \sim y$ iff $x - y \in \mathbb{Z}$)

* Rectangle to cylinder, by identifying two opposite sides.
* Torus, by identifying two rims of cylinder.
* Möbius strip, by identifying ~~two~~ a pair of opposite sides in opposite direction.

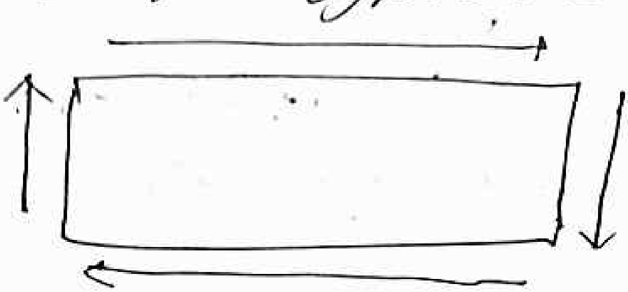


* Klein bottle, by identifying one

pair of opposite sides in the same direction & other in the opposite direction.

(84)

* Projective plane, by identifying each pair of sides in the opposite direction



* Sphere, by identifying boundary of rectangle as one point. There are other prominent example too.

The above these example, and many other, can be summerized into two methods for topologizing sets that play an increasingly important role in the topology.

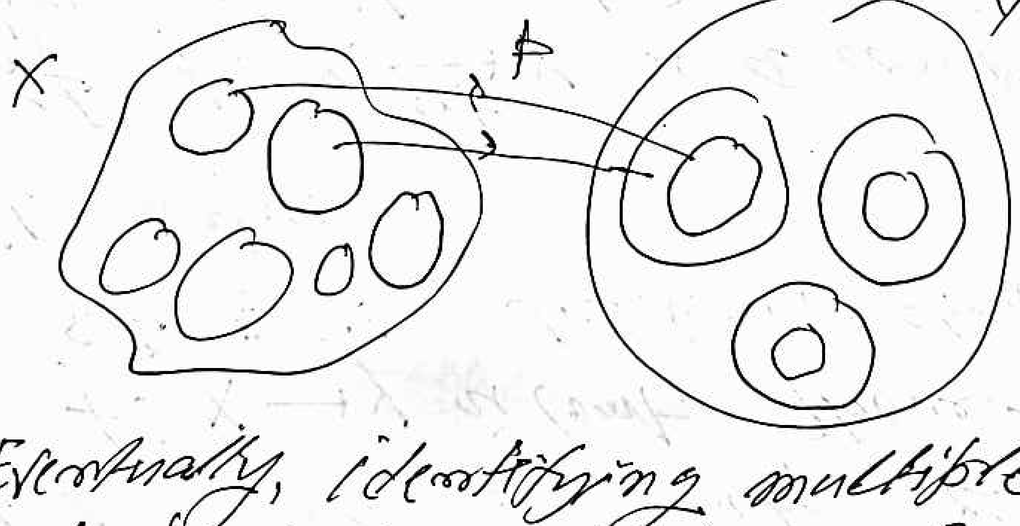
The first uses a map of a space into a set to topologize the set, which makes precise numerous constructions as mentioned some of them. the second constructs a space by "pasting" given spaces together along pre-assigned subsets. (85)

Quotient map:

Let X be a top. space & Y be a set. Let $p : X \longrightarrow Y$ be a surjective map. We construct a topology on Y by

$O \in \tau_Y$ iff $p^{-1}(O) \in \tau_X$.

i.e. $\tau_Y = \{ O \subseteq Y : p^{-1}(O) \in \tau_X \}$.



(Eventually, identifying multiple open sets in X to an open set in Y)

Notice that p has stronger property than a simple continuity. Because, if $f: X \to Y$ is cont, there may be set $A \subset Y$ (which is not open), but $f^{-1}(A)$ is open. (86)

Ex. If $f: X \xrightarrow{onto} Y$ is a continuous & open map, then f is a quotient map.

If $O \in \tau_Y$, then $f^{-1}(O) \in \tau_X$.

If $f^{-1}(O) = V$ for some set $O \subset Y$, then $f(V) = O$, and f is open, hence O is open.

Note that in the above discussion "open" set can be replaced with "closed" set.

However, there are quotient map, which is neither open nor closed.

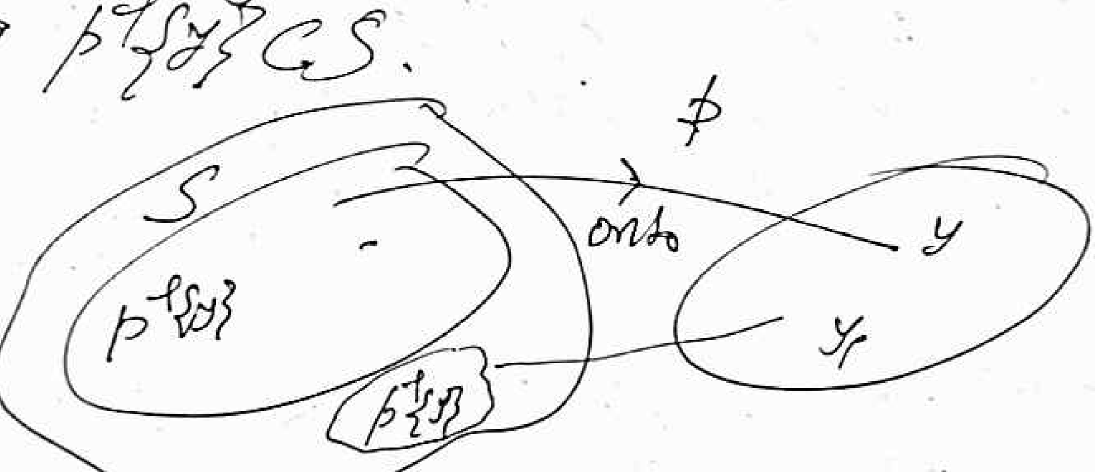
Ex. $X = [0,1] \cup [2,3]$, $Y = [0,2]$

$$p: X \longrightarrow Y \text{ by } p(x) = \begin{cases} x & x \in [0,1] \\ x-1 & x \in [2,3] \end{cases}$$

is a quotient map, but not open (27)
as $p([0,1]) = [0,1]$ not open in Y.
(it is also not closed with same argument)

Note that a map being cont. & open is not too sufficient for it to be a quotient map, and a small restriction to openness of map can prescribe quotient map.

Def^n: Let $p: X \longrightarrow Y$ be a surjective map. A set $S \subset X$ is said to be saturated if for any $y \in Y$ with $p^{-1}\{y\} \cap S \neq \emptyset$ $\Rightarrow p^{-1}\{y\} \subset S$.



Since $p^{-1}(Y) = X \Rightarrow X = p^{-1}(T) \cup p^{-1}(T^c)$ for some $T \subset Y$. Thus, $S = p^{-1}(T)$ if where we can assume that $p^{-1}(T) \cap S \neq \Phi$ only. (88)

That is S is saturated subset of X w.r.t. $p \Leftrightarrow S = p^{-1}(T)$ for some $T \subset X$.

<u>Proposition:</u> If $p : X \to Y$ is onto, then p is a quotient map iff p is cont. & p map saturated open set to open set.

Pf: Suppose p is a quotient map. Then we need to show that p sends saturated open set to open set.

Let $S \subseteq X$ be open & saturated.

Then $S = p^{-1}(T)$, for some $T \subset Y$.

Since p is quotient map, T must be open. That is, $p(S) = T$.

On the other hand, suppose p is cont & map saturated open set to open set.

Let $V = p^{-1}(O)$ - open. Then V is saturated & $p(V) = O$ is open.

Ex. let $X = [0,1] \cup [2,3]$, $Y = [0,2]$

& $p : X \to Y$ by $p(x) = \begin{cases} x & \text{if } x \in [0,1] \\ x-1 & \text{if } x \in [2,3] \end{cases}$

then p was a quotient map, but if

$A = [0,1) \cup [2,3]$ &

$q : A \to Y$, where $q = p/A$.

Then p is cont & onto but not a quotient map. because $[2,3]$ is open in A & saturated & open. but

$q([2,3]) = [1,2]$ is not open in Y.

Ex. let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\pi_1(x,y) = x$.

Then π_1 is open, cont & surjective, hence π_1 is a quotient map. But π_1 is not a closed map as

$\pi_1 \{(x,y) : xy = 1\} = \mathbb{R} \setminus \{0\}$ not closed.

Now, let $A = \{(x,y) : xy = 1\} \cup \{(0,0)\}$.

Then $q : A \to \mathbb{R}$, where $q = \pi_1/A$ is cont & surjective but not a quotient map.

Because $\{(0,0)\}$ is open & saturated in A ~~def~~ w.r.t. q but $q\{(0,0)\} = \{0\}$ is not open in $\mathbb{R}$.

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Now, we use quotient map to construct a top. on a set.

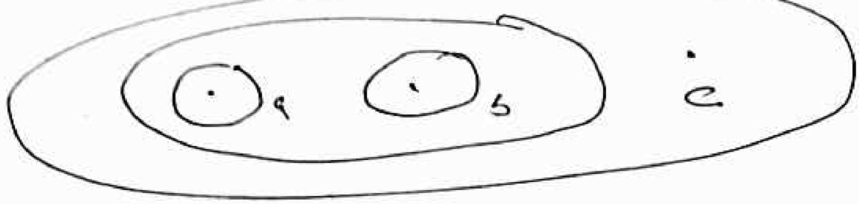
Notice that if $A \subset X$ &

$p: X \to A$ is surjective, then $\exists$ exactly one top. on A relative to which p is a quotient map. The top. induced on A is called quotient top.

i.e. $\mathcal{T} = \{O \subset A : p^{-1}(O) \text{ is open in } X\}$ is a top.

Ex. Let $A = \{a, b, c\}$ & $p: \mathbb{R} \to A$, where
$$p(x) = \begin{cases} a & \text{if } x > 0 \\ b & \text{if } x < 0 \\ c & \text{if } x = 0 \end{cases}$$

Then the quotient top. on A generated by p is $\mathcal{T}_A = \{\phi, \phi, \{a\}, \{b\}, \{a, b\}\}$

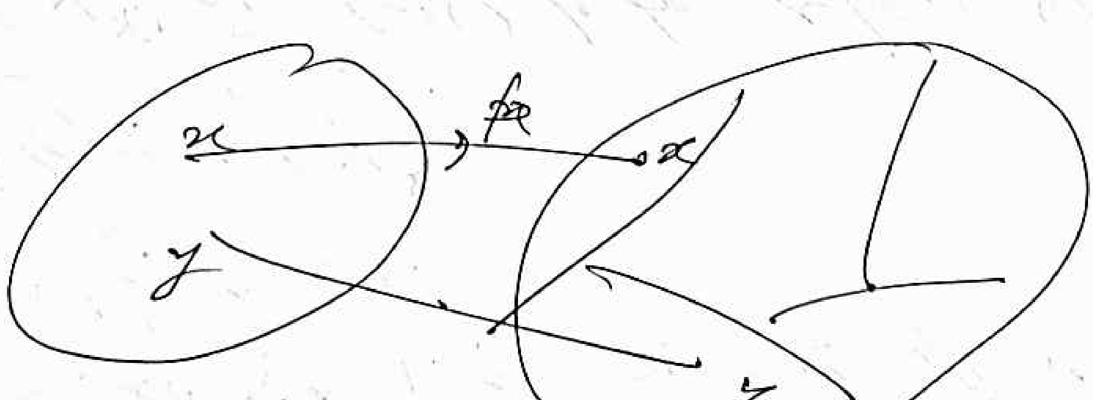


There are special situation in which quotient top. occur frequently.

Let X be a top. space & X^* be a partition of into disjoint sets.

i.e. $X = \bigcup_{i \in I} X_i$

Let $p : X \longrightarrow X^*$ be a surjective map carries each pt. $x \in X$ to a corresponding set $X_i \in X^*$, containing x.



Note that $X \longrightarrow X^*$, $\exists$ an equivalence relation on X^* s.t. $x \sim y$ iff $x, y \in X_i$ for exactly one i.

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<u>Topology of X^*</u>:

$O \subset X^*$ is said to be open if

$p^{-1}(O)$ = union of equivalence class is open.

This top. is denoted by $\tau(p)$.

Then $(X^*, \tau(p))$ is a quotient space

Now, we try to form a relationship between notions of quotient map & quotient space.

We know that subspaces do not behave well, as restriction of quotient map need not be a quotient map.

However, one has the following result.

<u>Thm</u>: Let $p: X \to Y$ be a quotient map & $A \subset X$ be saturated w.r.t. p. (93)
Let $q: A \longrightarrow p(A)$, where $q = p|_A$.

(i) If A is open then q is a quotient map

(ii) If p is open then q is a quotient map.

<u>Pf</u>: We verify 1st the following two equations:

(i) $q^{-1}(V) = p^{-1}(V)$ if $V \subset p(A)$

(ii) $p(U \cap A) = p(U) \cap p(A)$ if $U \subset X$

Pf: (i) if $V \subset p(A) \Rightarrow \; y \in V \Rightarrow y = p(a)$
$\Rightarrow \{a\} \subset p^{-1}\{y\}$, A is saturated
$\Rightarrow p^{-1}\{y\} \subset A$, $\forall y \in V$
ie $p^{-1}(V) \subset A$.
Since $p(A) = q(A)$.

Let $x \in p^{-1}(V) \subset A$

$\Rightarrow p(x) \in V \subset p(A) = q(A)$

$q(x) = p(x) = q(a) = p(a)$

$\Rightarrow x \in q^{-1}(V)$

$\Rightarrow q(x) \in V \subset p(A) = q(A)$

$\Rightarrow x \in A$

$\Rightarrow q(x) = p(x) \in V$

$\Rightarrow x \in p^{-1}(V)$.

(ii) For $U, A \subset X$,

$$p(U \cap A) \subset p(U) \cap p(A)$$

For reverse inclusion, let

$$y = p(u) = p(a), \quad u \in U, \ a \in A.$$

Since A is saturated, and $a \in p^{-1}\{p(a)\}$

$\Rightarrow p^{-1}\{p(a)\} \subset A$

$\Rightarrow p^{-1}\{p(u)\} \subset A \Rightarrow u \in A$

Thus $y = p(u)$, $u \in U \cap A$.

Now, Suppose A is open & p is open.

$q : A \longrightarrow p(A)$

Let $V \subset p(A)$ & assume $q^{-1}(V)$ is open in A. Claim V is open in $p(A)$

(a) ~~If~~ Case A is open, and $q^{-1}(V)$ is open in A, it follows that

$$q^{-1}(V) \text{ is open in } X$$

Since $q^{-1}(V) = p^{-1}(V)$ - open in X

$\Rightarrow V$ is open in $p(A)$.

Now, suppose p is open, and $q^{-1}(V)$ is open in A. Then, by $q^{-1}(V) = p^{-1}(V)$, $p^{-1}(V)$ is open in ~~A~~, hence it

$$\Rightarrow \quad p^{-1}(V) = U \cap A,$$

for some ~~b~~open set U in X

Now, $p(p^{-1}(V)) = V$, since p is onto.

But then $V = p(U \cap A) = p(U) \cap p(A)$.

Hence, $p(U)$ is open, as p is an open map.
Hence V is open in $p(A)$. 96

Cor: The above result can be imitated when p is closed or A is closed.

Ex. Composition of two quotient maps is a quotient map, because

$$p, q : X \xrightarrow{\text{quotient}} Y$$

$$\Rightarrow p^{-1}(q^{-1}(V)) = (q \circ p)^{-1}(V).$$

However, product of two quotient maps need not be a quotient map.
If both of them are open, then product is open cont. & onto, hence quotient map.

One of the most important result in the study of quotient space is to construct continuous functions on a quotient space.

<u>Theorem</u>: Let $p: X \longrightarrow Y$ be a quotient map & $g: X \longrightarrow Z$ be a map constant on each $p^{-1}\{y\}$ for $y \in Y$. Then g induces a map

$$f: Y \longrightarrow Z$$

such that $f \circ p = g$.

(i) f is continuous iff g is continuous.

(ii) f is a quotient map iff g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

<u>Pf</u>: Since g is constant on fiber $p^{-1}\{y\}$, it implies

$$g((p^{-1}\{y\})) = \text{singleton in } Z$$
$$= f(y) \text{ (say)}.$$

Then we can define a map

$$f: Y \longrightarrow Z \quad \text{via}$$
$$f(p(x)) = g(x) \Rightarrow f \circ p = g.$$

(i) If f is cont, then g is cont. (98)

Conversely, let g be cont. & $V \subset Z$ open. Then $g^{-1}(V)$ is open in X. But

$$g^{-1}(V) = p^{-1}(f^{-1}(V));$$

Since p is a quotient map, $f^{-1}(V)$ has to be open.

$\Rightarrow$ f is cont.

(ii) If f is a quotient map, then composition will be so. Hence g is a quotient map.

Conversely, suppose g is a quotient map. Since g is surjective, f is surjective.

Let $V \subset Z$. We claim that

$f^{-1}(V)$ is open $\Rightarrow$ V is open.

Now, set $p^{-1}(f^{-1}(V))$ = open in X as p is cont.

But $p^{-1}(f^{-1}(V)) = g^{-1}(V)$ - open

$\Rightarrow$ V is open as g is quotient.

Thus, f is a quotient map.

Cor: Let $g : X \xrightarrow[\text{Cont}]{\text{onto}} Z$, and

$$X^* = \{ g^{-1}\{z\} : z \in Z \}.$$

Give X^* the quotient topology generated by $p : X \to X^*$. Then

(i) g induces a continuous bijection $f : X^* \to Z$, which is a homeo. iff g is a quotient map

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"'] & Z \end{tikzcd}$$

(ii) If Z is Hausdorff, then so is X^*.

Pf: (i) By previous theorem, g induces a map $f : X^* \longrightarrow Z$. clearly f is 1-1, onto, because $X^* = \{g^{-1}(z) : z \in Z\}$. (100)

Suppose f is a homeo. Then both f & p are quotient maps, and hence their composition $f \circ p = g$ is a quotient map.

Conversely, Suppose g is a quotient map, then by previous thm f is a quotient map & bijection $\Rightarrow$ f is homeomorphism.

(ii) Suppose Z is a Hausdorff space. Given distinct pts of X^*, X_1 & X_2 $f(X_1) \neq f(X_2)$ in Z, and hence $\exists$ open disjoint sets U & V s.t $f(X_1) \subset U$ & $f(X_2) \subset V$.

But $U \cap V = \phi \Rightarrow f^{-1}(U) \cap f^{-1}(V) = \phi$.

Thus $f^{-1}(U)$ & $f^{-1}(V)$ are disjoint nbd of x_1 & x_2.

Remark: Some of properties, such as compactness, connectedness, and path-connectedness can pass to quotient space, but most of the other properties need not be passed onto quotient space. In fact, quotient spaces are simply not very tractable. It is important to consider several basic examples of quotient spaces rather thinking of general theory about quotient space.