Topology:

In this course we mainly discuss two facts, which can easily be motivated through metric spaces one is the class of all open sets in a metric space (known as metric topology; an abstract way of specifying 'nearness'). The second is class of continuous functions from a metric space to a metric space.

In fact, continuous function on a metric space can be understood by pre-images of open sets to be open sets.

Hence, it is evident from the above discussion that one can think to study the "theory of open" sets independently by discarding metric space

We Know that a set O in a metric (2)
space is open if for each $x \in O$, $\exists$
an open ball $B_\delta(x) \subset O$. Note that, from
above, it follows that every open set
is union of open balls (basic open sets)

Further, if $\mathcal{J}_d$ denotes the collection of
all open sets in (X,d), then

(i) $\phi, X \in \mathcal{J}_d$

(ii) arbitrary union of members of $\mathcal{J}_d$ is in $\mathcal{J}_d$

[i.e. $\{O_i : i \in I, O_i \in \mathcal{J}_d\}$
$\Rightarrow \bigcup_{i \in I} O_i \in \mathcal{J}_d$]

(iii) finite intersection of open sets is open

(i.e. $\{O_i \in \mathcal{J}_d : i = 1, 2, \ldots, n\}$
$\Rightarrow \bigcap_{i=1}^{n} O_i \in \mathcal{J}_d$)

Notice that for any set X, the
properties (i)–(iii) can independently

satisfied by many sub-collection τ ③
of $P(X)$.

Defn: Let X be a non-empty set and
τ be a sub-collection of $P(X)$ s.t

(i) $\phi, X \in \tau$

(ii) arbitrary union of members of
τ is in τ

(iii) finite intersection of members of
τ is in τ

then τ is known as topology, and
pair (X, τ) is known as topological
space.

The members of τ are known as
open sets.

Ex. $(X, \{\phi, X\})$ is a top space, known
as indiscrete top. space.

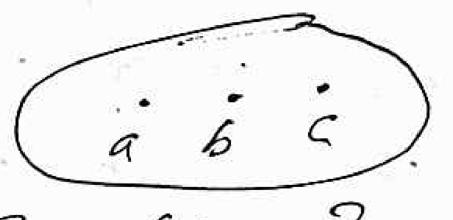
Ex. $(X, P(X))$ is a top. space, known
as discrete topo. space.

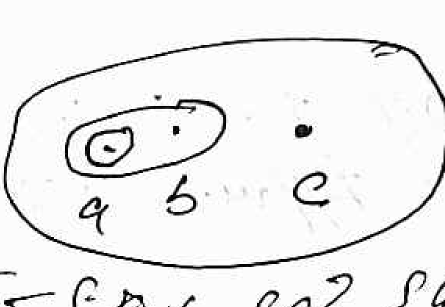
Notice that for set X having more than two elements have many topologies other than the above two extremes (discrete & indiscrete).

If τ_1, τ_2 be two topologies on X, we say τ_1 is <u>finer</u> than τ_2 if $\tau_1 \supset \tau_2$.

Hence discrete top. on any non-empty set X is ~~ϕ~~ strictly finer than any other top. on X.

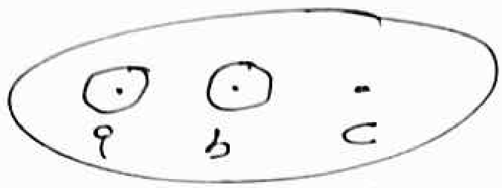
<u>Example</u>: Given a set $X = \{a, b, c\}$ there could be many topologies on X, we discuss a few of them.
However, the question ~~top~~ no. of topologies on a given finite set is still open.

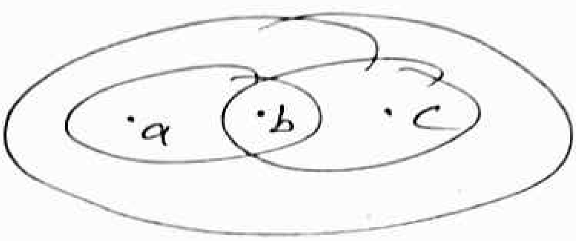




$\tau_0 = \{\phi, X\}$ - $\tau_1 = \{\phi, X, \{a\}, \{a,b\}\}$

on the other hand,





$\tau_2 = \{\varphi, X, \{a\}, \{b\}\}$ $\quad$ $\tau_3 = \{\varphi, X, \{a,b\}, \{b,c\}\}$

are not top. on X.

Ex. let X be a set, & τ be the collection of sets O in X s.t. either $X \setminus O$ is finite or all of X. Then τ is a top. on X, called Cofinite topology.

If $\{O_i : i \in I\} \subset \tau$, then

$$X \setminus \bigcup_{i \in I} O_i = \bigcap (X \setminus O_i) \text{ is finite.}$$

$$\Rightarrow \bigcup_{i \in I} O_i \in \tau$$

If $O_1, \ldots, O_n$ are non-empty sets in τ. Then

then $X \setminus \bigcap_{i=1}^{n} O_i = \bigcup_{i=1}^{n} (X \setminus O_i)$ - finite

Ex. Let X be a set, and τ be the collection of all sets O in X s.t. either $X \setminus O$ is countable or is all of X. Then τ is a top on X, called co-countable topology. (Similar to previous one).

Basis for a topology:

We know that any open set in $\mathbb{R}$ can be written as countable union of disjoint open intervals. In finite dim, an open set can be expressed as countable union of almost disjoint rectangles (only sides can overlap). But if is not a case in general of metric space.

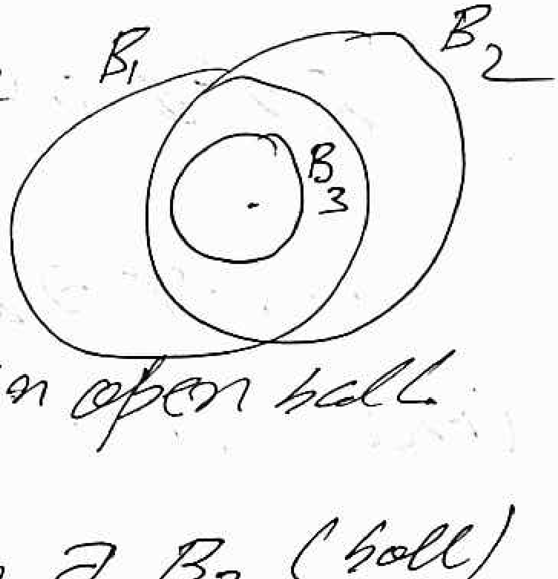
However, a set O in metric space (X, d) is open if $\forall x \in O$, $\exists B_r(x) \subset O$. Thus,

$$O = \bigcup_{x_i \in O} B_{r_i}(x_i), \quad r_i > 0.$$

That is, every open set in a metric space X can be expresssed as union of open balls.

Notice that intersection of two ball need not be a ball, however, if $x \in B_1 \cap B_2$, B_i – open ball, $i = 1, 2$, then $\exists$ an open ball B_3 s.t.

$$x \in B_3 \subset B_1 \cap B_2$$



Observe that

(i) for each $x \in X$, $\exists$ an open ball B s.t. $x \in B$,

(ii) if $x \in B_1 \cap B_2$, then $\exists$ B_3 (ball) s.t. $x \in B_3 \subset B_1 \cap B_2$:

This motivates to define basis for a top. space X.

Def^n: Let X be a set & $\mathcal{B} \subset P(X)$ s.t. (i) $\forall x \in X$, $\exists B \in \mathcal{B}$ st $x \in B$

(ii) if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ st $x \in B_3 \subset B_1 \cap B_2$.

is called basis for some top. on X.

The topology $\tau = \tau(\mathcal{B})$ generated by $\mathcal{B}$ is defined as follows:

$O \in \tau$ if for each $x \in O$, $\exists B \in \mathcal{B}$ st $x \in B \subset O$.

In this way each basis member $B \in \mathcal{B}$ is also a member of τ.

Claim $\tau = \tau(\mathcal{B})$ is a topology on X.

(i) $\phi \in \mathcal{T}$, as it satisfies condition for openess vacuously (9)

(ii) $X \in \mathcal{T}$, as for each $x \in X$, $\exists\, B \in \mathcal{B}$ st $x \in B \subset X$.

(iii) let $O = \bigcup_{i \in I} O_i$; $O_i \in \mathcal{T}$.

For $x \in O$, $\exists\, O_i \ni x \Rightarrow \exists\, B_i \in \mathcal{B}$ st $x \in B_i \subset O_i \subset O \Rightarrow O \in \mathcal{T}$.

(iv) let $x \in O_1 \cap O_2$. Then $x \in O_i$; $i = 1, 2$.

$\Rightarrow \exists\, B_i \subset O_i$ & $x \in B_i$.

$\Rightarrow x \in B_1 \cap B_2 \subset O_1 \cap O_2$

By defn of basis, $\exists\, B_3 \in \mathcal{B}$ st

$$x \in B_3 \subset B_1 \cap B_2 \subset O_1 \cap O_2$$

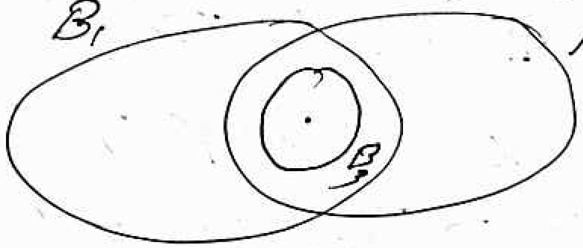
$\Rightarrow O_1 \cap O_2 \in \mathcal{T}$.

Finally, we show for the finite intersection By induction, suppose the result is true for $n-1$. Then

$$O_1 \cap \cdots O_{n-1} \cap O_n = (O_1 \cap \cdots \cap O_{n-1}) \cap O_n.$$

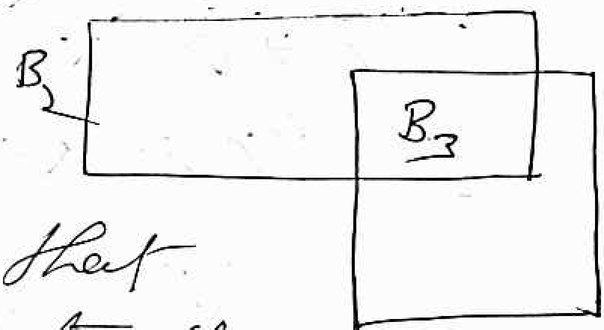
By the case for $n=2$, it follows that the result is true for n.

Ex. Let $\mathcal{B}$ be the collection of open discs in $\mathbb{R}^2$. Then $\mathcal{B}$ is a basis for the usual top. on $\mathbb{R}^2$.



Ex. Let $\mathcal{B}'$ be the collection of all open rectangles in the plane $\mathbb{R}^2$. Then $\mathcal{B}'$ is a basis.

In fact 2nd condition satisfied trivially.



We shall see later that both $\mathcal{B}$ & $\mathcal{B}'$ generate the same top on $\mathbb{R}^2$, the usual top.

<u>Lemma:</u> Let X be a set, and let $\mathcal{B} \subset P(X)$ be basis for a top. $\mathcal{T}$ on X. Then $\mathcal{T}$ is equal to all union of members of $\mathcal{B}$.

pf: Note that $\mathcal{T} = \mathcal{T}(\mathcal{B})$
$\Rightarrow \mathcal{B} \subset \mathcal{T}$,
Since $\mathcal{T}$ is top, the union of members of $\mathcal{B}$ is in $\mathcal{T}$.
On the other hand, let $O \in \mathcal{T}$. Then for each $x \in O$, $\exists\ B_x \in \mathcal{B} \subset \mathcal{T}$ s.t. $x \in B_x \subset O$. But then

$$O = \bigcup_{x \in O} B_x .$$

However, this rep^n need not be unique.

Notice that any subcollection $\mathcal{B} \subset P(X)$ is a basis if $\forall x \in X$, $\exists\ B \in \mathcal{B}$ st $x \in B$.
& if $x \in B_1 \cap B_2 \Rightarrow x \in B_3 \subset B_1 \cap B_2$.

Remark: Basis is an independent entity satisfies these two conditions. However it generates a top. via union of its members. (12)

Some time we need to go in the reverse direction. That is, to obtain a basis for a given topology.

Lemma: Let $(X, \mathcal{T})$ be a top. space. Let $\mathcal{C} \subset \mathcal{T}$ be such that for each open set $O \in \mathcal{T}$ & $\forall x \in O$, $\exists C \in \mathcal{C}$ s.t. $x \in C \subset O$. Then $\mathcal{C}$ is a basis for $\mathcal{T}$.

Pf: (1) Claim $\mathcal{C}$ is a basis. ~~for~~

Let $x \in X$, since X is open & $x \in X$, $\exists$ $C \in \mathcal{C}$ s.t. $x \in C \subset X$.

Let $G_1, G_2 \in \mathcal{G}$ & $x \in G_1 \cap G_2$.
Since G_1 & G_2 are open, by hypothesis, $\exists$ $G_3 \in \mathcal{G}$ s.t. $x \in G_3 \subset G_1 \cap G_2$.

Suppose $\tau' = \tau(\mathcal{G})$, the top. gen. by $\mathcal{G}$.

Claim $\tau = \tau'$.

If $O \in \tau$, then for each $x \in O$, $\exists$ $G_x \in \mathcal{G}$ s.t. $x \in G_x \subset O$.

$$\Rightarrow O = \bigcup_{x \in G} G_x.$$

Since $\mathcal{G}$ is a basis, it follows that $O \in \tau'$.

Conversely, if $w \in \tau'$, then

$$w = \bigcup_{i \in I} G_i, \text{ but then}$$

$w \in \tau$, as $G_i \in \tau$.

If we know the bases for some topologies, then it is useful to have criteria in terms of bases for

for comparing them.

<u>Lemma</u>: Let $\mathcal{B}$ & $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ & $\mathcal{T}'$ resp. on X. Then TFAE: (14)

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$ ($\mathcal{T}' \supset \mathcal{T}$)

(ii) for each $x \in X$ & each basis element $B \in \mathcal{B}$ containing x, $\exists$ $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

(finer top. has smaller size open sets)

<u>(ii) ⇒ (i)</u>: claim $\mathcal{T} \subset \mathcal{T}'$. That is, given

$O \in \mathcal{T} \Rightarrow O \in \mathcal{T}'$.

Let $x \in O$, then $\exists$ $B \in \mathcal{B}$ st $x \in B \subset O$.

By (ii), $\exists$ $B' \in \mathcal{B}'$ st $x \in B' \subset B \subset O$.

$\Rightarrow \forall x \in O, \exists B \in \mathcal{B}$ st $x \in B \subset O$.

Thus, $O \in \mathcal{T}$.

(i) $\Rightarrow$ (ii):

Let $x \in X$ & $B \in \mathcal{B}$ be such that $x \in B$. Since $B \in \tau$, by defn; & $\tau \subseteq \tau'$ by condition (i),

$\Rightarrow B \in \tau'$

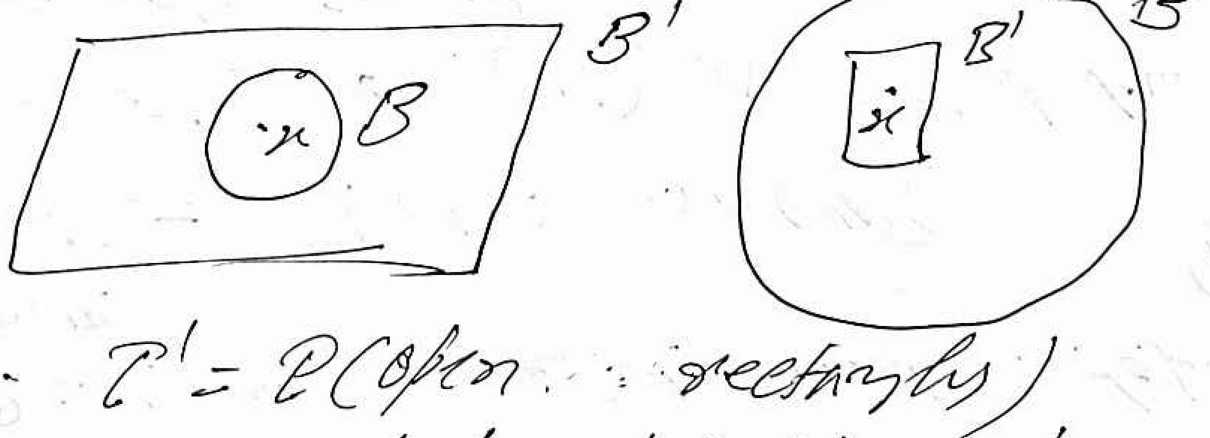
Since τ' is generated by $\mathcal{B}'$,

$\exists\, B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

(as $B = \bigcup_{x \in B} B'_x \in \tau'$).

(15)

Now, it follows from the above lemma that top. gen. by open discs & open rectangles are same



i.e. $\tau' = \tau$(open rectangles)

& $\tau = \tau$(open discs) $\Rightarrow \tau' = \tau$.

Ex. Let $\mathcal{B} = \{(a,b) : a,b \in \mathbb{R}\}$.
Then $\tau = \tau(\mathcal{B})$ is called standard top. on $\mathbb{R}$.

Let $\mathcal{B}' = \{[a,b) : a,b \in \mathbb{R}\}$.
Then $\tau' = \tau(\mathcal{B}')$ is called lower limit topo. on $\mathbb{R}$.

Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$, let

$$\mathcal{B}'' = \{(a,b) : a,b \in \mathbb{R}\} \cup \{(c,d) \setminus K : c,d \in \mathbb{R}\}$$
$$= \mathcal{B} \cup \{(c,d) \setminus K : c,d \in \mathbb{R}\}.$$

The top $\tau'' = \tau(\mathcal{B}'')$ is called K-top. on $\mathbb{R}$.

It is easy to see that $\mathcal{B}, \mathcal{B}'$ & $\mathcal{B}''$ are bases.

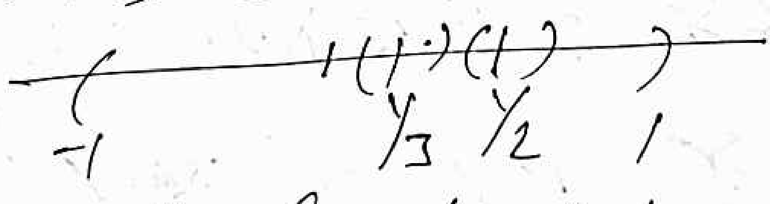
Ex. τ' & τ'' are strictly finer than τ, but τ' & τ'' are not comparable.

Given basis element $(a,b) \in \tau$ &
$x \in (a,b) \Rightarrow [x,b) \in \tau'$
$\Rightarrow \tau' \subseteq \tau$

On the other hand, given $[x,d) \in \tau'$,
$\nexists$ $(a,b) \in \mathcal{B}$ st $x \in (a,b)$ & $(a,b) \subset [x,d)$.
$\Rightarrow \tau' \subsetneq \tau$.

Since $\mathcal{B}'' = \mathcal{B} \cup \{(c,d) \setminus K : c,d \in \mathbb{R}\}$
any $(a,b) \subset \mathcal{B}$ is a basis element for
$\mathcal{B}'' \Rightarrow \tau'' \subsetneq \tau$.

On the other hand, for basis element
for $B'' = (-1,1) \setminus K \in \tau'' = \tau(\mathcal{B}'')$, $0 \in B''$,
but $\nexists$ any open interval containg "0"
and contains $\mathcal{B}''$.



That is, a single open interval cannot
contains $\mathcal{B}''$.

Subbasis:

We know that the top. gen. by a basis is all arbitrary union of members from the basis. In fact $\mathcal{B}$ is a basis on X if (i) $\forall x \in X \Rightarrow \exists B \in \mathcal{B}$ s.t. $x \in B$

(ii) if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

now, question is can we generate a top. when (ii) condition is relaxed?

Def^n: $\mathcal{G} \in P(X)$ is called subbasis if (i) $\bigcup_{S \in \mathcal{G}} S = X$

(ii) $\mathcal{B} = \mathcal{B}(\mathcal{G})$ is a basis.

notice that top. τ gen. by $\mathcal{G}$ is

$\tau = \tau(\mathcal{B}(\mathcal{G}))$

i.e. $O \in \tau$ iff O is union of finite intersections of members from $\mathcal{G}$.

We need to check $\tau = \tau(\mathcal{B}(\mathcal{S}))$ is a topology on X. (19)

It is enough to check that $\mathcal{B}(\mathcal{S})$ is a basis.

i.e. $\mathcal{B}(\mathcal{S}) = \{\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S}\}$

= the collection of all finite intersections

Given $x \in X \Rightarrow x \in S \in \mathcal{S}$, and S is an element of $\mathcal{B}$.

To check 2nd condition, let

$B_1 = S_1 \cap \cdots \cap S_m$, $B_2 = S_1' \cap \cdots \cap S_n'$.

be two members in $\mathcal{B}(\mathcal{S})$.

Then for $x \in B_1 \cap B_2 = (S_1 \cap \cdots \cap S_m) \cap (S_1' \cap \cdots \cap S_n')$

a member of $\mathcal{B}(\mathcal{S})$.

Thus $\mathcal{B} = \mathcal{B}(\tau)$ is a basis.

In a sharp contrast to "basis", we can see that <u>any</u> sub-collection $\mathcal{S} \subset P(X)$ generate a topology.

This follows by the fact that every

sub collection (need not contain ϕ & X) generates a basis, and hence a topology.

If $\mathcal{S} \subset P(X)$ is empty, then (20)

$$\Phi = \bigcup_{S_i \in \Phi} S_i \quad \& \quad X = \bigcap_{S_i \in \Phi} S_i \quad (\because \mathcal{S} = \Phi)$$

(Notice that intersection is larger when sets are fewer)

Thus $\mathcal{S} = \Phi$ generates only ϕ & X. Hence $\tau = \tau(\mathcal{B}(\Phi)) = \{\phi, X\}$.

If $\mathcal{S} \neq \Phi$. Let $\mathcal{B}$ be the collection of all finite intersections of members from $\mathcal{S}$. Then $\mathcal{B}$ is a basis.

It is clear that $X \in \mathcal{B}$. Hence, $\forall x \in X$ is in some member of $\mathcal{B}$.

Let $x \in B_1 \cap B_2$, $B_1, B_2 \in \mathcal{B}$

then $x \in B_1 \cap B_2 =$ finite intersection.

$\Rightarrow B_1 \cap B_2 \in \mathcal{B}$.

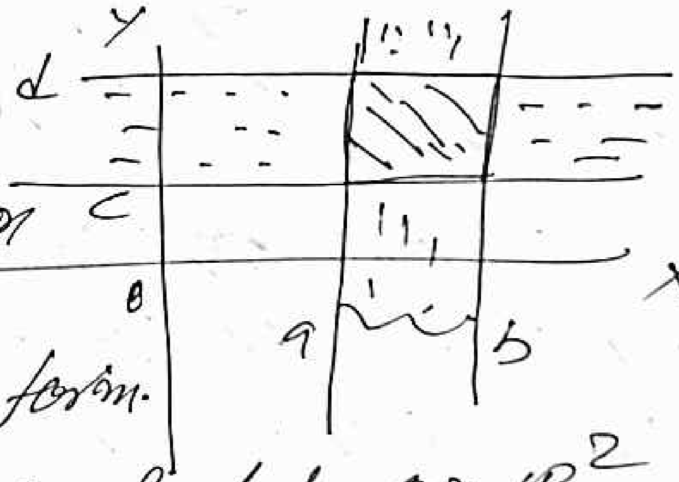
Thus, $\mathcal{B}$ is a basis

Ex. We know that $\mathcal{B} = \{(a,b) : a,b \in \mathbb{R}\}$ is a basis for usual top. on $\mathbb{R}$.

$\mathcal{S} = \{(a, +\infty), (-\infty, b) : a,b \in \mathbb{R}\}$ is an open sub-basis for usual top. $\mathcal{U}$ on $\mathbb{R}$.

Ex. We know that open rectangles are basis for usual top on $\mathbb{R}^2$.

But any open rectangle can be consider as intersection of two strips.



Thus, open strips form a subbasis for usual top. on $\mathbb{R}^2$.

$$(a,b) \times (c,d) = ((a,b) \times \mathbb{R}) \cap (\mathbb{R} \times (c,d)).$$

$\mathcal{S} = \{(a,b) \times \mathbb{R} : a,b \in \mathbb{R}\} \cup \{\mathbb{R} \times (c,d) : c,d \in \mathbb{R}\}$ is a subbasis for $(\mathbb{R}^2, \mathcal{U})$.

Product topology on $X \times Y$:

There is two (common) ways to create topologies from a given topology. One is subspace topology as intersection of open sets with small set, and other is through Cartesian product. We first discuss the later one.

Let $(X, \mathcal{T}_X)$ & $(Y, \mathcal{T}_Y)$ be two topological spaces, and let

$$\mathcal{B} = \mathcal{B}(\mathcal{T}_X \times \mathcal{T}_Y)$$
$$= \{ O \times W : O \in \mathcal{T}_X, W \in \mathcal{T}_Y \}.$$

We can see that $\mathcal{B}$ is a basis on X.

(i) $\forall (x,y) \in X \times Y \Rightarrow x \in O$ & $y \in W$

$\Rightarrow (x,y) \in O \times W$.

(ii) $(O_1 \times W_1) \cap (O_2 \times W_2) = \underset{\in \mathcal{B}}{(O_1 \cap O_2) \times (W_1 \times W_2)}$.

$\Rightarrow \mathcal{B}$ is a basis.

The top. generated by $\mathcal{B}$ on $X \times Y$ is called product top. on $X \times Y$. (23)

It is somehow useful to express product top. in terms of sub-basis.

$$\left\{ \begin{array}{l} \pi_1 : X \times Y \longrightarrow X \, ; \, \pi_1(x, y) = x \\ \pi_2 : X \times Y \longrightarrow Y \, ; \, \pi_2(x, y) = y \end{array} \right\}$$

are called projections of $X \times Y$ onto X & Y resp.

Notice that the maps are onto unless one of X or Y is empty. In that case $X \times Y$ is empty.

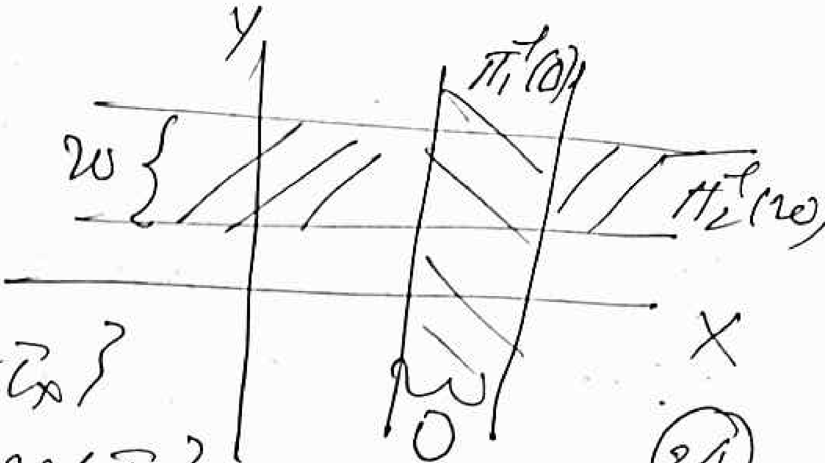
Observe that

$$\pi_1^{-1}(O) = O \times Y, \quad \forall \, O \in \mathcal{T}_X$$

$$\& \; \pi_2^{-1}(W) = X \times W, \quad \forall \, W \in \mathcal{T}_Y.$$

Also, $\pi_1^{-1}(O) \cap \pi_2^{-1}(W) = O \times W$.

Theorem:



(24)

Let

$\mathcal{J} = \{\pi_1^{-1}(O) : O \in \tau_X\}$
$\quad \cup \{\pi_2^{-1}(W) : W \in \tau_Y\}$,

Then $\mathcal{J}$ is a subbasis for the product topology on $X \times Y$.

Pf: Let τ denote the product topology on $X \times Y$, let $\tau' = \tau(\mathcal{B}(\mathcal{J}))$.

Since $\mathcal{J} \subset \tau$

$\Rightarrow \tau' = \tau(\mathcal{B}(\mathcal{J})) \subset \tau$.

On the other hand, every basis open set $O \times W = \pi_1^{-1}(O) \cap \pi_2^{-1}(W)$

= finite intersection of members from $\mathcal{J}$.

$\Rightarrow O \times W \in \tau'$.

$\Rightarrow \tau(\{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}) \subset \tau'$

(25)

$\Rightarrow \tau = \tau' = \tau(\mathcal{B}(\mathcal{J}))$.

Ex. Show that projection are open maps, sending open set to open set.

Closed set

A set A in a topological space X is said to be closed if its complement $X \setminus A$ is open.

Ex $\mathbb{R} \setminus [a,b] = (-\infty, a) \cup (b, +\infty)$ - open

Ex. In the co-finite topological space X, the closed sets are X & all finite sets.

Ex. In the discrete topological space, every set is open + closed.

Ex. Let (X, τ) be a top space.
Then (i) ϕ & X are closed sets
(ii) finite union of closed sets is closed
(iii) arbitrary intersection of closed set is closed.

Subspace topology:

Let Y be a subset of top. space (X, τ). Define

$$\tau_Y = \{O \cap Y : O \in \tau\}$$

Then it is easy to see that τ_Y is a top. on Y. This is known as subspace topology.

Remark: Every subset not just create new topology, many properties of topology τ is inherited to τ_Y.

This large top-space event/result can transfer to small top; will be convenient way of dealing with large topological space. (27)

However, some features of parent top. may fail to happen on small top-space or sections.

Ex. Let $Y = [0,1] \cup (2,3)$

(i) $[0,1] = (-\frac{1}{2}, \frac{3}{2}) \cap ([0,1] \cup (2,3))$

$\Rightarrow$ $[0,1]$ is open in subspace top. $(Y, \mathcal{T}_Y)$.

Similarly, $(2,3)$ is open in Y, in fact open in $\mathbb{R}$.

Since $[0,1]$ & $(2,3)$ are complements of each other, both of them are open & closed in Y.

(note complement is taken in Y)

Ex. Let Y be a subspace of top. space X. Then A is closed in Y iff $A = C \cap Y$ for some closed set C in X. (28)

Let A be closed in Y. Then

$Y - A$ is open in Y

$\Rightarrow Y \setminus A = Y \cap O$, O - open in X

Since $X \setminus O$ is closed in X,

$$A = Y \setminus Y \cap O$$
$$= Y \cap (X \setminus O)$$
$$= Y \cap C, \quad C = X \setminus O.$$

On the other hand, let

$A = Y \cap C$, C - is closed in X.

Since $X \setminus C$ is open in X,

$\Rightarrow (X \setminus C) \cap Y$ is open in Y

$$= Y \setminus C \cap Y$$
$$= Y \setminus A$$

$\Rightarrow$ A is closed in Y.

Closure of a set:

Closure of a set A in top. space X is the smallest closed set containing A.

i.e. $\bar{A} = \bigcap \{F : F \text{ closed } \& \ F \supseteq A\}$.

Theorem: Let Y be a subspace of X. Let $A \subset Y$ & $\bar{A}$ be the closure of A in X. Then closure of A in Y is equal to $\bar{A} \cap Y$.

Pf: Let $B = \overline{A}|_Y$ (closure of A in Y).

Since $\bar{A}$ is closed in X,

$\bar{A} \cap Y$ is closed in Y

By def^n of closure,

$$B \subset \bar{A} \cap Y$$

On the other hand, B is closed in Y, so $\exists$ closed set C in X s.t.

$B = C \cap Y$

$\Rightarrow A \subset C$ ($\because B = \overline{A/_Y}$)

$\Rightarrow \bar{A} \subset C$

($\because \bar{A}$ is the intersection of all closed set containg A)

$\Rightarrow \bar{A} \cap Y \subset C \cap Y = B$

$\Rightarrow \bar{A} \cap Y = \overline{A/_Y}$.

Interior of a set:

Interior of a set A in a top. space X is the largest open set A^o contained in A.

i.e. $A^o = \cup \{O : O\text{-open}, O \subset A\}$.

Ex. $Q^o = \emptyset$ in $(\mathbb{R}, U)$

Ex. Interior of Cantor set is empty.

Ex. $(\{1\} \cup [0,1))^o = (0,1)$, etc.

We know that considering intersection of all closed sets & union all open sets, etc is a huge process, and does not give a specified way to get closure & interior.

The following result dealt with the closure of a set in terms of basic sets.

<u>Theorem</u>: Let A be a subset of a top. space $(X, \mathcal{T})$. Then

(i) $x \in \bar{A}$ iff $\forall$ open set $O \ni x$ $\Rightarrow O \cap A \neq \Phi$.

(ii) Let $\mathcal{T} = \mathcal{T}(\mathcal{B})$, $\mathcal{B}$ basis for $\mathcal{T}$. Then $x \in \bar{A}$ iff $\forall$ $B \in \mathcal{B}$ & $x \in B$, $\Rightarrow B \cap A \neq \Phi$

(i.e. (i) is true when O is replaced by B).

Proof: (i) $x \notin \bar{A}$ iff $\exists$ an open set (32)
$O \ni x$ s.t. $O \cap A = \Phi$.

If $x \notin \bar{A}$, then $x \in O = X \setminus \bar{A}$ & $O \cap A = \Phi$.

Conversely, if $\exists$ open set $O \ni x$ s.t.
$O \cap A = \Phi$, then
$A \subset X \setminus O$ - closed
$\Rightarrow \bar{A} \subset X \setminus O$ (by defn of closure)
$\Rightarrow x \notin \bar{A}$.

(ii) If every open set containing x intersect A, then so does every $B \in \mathcal{B}$ containing x.

Conversely, let every $B \ni x \Rightarrow$
$B \cap A \neq \Phi$

Let O be open set & $x \in O$,
then $\exists$ $B \in \mathcal{B}$ containing x s.t.
$x \in B \subset O \Rightarrow O \cap A \neq \Phi$.
($\because \mathcal{T} = \mathcal{T}(\mathcal{B})$)

Ex. Let A be a subset of metric space X. Show that $x \in \bar{A}$ iff
$\forall r > 0, B_r(x) \cap A \neq \Phi$.
(Hint: If $x \in \bar{A} \Rightarrow \exists x_n \in A$ s.t. $x_n \to x$
$\Rightarrow x_n \in B_\epsilon(x)$ for $n \geq N$
$\Rightarrow A \cap B_\epsilon(x) \neq \Phi \quad \forall \epsilon > 0$)

We relax the condition "open $O \ni x$, intersect A" to "neighbourhood of x".

In general, we consider open nhd, unless specified.

Cor: If $A \subset X$ (top. space), then $x \in \bar{A}$ iff every nhd of x intersects A.

Limit point:

Let $A \subset X$ (top. space). A pt $x \in X$ is called limit pt of A if every nhd containing x contains at least one point of A other than x.

i.e. $N_x \cap A \setminus \{x\} \neq \Phi$.

The set of all limit pts of A is denoted by A' or $D(A)$.

Ex. $x \in A'$ iff $x \in \overline{A \setminus \{x\}}$.

(Hint: if $x \in A'$, then $\forall N_x \cap A \setminus \{x\} \neq \Phi$
$\Rightarrow x \in \overline{A \setminus \{x\}}$, etc)

<u>Theorem</u>: Let $A \subset X$ (top. space).

Then $\overline{A} = A \cup A'$

Pf: If $x \in A'$, then $\forall N_x \cap (A \setminus \{x\}) \neq \Phi$

$\Rightarrow$ each $N_x \cap A \supset N_x \cap (A \setminus \{x\}) \neq \Phi$

$\Rightarrow x \in \overline{A}$.

$\Rightarrow A \cup A' \subset \overline{A}$.

On the other hand, if $x \in \overline{A}$, claim $x \in A \cup A'$.

If $x \in A$, then trivial.

Let $x \notin A$, but $x \in \overline{A}$

$\Rightarrow \forall\ N_x \cap A \neq \Phi$. Since $x \notin A$

$\Rightarrow N_x \cap (A \setminus \{x\}) \neq \Phi$

$\Rightarrow x \in A'$.

Thus $\bar{A} \subset A \cup A'$

i.e. $\bar{A} = A \cup A'$.

Ex. If $A, B \subset X$ (top. space), then show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$

Notice that $A \cup B \subset \bar{A} \cup \bar{B}$.

Since $\bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$, it follows that

$$\overline{A \cup B} \subset \bar{A} \cup \bar{B}.$$

Also, $A \subset A \cup B \subset \overline{A \cup B}$

$\Rightarrow \bar{A} \subset \overline{A \cup B}$

$\Rightarrow \bar{A} \cup \bar{B} \subset \overline{A \cup B}$

$\Rightarrow \bar{A} \cup \bar{B} = \overline{A \cup B}$.

However, the inclusion

$$\overline{A \cap B} \subsetneq \bar{A} \cap \bar{B}$$

may be strict. e.g.

$A = [0,1)$, $B = [1,2]$

$\Rightarrow \bar{\phi} \subset \{1\}$

Notice that the rep^n $\bar{A} = A \cup A'$ need not disjoint. However, we can define a new set, called isolated set to get a disjoint rep^n.

Def^n: Let $A \subset X$ (top. space). A point $x \in A$ is called isolated pt of A if $\exists$ a nbd N_x which intersect A only at x.

ie $N_x \cap A = \{x\}$.

Set of all isolated pts is denoted by iso(A).

<u>Theorem</u>: let X be a top. space, and $A \subset X$. Then $\bar{A} = \text{iso}(A) \cup A'$

= disjoint union

Pf: Let $x \in \bar{A}$. Then $\forall N_x \cap A \neq \Phi$.
This implies two choices for a given nbd N_x. (37)

(i) $N_x \cap A = \{x\}$ or
(ii) $N_x \cap A = \{x, \ldots\}$ } —(*)

Hence, (i) $\Rightarrow x \in iso(A)$
& (ii) $\Rightarrow N_x \cap A \setminus \{x\} \neq \Phi$
$\Rightarrow x \in A'$

Thus $\bar{A} \subseteq iso(A) \dot{\cup} A'$,
other inclusion is trivial.
Thus $\bar{A} = iso(A) \dot{\cup} A'$.
(Notice that disjointness is clear from (*))

Notice that $Int(A)$ is an open set, hence $x \in Int(A)$ iff $\exists$ an (open) nbd N_x of x s.t. $N_x \subset A$.

The boundary of a set A is defined by $b(A) = \bar{A} \cap \overline{A^c}$.

It is clear that $b(A)$ is closed, and $x \in X$ is a pt in $b(A)$ iff every nbd N_x of x intersect both A & A^c.

Ex. $b(\mathbb{N}) = \overline{\mathbb{N}} \cap \overline{\mathbb{N}^c} = \mathbb{N} \cap \mathbb{R} = \mathbb{N}$

Ex. $b(\mathbb{Q}) = \overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^c} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

(So the boundary of a set can be larger than the set itself)

Ex. $b(\{0\}) = \overline{\{0\}} \cap \overline{\{\mathbb{R} \setminus \{0\}\}} = \{0\} \cap \mathbb{R} = \{0\}$

Ex $b\{(0,1)\} = \{0,1\}$

Ex. Show that $A \subset X$ is closed iff $A \supset b(A)$.

Ex. Show that $b(A) = \Phi$ iff A is both closed & open.

Ex. If $A \subset X$ (top-space), then $\overline{A} = A^{o} \cup b(A)$.

Defⁿ: A set $A \subset X$ (top. space) is said to be perfect if $A = A'$. (39)

Ex: Cantor's set is a perfect set

Ex. $A = [0,1] = [0,1]'$ in $(\mathbb{R}, \mathcal{U})$.
(as any nbd of each pt intersect A other than the pt.)

Defⁿ: A set $A \subset X$ (top. space) is said to be dense if $\bar{A} = X$
i.e. every pt $x \in X$ has nbd N_x which intersect A. i.e. $N_x \cap A \neq \Phi$

Ex. Show that
$$Int(A^c) = (\bar{A})^c.$$

Defⁿ: A subset A of top. space X is said to be nowhere dense if
$$(\bar{A})^0 = \Phi$$
(i.e. closure has no interior)

Ex. Cantor's set is nowhere dense in $[0,1]$. $(\bar{C})^0 = C^0 = \Phi$.

Ex. Let $A \subset X$ (Top. space) be closed. Then A is nowhere dense iff $\overline{A^c} = X$. (40)

Ex. Show that boundary of a closed set is nowhere dense. Is this true for an arbitrary set?

Ex. If $A \subset X$ (metric space). Then $\overline{A}$ is the set of pts of X which are a zero distance from A.

i.e. $\overline{A} = \{x \in X : d(x, A) = 0\}$.

And $b(A) = \{x \in X : d(x, A) = 0$ & $d(x, X \setminus A) = 0\}$.

Ex. $b(A) \cup (X \setminus A) = \overline{X \setminus A}$, and

$X \setminus A^0 = \overline{X \setminus A}$

(Hint: $x \in b(A) \Rightarrow \forall N_x \cap A \neq \phi$ & $N_x \cap X \setminus A \neq \phi$ $\Rightarrow x \in \overline{X \setminus A}$.)

Ex. Let $b(A) = \overline{A} \cap \overline{A^c}$. Then

(i) $\overline{A} = A \cup b(A)$

(ii) $A^{\circ} = A \setminus b(A)$

(iii) $X = A^{\circ} \cup b(A) \cup (A^{c})^{\circ}$.

In general, a basis is useful only if its sets are simple or few in number.

For instance, a space which has a countable basis, has many pleasant properties, and such space is known second countable.

A central fact about 2nd countable space is a follows:

Thm: Let (X, τ) be a 2nd countable top. space. If a non-empty open set is represented by

$$O = \bigcup_{i \in I} O_i \,;\; O_i \in \tau,$$

then $O = \bigcup_{i=1}^{\infty} O_i$

= countable union.

This is known as Lindelöf's theorem.

Proof: Let $\{B_n\}_{n=1}^{\infty}$ be a countable base for top. τ.

Let $x \in O$, then $x \in O_i$ for some $i \in I$.

Then (by def^n of basic top.), $\exists$ B_n s.t.

$$x \in B_n \subset O_i \subset O$$

$$\Rightarrow O = \bigcup_{i=1}^{\infty} O_i$$

(we need as many O_i as many B_n)

Cor: Let X be a 2nd countable top. space. Then any can be reduce to countable base.

Pf: Let $\{B_n\}_{n=1}^{\infty}$ be a countable base for (X, τ), and $\{B_i : i \in I\}$ be a base for (X, τ).

Since each B_n is union of B_i's

$\Rightarrow$ B_n is union of countably many B_i

i.e. $B_n = \bigcup_{i=1}^{\infty} B_i$.

In this way we obtain a countable family of countable union of B_i's, and this family is a base for τ. (43)

Cor: If top-space X has countable base $\{B_n\}$, then it has also a countable dense subset.

Pf: Let $\{x_n \in B_n : n = 1, 2, \ldots\} = A$.
Then A is countable. & $\forall x \in X$,
$\exists B_n$ s.t. $x \in B_n$ & $x_n \in B_n \cap A \neq \Phi$.
Hence $\bar{A} = X$.

Def^n: A topological space X is said to be separable if X has a countable dense set

Ex. If (X, d) is metric space, it is separable if $\exists$ countable set $A \subset X$ s.t. $\bigcup_{x_i \in A} B_\epsilon(x_i) = X$, $\forall \epsilon > 0$.
i.e. $\forall \epsilon > 0$, $\exists$ $B_\epsilon(x_i)$'s s.t. $X = \bigcup_{i=1}^{\infty} B_\epsilon(x_i)$.

i.e. (X,d) is separable if it is patched by countably many open balls of arbitrarily small radius.

<u>Theorem</u>: Every separable metric space is 2nd countable.

pf: Let X be a separable metric space & A be a countable dense set in X.

Let $\mathcal{B} = \{ B_{r_i}(x_i) : r_i \in \mathbb{Q}, x_i \in A \}$

claim $\mathcal{B}$ is a base for $(X, \mathcal{T}_d)$.

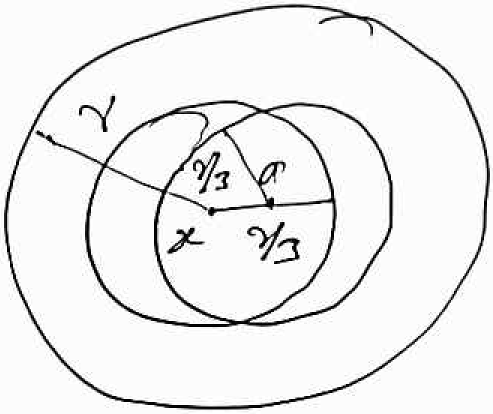
Let O be a non-empty open set & $x \in O$.

We need to find an open sphere $B_{r_i}(x_i)$ s.t. $x \in B_{r_i}(x_i) \subset O$.

Let $S_r(x) \subset O$. Since $\bar{A} = X$, $\exists\ a \in A$ s.t. $a \in S_{r/3}(x) \Rightarrow x \in S_{r/3}(a)$

Let r_1 be a rational no. s.t.
$r/3 < r_1 < 2r/3$.

Then $B_{r_1}(a) \subset S_r(x) \subseteq O$.
Thus $\mathcal{B}$ is a basis for $\mathcal{T}$, which is countable.



Question: (i) Is it possible to find a sub-basis from a basis?
(ii) Is every separable top. space is 2nd countable?