Continuous function:

Continuous function is one of the important tool in mathematics and perhaps the most important concept. Recall that a continuous function on a metric space to metric space can be understood by considering pre-image of open set in range space to be open in domain space. For this, let

$$f : (X, d) \xrightarrow{\text{cont}} (Y, \rho)$$

Then for $x \in X$ & $\forall \epsilon > 0$, $\exists \delta_x > 0$ s.t. $d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \epsilon$

$$\Rightarrow y \in B^d_{\delta_x}(x) \Rightarrow f(y) \in B^\rho_\epsilon(f(x))$$

$$\Rightarrow f(B^d_{\delta_x}(x)) \subseteq B^\rho_\epsilon(f(x)).$$

Now, let $O \subset Y$ be open & $x \in f^{-1}(O)$.

Then $f(x) \in O$, & open.

$$\Rightarrow \exists \epsilon > 0, \text{ s.t } B^\rho_\epsilon(f(x)) \subseteq O.$$

By continuity of f, $\exists\ \delta_x > 0$ s.t.

$$f(B^d_{\delta_x}(x)) \subset B^\rho_\epsilon(f(x)) \subset O$$

$$\Rightarrow B^d_{\delta_x}(x) \subset f^{-1}(O).$$

Hence $f^{-1}(O)$ is open.

The converse part, (that is, if $f^{-1}(O)$ is open in X, $\forall$ open set $O \subset Y$, then f is continuous in ϵ-δ def^n) will be followed in similar fashion.

In fact, it can be deduced from the previous discussion that

$f : X \to Y$ is continuous at $x \in X$ iff for every nhd $N_{f(x)}$ of $f(x)$ in Y, $\exists$ a nhd N_x of x in X

s.t. $\quad f(N_x) \subset N_{f(x)}$

i.e. $\quad N_x \subset f^{-1}(N_{f(x)})$

i.e. pre-image of every nhd of $f(x)$

contains a nbd of x.

observe that these two definitions of continuity are meaningful in general topological setups.

i.e $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is cont. if for each open set $O \in \tau_Y$, $f^{-1}(O) \in \tau_X$.

And f is cont at $x \in X$, if each nbd of $f(x)$ in Y produces a nbd of N_x in X & f:

$$f(N_x) \subset N_{f(x)} \text{ (nbd of } f(x)).$$

Notice that the continuity of function f is not depending only on function itself rather top. space X & Y too.

As usual, like other previous cases, the continuity of function can be completely understood by basis open sets, instead of all open sets in Y.

Let $\mathcal{B}_Y$ be a basis for Y. Then $O \in \mathcal{T}_Y$ can be represented by

(49)

$$O = \bigcup_{i \in J \subset I} \{B_i : B_i \in \mathcal{B}_Y\}$$

= an union of members of $\mathcal{B}$.

$$\Rightarrow f^{-1}(O) = \bigcup_{i \in J} f^{-1}(B_i)$$

Hence $f^{-1}(O)$ is open for <u>each</u> $O \in \mathcal{T}_Y$ iff $f^{-1}(B_i)$ is open for each $i \in I$.

Further, if $\mathcal{S}$ is a subbasis for $\mathcal{T}_Y$, then sets of the form

$$B = \bigcap_{i=1}^{n} S_i = \text{intersection of finitely many } S_i \in \mathcal{S}.$$

a basis for $\mathcal{T}_Y$, and

$$f^{-1}(B) = \bigcap_{i=1}^{n} f^{-1}(S_i).$$

Hence $f^{-1}(O)$ is open iff $f^{-1}(S_i)$ is open for each $i \in I$

(Note that choice of O is arbitrary).

Theorem: let $f: X \to Y$ i.e. $f:(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$. then FAE:

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(i) f is cont on X

(ii) If $\mathcal{B}$ is a basis for Y, then for each $B \in \mathcal{B} \Rightarrow f^{-1}(B) \in \mathcal{T}_X$

(iii) If $\mathcal{S}$ is a subbasis for $\mathcal{T}_Y$, then $\forall S \in \mathcal{S} \Rightarrow f^{-1}(S) \in \mathcal{T}_X$.

Ex. $f(x) = x$ is not cont if f is considered $f: (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{T}_l)$. Because,

$$f^{-1}([a,b)) = [a,b) \notin \mathcal{U},$$

however $g: (\mathbb{R}, \mathcal{T}_l) \to (\mathbb{R}, \mathcal{U})$, defined by $g(x) = x$, is cont, since

$$g^{-1}\{(a,b)\} = (a,b) \in \mathcal{T}_l$$

$$\left(\because (a,b) = \bigcup_{n=1}^{\infty} [a-\tfrac{1}{n}, b)\right).$$

The following characterization of continuous function is very much useful as long as the topological setup is concerned.

<u>Thm:</u> Let X & Y be top. spaces & $f: X \to Y$ be a map. Then TFAE:

(i) f is continuous on X.

(ii) $\forall$ closed set F in Y, $f^{-1}(F)$ is closed in X.

(iii) $\forall$ $x \in X$ & every nhd $N_{f(x)}$ of $f(x)$ in Y, $\exists$ nhd N_x of x in X s.t.

$$f(N_x) \subset N_{f(x)}.$$

(iv) $\forall A \subset X \Rightarrow f(\bar{A}) \subset \overline{f(A)}$

(v) $\forall B \subset Y \Rightarrow \overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$.

Pf: Sf. F is closed in $Y \Leftrightarrow O = Y \setminus F$ is open. Hence (i) $\Leftrightarrow$ (ii) holds.

(i) $\Rightarrow$ (iii): Let $N_{f(x)}$ be a nhd of $f(x)$. Then $f^{-1}(N_{f(x)})$ is open & containing x.
$\Rightarrow \exists N_x \subset f^{-1}(N_{f(x)})$.

Conversely, (iii) $\Rightarrow$ (i) follows. For this, let $O \subset Y$ be open. Then for $x \in f^{-1}(O)$ i.e. $f(x) \subset O \Rightarrow N_{f(x)} \subset O$, but then $\exists N_x$ s.t. $f(N_x) \subset N_{f(x)} \subset O$. (52)

$\Rightarrow N_x \subset f^{-1}(O)$.

Hence $f^{-1}(O)$ is open in X.

Thus, (i), (ii) & (iii) are equivalent.

<u>(i) $\Rightarrow$ (iv):</u>

Claim $f(\bar{A}) \subset \overline{f(A)}$

Let $x \in \bar{A}$. Claim $f(x) \in \overline{f(A)}$.

Let $N_{f(x)}$ be a nhd of $f(x)$ in Y.

Since $f^{-1}(N_{f(x)})$ is an open set containing x, it follows that

$$f^{-1}(N_{f(x)}) \cap A \neq \Phi,$$

Let $y \in f^{-1}(N_{f(x)}) \cap A$. then

$$f(y) \in N_{f(x)} \ \& \ f(y) \subset f(A)$$

$$\Rightarrow f(y) \in N_{f(x)} \cap f(A)$$

Hence $f(x) \in \overline{f(A)}$.

(iv) $\Rightarrow$ (v): let $A = f^{-1}(B)$.

claim $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

Since $f(\overline{A}) \subset \overline{f(A)}$.

$\Rightarrow f(\overline{A}) \subset \overline{f(f^{-1}(B))}$
$= \overline{B \cap f(X)} \subset \overline{B}$

$\Rightarrow \overline{A} \subseteq f^{-1}(\overline{B})$

i.e. $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

(v) $\Rightarrow$ (i): let $B \subset Y$ be closed.

Then $\overline{f^{-1}(B)} \subset f^{-1}(B) \subset f^{-1}(\overline{B})$

$\Rightarrow \overline{f^{-1}(B)} = f^{-1}(B)$

$\Rightarrow f$ is Continuous

i.e (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

For set $A \subset X$ (top. space), we define $f/A : A \to Y$ by $f/A(a) = f(a)$, where $f: X \to Y$. f/A is called restriction of f to A.

Lemma: If $f: X \to Y$ is continuous, then $f|A$ is continuous.

Pf: Let $O \subset Y$ be open, then

$$(f|A)^{-1}(O) = f^{-1}(O) \cap A$$

i.e $(f|A)^{-1}(O)$ is open in the relative top. of A.

This prompts us to consider a sort of converse if f is continuous on properly fitting pieces of X, then f is continuous on X. This is known as pasting lemma.

Theorem (Pasting Lemma):

Let $X = A \cup B$, where A & B both open (or closed) in X. If $f: X \to Y$ be such that $f|A$ & $f|B$ both are continuous, then f is continuous.

Pf: Let $O \subset Y$ be open, then

$$f^{-1}(O) = (f/A)^{-1}(O) \cup (f/B)^{-1}(O)$$ (55)

$$= (O_1 \cap A) \cup (O_2 \cap B),$$

where O_1 & O_2 are open in X., because f/A & f/B are continuous. Thus,

If A & B are open, then $f^{-1}(O)$ is open in X. Hence f is continuous. The case A & B closed is followed by closed set criteria for continuity.

<u>Theorem:</u> Let X, Y & Z be top. spaces & $f: X \to Y$ & $g: Y \to Z$ be continuous. Then $g \circ f: X \to Z$ is continuous.

Pf: Let O be open in Z. Then $g^{-1}(O)$ is open in Y (since g cont), and by continuity of f, it follows that

$$f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O) \text{ is open in } X.$$

Hence $g \circ f$ is continuous.

Next, we see that pasting lemma holds for arbitrary decomposition of a top. space through open sets. (56)

Defn: A family $\{A_i : i \in I\}$ is said to be locally finite (or nbd-finite) if each pt $x \in X$ has a nbd N_x which intersect finitely many A_i's.
i.e. $\#\{i \in I : N_x \cap A_i \neq \Phi\}$ = finite.

Lemma: (i) If $\{A_i : i \in I\}$ is locally finite, then $\{\overline{A_i} : i \in I\}$ is locally finite.
(ii) If $\{A_i\}_{i \in I}$ is locally finite & each A_i is closed then $\cup A_i$ is closed.

Pf: (i) For $x \in X$, $\exists$ N_x s.t.

$$N_x \cap A_i \neq \Phi \text{ for } i = 1, 2, \ldots, n$$

$$\Rightarrow N_x \cap \overline{A_i} \neq \Phi$$

Also, $N_x \cap A_i = \Phi$ for $i \neq 1,2,\dots,n$. (58)

$\Rightarrow A_i \subset X \setminus N_x \Rightarrow \bar{A_i} \subset X \setminus N_x$

$\Rightarrow \bar{A_i} \cap N_x = \Phi$ for $i \neq 1,2,\dots,n$.

(ii) Let $A = \bigcup_{i \in I} A_i$, & let $x \in X \setminus A$.

Then $\exists$ nbd. N_x s.t.

$N_x \cap A_i \neq \Phi \quad i = 1,2,\dots,n$

& $N_x \cap A_i = \Phi \quad \forall i \neq 1,2,\dots,n$.

$N_x \cap \left(\bigcup_{i \neq 1,2,\dots,n} A_i\right) = \Phi \quad i \neq 1,2,\dots,n$

$N_x \subset X \setminus \bigcup_{i \neq 1,2,\dots,n} A_i$

But $\underbrace{N_x \left(\bigcap_{i=1}^{n} A_i^c\right)}_{\text{open nbd of } x} \subset \underbrace{\left(X \setminus \bigcup_{i \neq 1,2,\dots} A_i\right) \left(\bigcap_{i=1}^{n} A_i^c\right)}$

$= X \setminus \bigcup_{i \in I} A_i$

$= X - A$.

i.e. $\forall x \in X - A$, $\exists$ nbd

$N_x' = N_x \bigcap_{i=1}^{n} A_i^c \subset X \setminus A$.

Theorem: Let $(A_i)_{i\in I}$ be a cover of X (i.e. $X = \bigcup A_i$).

Assume either

(i) All A_i's are open

or (ii) $(A_i)_{i\in I}$ be a locally finite family of closed sets.

Then $B \subset X$ is open (resp. closed) iff each $B \cap A_i$ is open (resp. closed) in the subspace A_i.

Pf: (i) Let each $B \cap A_i$ be open in A_i; then $B \cap A_\alpha$ is open in X

$\Rightarrow B = B \cap X = \bigcup_{i \in I}(B \cap A_i)$ — open in X.

(ii) Assume each $B \cap A_i$ is closed in A_i. Then $B \cap A_\alpha$ is closed in X.

Since $\{A_i\}_{i\in I}$ is locally finite,

$\Rightarrow \{A_i \cap B\}_{i\in I}$ is locally finite

Theorem: Let $(A_\alpha)_{\alpha\in I}$ be a cover of top. space X such that either (59)

(i) all A_α are open or

(ii) all A_α are closed with $\{A_\alpha\}_{\alpha\in I}$ is locally finite.

Suppose $f_\alpha : A_\alpha \to Y$ be continuous for each $\alpha \in I$ & $f_\alpha|_{A_\alpha\cap A_\beta} = f_\beta|_{A_\alpha \cap A_\beta}$

Then $\exists!$ continuous map

$$f : X \to Y$$

s.t. $f|_{A_\alpha} = f_\alpha$.

Pf: Let $x \in X = \bigcup_{\alpha\in I} A_\alpha \Rightarrow x \in A_\alpha$ for some $\alpha \in I$. Write

$$f(x) = f_\alpha(x).$$

Then f is well-defined. If $f(x) = f_\beta(x) \Rightarrow x \in A_\beta \cap A_\alpha$

$\Rightarrow f(x) = f_\alpha(x) = f_\beta(x)$.

The map $f: X \to Y$ is unique.
For this, let $g: X \to Y$ be such that $g|_{A_\alpha} = f_\alpha$.
Then $g(x) = f_\alpha(x) = f(x)$.

It remains to show that f is continuous. Let $O \subset Y$ be open.
Then $f^{-1}(O) \cap A_\alpha = f_\alpha^{-1}(O)$ — open ~~for~~ in A_α for each α.
Hence $f^{-1}(O)$ is open in X
(by previous theorem)

<u>Open map</u>: If $f: X \to Y$ is a continuous map, we may loose information in two ways: the pts of X mapped to fewer pts of Y & f may map open sets in X to lesser no. of

open sets. However, this issue can be resolved by assuming f is bijection, and f sends open set to open set. (61)

This implies if $O \subset X$ open, then

$$(f^{-1})^{-1}(O) = f(O) - \text{open}$$

$\Rightarrow f^{-1}$ is continuous.

A map sends open set to open set is called open map & similarly closed map.

The following theorem will summerise the above discussion.

<u>Theorem:</u> Let $f: X \longrightarrow Y$ be a bijection. Then FAE:

(i) f & f^{-1} both continuous

(ii) f is cont & open

(iii) f is closed & open

(iv) $\forall A \subset X \Rightarrow f(\bar{A}) = \overline{f(A)}$.

A map satisfies any of them (and hence all) is known as homeomorphism. (62)

Pf: (i) $\Leftrightarrow$ (ii) & (i) $\Leftrightarrow$ (iii). Hence, (i), (ii) & (iii) are equivalent.

Next, (iii) $\Leftrightarrow$ (iv):

claim $f(\bar{A}) = \overline{f(A)}$.

Since f is cont, $f(\bar{A}) \subset \overline{f(A)}$

Since f is closed,

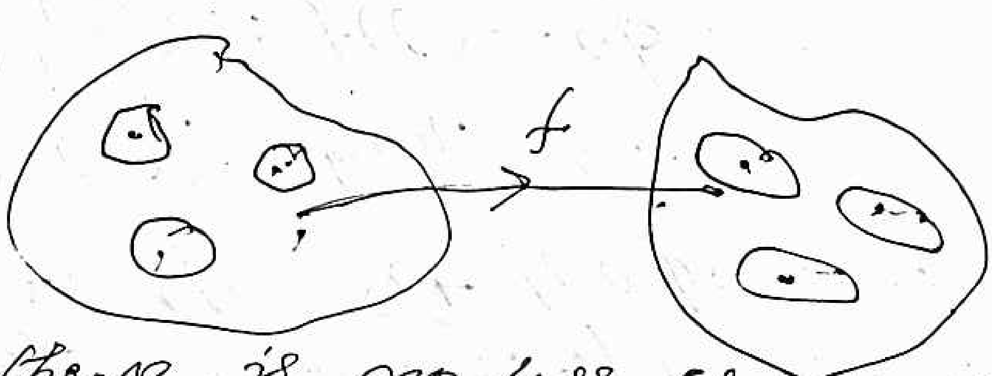
$f(A) \subset f(\bar{A})$ — closed

$\Rightarrow \overline{f(A)} \subset f(\bar{A})$.

On the other hand, if $f(\bar{A}) = \overline{f(A)}$ it implies f is cont. For A closed

$f(A) = \overline{f(A)} \Rightarrow f(A)$ is closed.

Eventually, homeo is a re-positioning of open sets [or/and deformation without tearing].



that is, there is no loss of pts and open sets due to homeomorphism.

Theorem: let $f : X \longrightarrow Y$. Then FAE: (63)

(i) f is an open map.

(ii) $f(A^{o}) \subset (f(A))^{o}$

(iii) If $\mathcal{B}$ is a basis for $\mathcal{T}_X$, then each $f(B)$ is open in Y

(iv) If $\mathcal{S}$ is a subbasis for $\mathcal{T}_X$, then each $f(\mathcal{S})$ is open in Y

(v) for each pt $x \in X$ & $\forall$ nhd N_x of x, $\exists$ nhd $W_{f(x)}$ in Y s.t

$f(x) \subset W_{f(x)} \subset f(N_x)$.

Pf: (i) $\Rightarrow$ (ii): $A^{o} \subset A \Rightarrow f(A^{o}) \subset f(A)$

Since f is an open map, & A^{o} is open $\Rightarrow f(A^{o})$ is open containing $f(A)$.

Hence $f(A^{o}) \subset (f(A))^{o}$.

(ii) $\Rightarrow$ (iii): let $B \in \mathcal{B} \subset \mathcal{T}_X$ &

$\Rightarrow B = B^{o}$

$(f(B))^{o} \subseteq f(B) = f(B^{o}) \subset (f(B))^{o}$.

$\Rightarrow (f(B))^{\circ} = f(B) \Rightarrow f(B)$ is open.

(iii) $\Leftrightarrow$ (iv) Easy. We need to prove (iii) $\Rightarrow$ (v).
and (v) $\Rightarrow$ (i).

(a) (iii) $\Rightarrow$ (v). Notice that each pt $x \in B$
for some $B \in \mathcal{B}$, s.t. $x \in B \subset N_x$ (64)

$\Rightarrow f(x) \subset f(B) \subset f(N_x)$

Let $W_{f(x)} = f(B)$.

(b) (v) $\Rightarrow$ (i). Let $O \subset Y$ be an open set
then for $y \in f(O)$ has a nbd W_y st
$W_y \subset f(O)$.

But then $f(O) = \cup \{W_y : y \in f(O)\}$
is open in Y.

Continuous map into $\mathbb{R}$:

Let $f: X \to \mathbb{R}$ be map. We know
that $(a,b) = (-\infty, b) \cap (a, \infty)$.

$\Rightarrow f^{-1}\{(a,b)\} = f^{-1}\{(-\infty,b)\} \cap f^{-1}\{(a,\infty)\}$

Also $(-\infty, b) = \bigcup_{n=1}^{\infty}(-n, b)$ &

$$(a,\infty) = \bigcup_{n=1}^{\infty} (a,n).$$

Hence it follows that $f^{-1}(a,b)$ is open for each $(a,b) \subseteq \mathbb{R}$ iff $\{x : f(x) < b\}$ and $\{x : f(x) > a\}$ both open for each $a, b \in \mathbb{R}$.

Notice that the requirement of both types of sets are open cannot be relaxed. For example,

$A = (0,1)$, $C_A : \mathbb{R} \to \mathbb{R}$ be defined by $C_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

Then $\{x : C_A(x) < 1\} = (0,1)^c$ is not open, whereas $\{x : C_A(x) > a\}$ is open for each $a \in \mathbb{R}$, since

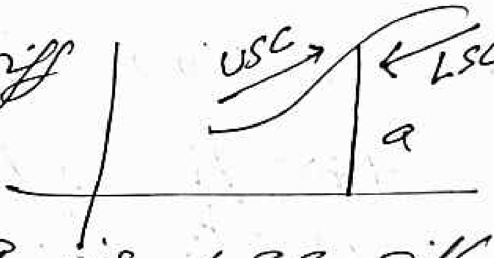
$$\{x : C_A(x) > a\} = \begin{cases} \phi & a \geq 1 \\ A & 0 \leq a < 1 \\ \mathbb{R} & a < 0 \end{cases}$$

Thus, C_A is not continuous.

This indicates a splitting of the notion of continuity.

Defn: $f : X \to \mathbb{R}$ is said to be lower semi-continuous if $\{x : f(x) > a\}$ is open for each $a \in \mathbb{R}$.
Similarly, upper semi-continuous if $\{x : f(x) < a\}$ is open for each $a \in \mathbb{R}$.

Hence, f is cont. on X iff f is both LSC & USC.



Ex. Show that $f : \mathbb{R} \to \mathbb{R}$ is LSC iff

$$f(x) \le \liminf_{n \to \infty} f(x_n)$$

$\forall\, x_n \to x$. [ref^n to M642, page-163]

Ex. Show that $f : \mathbb{R} \to \mathbb{R}$ USC iff

$$\limsup_{n \to \infty} f(x_n) \le f(x), \quad \forall\, x_n \to x.$$

Notice that it follows from the above two exercises that f is cont iff $\forall\, x_n \to x$

$$\Rightarrow \lim_{n \to \infty} f(x_n) = f(x).$$

Also, $\{x : f(x) > a\} = \{x : -f(x) < -a\}$.
Hence f is cont iff f & $-f$ both LSC & both USC.

The following theorem is not the simplex, rather general to other situation (e.g. measure theory, measurable functions etc).

Theorem: let $f, g : X \longrightarrow \mathbb{R}$ be cont. Then
(i) $|f|^{\alpha}$ is cont. $\forall\ \alpha > 0$
(ii) $af + bg$ is cont $\forall\ a, b \in \mathbb{R}$.
(iii) fg is continuous
(iv) If $f(x) \neq 0$ on X, then $1/f$ is continuous
(i.e $1/f$ is continuous, wherever it defined)

These results will be proved using the above two types of continuities.

Pf: (i) $\{x : |f|^{\alpha}(x) > a\} = \{x : f(x) > a^{1/\alpha}\}$
$\cup \{x : f(x) < -a^{1/\alpha}\}$. etc.

$\{x : |f(x)|^{\alpha} < a\} = \{x : f(x) < a^{1/\alpha}\}$
$\cap \{x : f(x) > -a^{1/\alpha}\}$. (68)

(ii) $\{x : f(x)+g(x) > a\}$

$= \bigcup_{\lambda \in \mathbb{R}} \left(\{x : f(x) > a-\lambda\} \cap \{x : g(x) > \lambda\}\right)$

Similarly, $\{x : f(x)+g(x) < a\}$ is open.

(iii) $fg = \frac{1}{4}\left[|f+g|^2 - |f-g|^2\right]$

(iv) If $f(x) \neq 0$, $\forall x \in X$, then

$\{x : 1/f(x) > a\}$

$= [\{x : f(x) > 0\} \cap \{x : af(x) < 1\}$.

~~More~~ Similarly

$\{x : 1/f(x) < a\}$ is open.

Ex. Let $f \in C^1[0,1]$, and define

$$L(f) = \int_0^1 \sqrt{1+(f'(t))^2}\, dt$$

is lower-semi continuous on $C^1[0,1]$ to $\mathbb{R}$.

Ex. Let $\{f_\alpha\}_{\alpha \in I}$ be a family of LSC functions on X, for which, for each $x \in X$, $\{f_\alpha(x) : \alpha \in I\}$ has an upper bound. Let $g(x) = \sup\{f_\alpha(x) : \alpha \in I\}$. Then g is LSC. (69)

Note that $g(x) > a$ iff $\exists$ at least one $f_\alpha(x) > a$. Hence

$$\{x : g(x) > a\} = \bigcup_{\alpha \in I} \{x : f_\alpha(x) > a\}$$

$\Rightarrow$ g is LSC.

Ex. Similar result is true for USC.

Ex. Suppose $f : \mathbb{R} \to \mathbb{R}$ is cont from the right, that is,

$$\lim_{x \to a^+} f(x) = f(a), \ \forall\, a \in \mathbb{R}.$$

Show that f is continuous, when considered $f : (\mathbb{R}, \mathcal{U}_\ell) \to (\mathbb{R}, \mathcal{U})$.

For given $\epsilon > 0$; $\exists\ \delta_a > 0$ s.t.
for $x \in (a, a+\delta_x) \Rightarrow |f(x) - f(a)| < \epsilon$
(*) i.e. $f\{(a, a+\delta_x)\} \subset (f(a)-\epsilon, f(a)+\epsilon)$
i.e. for every nhd of $f(a)$, in $(\mathbb{R}, u)$,
$\exists$ nhd $(a, a+\delta_x) \subset \mathcal{I}_\ell$ s.t.
(*) holds.

Ex. let $f: X \longrightarrow Y$ be a map.
TFAE:
(i) f is cont. on X
(ii) $f(A') \subset \overline{f(A)}$, $\forall\ A \subset X$
(iii) $\partial[f^{-1}(B)] \subset f^{-1}[\partial(B)]$

Ex. (i) $\Rightarrow$ (ii)

$f(A') \subset f(\bar{A}) \subset \overline{f(A)}$ (by (i))
and previous result about chara. of continuous function.

(ii) $\Rightarrow$ (iii)

Let $x \in \partial(f^{-1}(B))$. Then $x \in \overline{f^{-1}(B)} \cap \overline{(X \setminus f^{-1}(B))}$

~~or~~ If $x \in f^{-1}(B)$, then $x \in \underline{(X \setminus f^{-1}(B))'}$. By

(ii) $f(x) \in f(\overline{X \setminus f^{-1}(B)}) \subset \overline{Y \setminus B}$

$\Rightarrow f(x) \in \partial B \Rightarrow x \in f^{-1}(\partial B)$. (7.1)

If $x \notin f^{-1}(B) \Rightarrow x \in X \setminus f^{-1}(B)$

~~$\Rightarrow x \in (f^{-1}(B))'$~~

But $x \in X \setminus f^{-1}(B) \Rightarrow f(x) \in Y \setminus B$

by (ii) $f(x) \in f((f^{-1}(B))') \subset \overline{f(f^{-1}(B))}$

$\subset \overline{B}$

$\Rightarrow f(x) \in \partial B$

$\Rightarrow x \in f^{-1}(\partial B)$

~~Thus~~

Thus $\partial(f^{-1}(B)) \subset f^{-1}(\partial B)$.

(iii) $\Rightarrow$ (i):

Let $O \subset Y$ be open, then

$\partial(f^{-1}(Y \setminus O)) \subset f^{-1}(\partial(Y \setminus O))$.

We use the fact that A 28

closed iff $\partial A \subset A$. This implies that $\partial(Y \setminus O) \subset Y \setminus O$.

Thus, $\partial(f^{-1}(Y \setminus O)) \subset f^{-1}(Y \setminus O) = X \setminus f^{-1}(O)$.

$\Rightarrow X \setminus f^{-1}(O)$ is closed & hence $f^{-1}(O)$ is open.

Ex. Let $f, g: X \longrightarrow \mathbb{R}$ be cont.

(a) Show that $A = \{x : f(x) \leq g(x)\}$ is closed.

(Hint: $A^c = \{x : f(x) - g(x) > 0\}$

But $f - g$ is cont $\Rightarrow$ LSC $\Rightarrow A^c$ open)

(b) Let $h: X \longrightarrow \mathbb{R}$ be defined by

$h(x) = \min\{f(x), g(x)\}$.

Show that h is continuous.

We can re-write

$$h(x) = C_B(x) f(x) + C_D(x) g(x),$$

where C_B & C_D are characteristic functions of $B = \{x : f(x) \leq g(x)\}$.

and $D = \{x : f(x) \geq g(x)\}$ resp.
If $x \in B \cap D$, then $f(x) = g(x)$.
Since B & D are closed by pasting lemma, $\exists$ a unique

$$K : X \xrightarrow{\text{Cont}} \mathbb{R},$$

which is nothing but h itself due to uniqueness.

Return to product topology

We know that product top. of topological spaces (X, τ_X) & (Y, τ_Y) can be defined through basis

$$\mathcal{B} = \{O_X \times O_Y : O_X \in \tau_X, O_Y \in \tau_Y\}$$
$$\subseteq \tau_X \times \tau_Y.$$

Also, we can generate the same topology on $X \times Y$ via projections.

Let $\pi_1 : X \times Y \longrightarrow X$, $\pi_1(x,y) = x$
& $\pi_2 : X \times Y \longrightarrow Y$, $\pi_2(x,y) = y$.

Then $\mathcal{S} = \{\pi_1^{-1}(O_X),\ \pi_2^{-1}(O_Y) : O_X \in \mathcal{T}_X,\ O_Y \in \mathcal{T}_Y\}$ (74)

defines a subbasis for the product topology.

Note that $\pi_1^{-1}(O_X) = O_X \times Y$ & $\pi_2^{-1}(O_Y) = X \times O_Y$.

and this implies that

$$\pi_1^{-1}(O_X) \cap \pi_2^{-1}(O_Y) = O_X \times O_Y \in \mathcal{B}.$$

Hence $\mathcal{S}$ & $\mathcal{B}$ generate the same top. on $X \times Y$.

The above conclusion is true for finite product of top. space.

Let $X = X_1 \times \cdots \times X_n = \prod_{i=1}^{n} X_i$. Then

$$\mathcal{B}_n = \{\prod_{i=1}^{n} O_i : O_i \in \mathcal{T}_i\}$$

forms a basis for a top. on X, and $\bigcap_{i=1}^{n} \pi_i^{-1}(O_i) = \prod_{i=1}^{n} O_i \in \mathcal{B}_n$.

Let $\mathcal{S}_n = \{\pi_i^{-1}(O_i) : i = 1, 2, \ldots, n,\ O_i \in \mathcal{T}_i\}$

Then $\mathcal{B}_n$ & $\mathcal{S}_n$ generate the same top. on X.

However, in general, the top. generated by $\mathcal{B}_n$ is called box top. on $\prod_{i=1}^{n} X_i$, whereas top. produced by $\mathcal{S}_n$ is known as product top. on $\prod_{1}^{n} X_i$.

Let $X = \prod_{i \in I} X_i$ and

$$\mathcal{S} = \{ \pi_i^{-1}(O_i) : i \in I \}.$$

Then $\mathcal{S}$ is a sub-basis for some top. on X, and it can be realised through basis

$$\mathcal{B}_p = \{ \bigcap_{i \in F} \pi_i^{-1}(O_i) : F \overset{\text{finite}}{\subset} I \}$$

= all such intersections of (finite) n-tuples in I.

Note that a typical member of $\mathcal{B}_p$ looks like

$$B = \prod_{i \in I} O_i,$$

where all but finitely many

$O_i = X_i$

Now, let

$$\mathcal{B}_b = \{ \prod_{i \in I} O_i : O_i \in I \}.$$

Then $\mathcal{B}_p \subsetneq \mathcal{B}_b$. Thus box top. on $\prod_{i \in I} X_i$ is finer than product top. on $\prod_{i \in I} X_i$.

Notice that a typical point of $\prod_{i \in I} X_i$ is represented by

$$x = (x_i)_{i \in I}, \text{ where } x_i \in X_i$$

Once again, we know that

$$\tau(\mathcal{B}_p) \subsetneq \tau(\mathcal{B}_b)$$

Hence, a function f continuous on $(X, \tau(\mathcal{B}_b))$ need not be continuous in $(X, \tau(\mathcal{B}_p))$.

Note that box top. is not as interesting as to the product top, on which

we try to generalize finite dimen-
-sional results. Besides this, there
will be clear distinction between
product top & box top. as long as
continuity, and many more to be
discuss in the due course of time.

Theorem: let $f : A \to \prod_{\alpha \in I} X_\alpha$ be given
by $f(a) = (f_\alpha(a))_{\alpha \in I}$, where
$f_\alpha : A \to X_\alpha$. let $\prod X_\alpha$ be given
product topology. Then f is continuous
iff each f_α is continuous.

Pf:- Suppose f is cont on A.
notice that each projection π_α
is continuous. For this, let $O_\alpha \in \tau_\alpha$.
The set $\pi_\alpha^{-1}(O_\alpha)$ is a subbasis element
for the product top. on $\prod X_\alpha$.
Hence $\pi_\alpha^{-1}(O_\alpha)$ is open in A.

This implies

(78)

$$\pi_\alpha \circ f(a) = f_\alpha(a)$$

$\Rightarrow f_\alpha$ is cont. on A.

Conversely, if each f_α is continuous, then for a typical subbasis element $\pi_\alpha^{-1}(O_\alpha)$, it follows that

$$f^{-1}(\pi_\alpha^{-1}(O_\alpha)) = (\pi_\alpha \circ f)^{-1}(O_\alpha)$$
$$= f_\alpha^{-1}(O_\alpha) \text{ - open}$$

$\Rightarrow f$ is continuous.

Notice that each projection π_α is continuous on box top. too, since box top is finer than product top. However, the above result does not hold for box top. on $\prod X_\alpha$.

For this, let $\mathbb{R}^\omega$ be denoting countable infinite product of $\mathbb{R}$.

i.e. $\mathbb{R}^\omega = \prod_{n=1}^{\infty} X_n$, $X_n = \mathbb{R}$.

Define $f: \mathbb{R} \to \mathbb{R}^{\omega}$ by

$f(t) = (t, t, \cdots)$. Then each coordinate function $f_n(t) = t$ is cont. $\Rightarrow$ f is cont. in product top. on $\mathbb{R}^{\omega}$. However, f is not continuous when $\mathbb{R}^{\omega}$ is endowed with box top. For this, let us consider a basis open set

$$B = (-1,1) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times \cdots \times (-\tfrac{1}{n}, \tfrac{1}{n}) \times \cdots$$

in the box top. on $\mathbb{R}^{\omega}$. Then $f^{-1}(B)$ is not open in $\mathbb{R}$. ~~If~~

$f^{-1}(B)$ is open in $\mathbb{R}$, then as $0 \in f^{-1}(B)$, $\exists$ small interval (open) in $f^{-1}(B)$.

i.e $(-\delta, \delta) \subset f^{-1}(B)$

$\Rightarrow f(-\delta, \delta) \subset B$

$\Rightarrow \pi_n(-\delta, \delta) \subset \pi_n(B)$

$\Rightarrow (-\delta, \delta) \subset (-\frac{1}{n}, \frac{1}{n})$, $\forall n \geq 1$

which is not possible for large n.

Ex. Let $(a_1, a_2, \dots)$ & $(b_1, b_2, \dots) \in \mathbb{R}^{\omega}$ and define (80)

$h : \mathbb{R}^{\omega} \to \mathbb{R}^{\omega}$ by

$h(x_1, x_2, \dots) = (a_1x_1+b_1, a_2x_2+b_2, \dots)$

then show that h is a homeo. on $\mathbb{R}^{\omega}$ onto $\mathbb{R}^{\omega}$ w.r.t product top. on $\mathbb{R}^{\omega}$.

(Hint: $\pi_n \circ h(x_1, x_2, \dots) = a_nx_n+b_n$

$\Rightarrow$ each $\pi_n \circ h$ is cont $\Rightarrow$ h is cont)

Also $h^{-1}(y_1, y_2, \dots) = \left(\frac{y_1-b_1}{a_1}, \frac{y_2-b_2}{a_2}, \dots\right)$

$\Rightarrow h^{-1}$ is conti.)

Q. What if $\mathbb{R}^{\omega}$ is given box top?

(Hint: It is enough to show that

$I : \mathbb{R}^{\omega} \to \mathbb{R}^{\omega}$, $I(x_1, x_2, \dots) = (x_1, x_2, \dots)$

is a homeo w.r.t box top or not?)

<u>Theorem:</u> let $\{X_\alpha\}_{\alpha\in I}$ be a family of topological spaces, and let $A_\alpha \subset X_\alpha$. If $\prod X_\alpha$ is given either product top or box top., then

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}.$$

Pf: let $x = (x_\alpha)_{\alpha\in I} \in \prod \overline{A_\alpha}$,
and $O = \bigcup_{\alpha\in I} O_\alpha$ be a basis element for either product top or box top. containing x.
Since $x_\alpha \in \overline{A_\alpha}$, choose $y_\alpha \in O_\alpha \cap A_\alpha$.
Then $y = (y_\alpha) \subset O$ & $y \in \prod A_\alpha$
i.e $y \in O \cap (\prod_{\alpha\in I} A_\alpha)$ ($\neq \phi$).
$\Rightarrow x \in \overline{\prod A_\alpha}$.

On the other hand, let
$x = (x_\beta) \in \overline{\prod A_\alpha}$ (in either top. let O_β be an open set containing x_β.

Since $\pi_\beta^{-1}(O_\beta)$ is open in either top. on $\prod X_\alpha$, it must contain a pt. $y = (y_\beta) \in \prod_{\alpha \in I} A_\alpha$ (82)

$\Rightarrow y_\beta \in O_\beta \cap A_\beta$

$\Rightarrow x_\beta \in \bar{A}_\beta;\ \forall \beta.$

(Notice that in either topology,

$\pi_\beta^{-1}(O_\beta) = \prod_{i \in I} O_i$; where

$O_i = X_i$ if $i \neq \beta$ & $O_i = O_\beta$)