Department of Mathematics
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Complex Analysis

Lecture Notes

MA547

Holomorphic functions, contour integration, and conformal methods

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Undergraduate lecture notes

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Contents

Introduction

These notes provide a rigorous graduate-level introduction to classical complex analysis, guided by two complementary aims: conceptual clarity and mathematical precision. The exposition emphasizes the remarkable rigidity of holomorphic functions and develops the subject in a syllabus-aligned progression from algebraic and topological foundations to Cauchy theory, analytic continuation, conformal mapping, singularities, residues, and applications to definite integrals.

Complex analysis is one of the central achievements of classical mathematics. It begins with the seemingly modest extension of the real field by a square root of -1, yet this algebraic enlargement produces a theory of functions whose local regularity is dramatically stronger than anything familiar from ordinary real-variable calculus. A single complex derivative forces a function to satisfy the Cauchy–Riemann equations, to be infinitely differentiable, to admit local power-series expansions, and to obey global principles that connect interior behavior with boundary information. The subject therefore combines algebra, geometry, topology, and analysis in a remarkably tight and elegant way.

A second unifying theme of the course is the interplay between local structure and global consequence. Local differentiability leads to integral formulas; integral formulas lead to power-series representations; power-series representations lead to uniqueness theorems, maximum principles, and analytic continuation; and these, in turn, culminate in the geometric theory of conformal maps and the meromorphic theory of zeros, poles, and residues. The aim of these notes is to make that logical architecture as transparent as possible while maintaining the level of precision and seriousness expected in a graduate treatment.

Prerequisites. The reader is assumed to be comfortable with core real analysis, elementary point-set topology, multivariable differentiation, and line integrals. Familiarity with compactness, connectedness, continuity, sequences, and basic differential calculus in $\mathbb{R}^2$ is used throughout without repeated review.

Standing conventions. Unless stated otherwise, Ω denotes an open subset of $\mathbb{C}$, and a *domain* means a connected open set. Closed disks are written $\overline{D(z_0, r)}$, positively oriented contours are traversed counterclockwise, and the notation $\mathrm{Hol}(\Omega)$ refers to the class of holomorphic functions on Ω. When a statement is first proved under additional smoothness or geometric hypotheses, the later goal is usually to identify the exact form in which those hypotheses can be weakened without changing the conclusion.

Syllabus-aligned organization. The notes follow a direct chapter-by-chapter progression. After the algebra and geometry of complex numbers, we develop the topology of the complex plane and the differential theory of holomorphic functions in a systematic order. We then introduce the elementary transcendental functions, complex integration, and Cauchy theory, followed by power series and analytic continuation. With this analytic core established, the exposition turns to harmonic functions, conformal mappings, Schwarz's lemma, and automorphisms of the disk, and concludes with Laurent theory, isolated singularities, residue calculus, and applications to definite and principal-value integrals.

How to use these notes. Definitions and structural lemmas precede the main theorems of each chapter, and chapter abstracts signal the conceptual role of the material before the technical development begins. Wherever possible, the proofs have been arranged so that the strategic idea is visible before the computational details. The problem sets at the end are designed not merely as routine exercises but as extensions of the exposition: they move from foundational checks to synthesis problems that connect several chapters at once. A bibliography is included at the end of the notes for further reading, historical perspective, and alternative treatments of the main topics.

Chapter 1

Complex Numbers

This opening chapter establishes the algebraic and geometric language of the complex plane. We develop the modulus, conjugation, argument, and polar representation, and we emphasize the geometric meaning of complex multiplication as the foundational mechanism behind later analytic phenomena.

Learning objectives.

- Understand the algebraic and geometric models of $\mathbb{C}$ and move fluently between rectangular and polar coordinates.
- Use modulus, conjugation, and argument to derive estimates that later become analytic tools.
- Recognize the first rigid geometric features of complex multiplication.

Chapter roadmap.

- We begin with the field structure of $\mathbb{C}$ and its Euclidean geometry.
- We then isolate the quantitative estimates—especially the triangle inequality and polar representation—that recur throughout the course.
- The chapter closes by preparing the language of argument and elementary geometry needed for logarithms, power series, and contour integration.

1.1 Complex Number System

We begin with the algebraic enlargement of $\mathbb{R}$ that makes the theory possible. The elementary notions introduced here—especially modulus, conjugation, and polar representation—already encode the Euclidean geometry of the plane and provide the estimates that will recur throughout the subject.

The quadratic equation $x^2 + 1 = 0$ has no solution in $\mathbb{R}$. We adjoin a symbol i satisfying

$$i^2 = -1,$$

and define the *complex numbers* to be expressions of the form

$$z = x + iy, \qquad x, y \in \mathbb{R}.$$

The set of all complex numbers is denoted by

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\},$$

and becomes a field under the usual addition and multiplication.

It is convenient to identify $z = x + iy$ with the point $(x, y) \in \mathbb{R}^2$, and to use the matrix model

$$x + iy \longleftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

under which multiplication of complex numbers corresponds to matrix multiplication.

For $z = x + iy$, we write $\Re z = x$ and $\Im z = y$. The *complex conjugate* of z is the reflection across the real axis,

$$\overline{z} := x - iy.$$

The *modulus* of z is

$$|z| := \sqrt{x^2 + y^2}.$$

These notions satisfy the elementary identities

$$\Re z = \tfrac{1}{2}(z + \overline{z}), \qquad \Im z = \tfrac{1}{2i}(z - \overline{z}), \qquad z\,\overline{z} = |z|^2,$$

and the conjugation rules $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$ and $\overline{z_1 z_2} = \overline{z}_1\,\overline{z}_2$.

If $z = x + iy \neq 0$, then

$$\det \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x^2 + y^2 \neq 0,$$

hence the inverse matrix exists and one obtains the familiar formula

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{|z|^2}.$$

Triangle inequality and related estimates

For $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)\,\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) \\ &= |z_1|^2 + |z_2|^2 + z_1\overline{z}_2 + \overline{z}_1 z_2 = |z_1|^2 + |z_2|^2 + 2\Re(z_1\overline{z}_2) \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2, \end{aligned}$$

which yields the *triangle inequality*

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

A standard consequence is the *reverse triangle inequality*

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2|.$$

In particular, the map $z \mapsto |z|$ is uniformly continuous on $\mathbb{C}$.

Polar representation

A point on the unit circle may be parametrized by

$$\cos\theta + i\sin\theta, \qquad \theta \in [0, 2\pi).$$

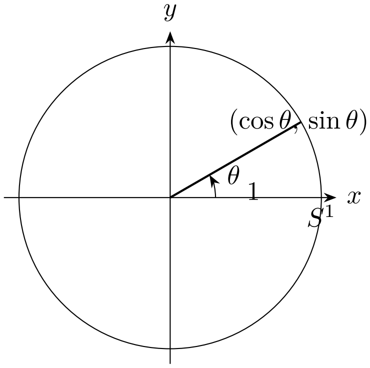

Figure 1.1: *Unit circle and the polar angle θ.*

If $z \neq 0$, then $\left|\frac{z}{|z|}\right| = 1$, so $\frac{z}{|z|}$ lies on the unit circle and hence

$$\frac{z}{|z|} = \cos\theta + i\sin\theta \qquad \text{for some } \theta \in [0, 2\pi).$$

Writing $r := |z| > 0$, we obtain the *polar form*

$$z = r(\cos\theta + i\sin\theta).$$

The angle θ is called an *argument* of z; the systematic discussion of arguments is taken up in the next section.

1.2 Argument of a Complex Number

> *The passage from rectangular to polar form reveals that multiplication in $\mathbb{C}$ combines dilation with rotation. The notion of argument records this angular information and prepares the way for roots, powers, branches of the logarithm, and the topological phenomena that appear later.*

Let $z \in \mathbb{C} \setminus \{0\}$. If

$$z = re^{i\theta} \qquad (r = |z| > 0),$$

then every number of the form $\theta + 2\pi k$ with $k \in \mathbb{Z}$ determines the same point z. This motivates the *argument set*

$$\arg(z) := \{\theta + 2\pi k : \ z = re^{i\theta}, \ k \in \mathbb{Z}\}.$$

Thus $\arg(z)$ is inherently *multivalued.*

To obtain a single-valued choice, one usually selects the *principal argument*

$$\operatorname{Arg}(z) \in (-\pi, \pi],$$

and then writes

$$\arg(z) = \operatorname{Arg}(z) + 2\pi k, \qquad k \in \mathbb{Z}.$$

Example 1.1. For $z = i$, one has

$$\arg(i) = \left\{ \frac{\pi}{2} + 2\pi k : \ k \in \mathbb{Z} \right\}, \qquad \operatorname{Arg}(i) = \frac{\pi}{2}.$$

The argument behaves naturally under multiplication. If

$$z_1 = r_1 e^{i\theta_1}, \qquad z_2 = r_2 e^{i\theta_2},$$

then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}. \tag{1.1}$$

Consequently,

$$|z_1 z_2| = |z_1|\,|z_2|,$$

and, for nonzero z_1, z_2,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2),$$

where the equality is understood modulo integer multiples of 2π.

More generally, if $z_j = r_j e^{i\theta_j}$ for $j = 1, \ldots, n$, then

$$z_1 \cdots z_n = r_1 \cdots r_n \, e^{i(\theta_1 + \cdots + \theta_n)}.$$

In particular, if $z = re^{i\theta}$, then

$$z^n = r^n e^{in\theta}.$$

Equivalently,

$$z^n = \big[r(\cos\theta + i\sin\theta)\big]^n = r^n(\cos n\theta + i\sin n\theta), \tag{1.2}$$

which yields *de Moivre's formula*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

We now describe the solutions of the equation $z^n = z_0$ for a given $z_0 \in \mathbb{C}$.

Proposition 1.2 (nth roots)**.** *Let $z_0 \neq 0$ and write*

$$z_0 = |z_0|(\cos\alpha + i\sin\alpha).$$

Then the solutions of $z^n = z_0$ *are precisely*

$$|z_0|^{1/n}\left(\cos\frac{\alpha+2\pi k}{n} + i\sin\frac{\alpha+2\pi k}{n}\right), \qquad k = 0, 1, \ldots, n-1.$$

Hence every nonzero complex number has exactly n *distinct* n*th roots.*

Proof. If $z^n = z_0$ and $z = \rho e^{i\theta}$, then

$$\rho^n = |z_0|, \qquad n\theta \in \arg(z_0),$$

so $\rho = |z_0|^{1/n}$ and

$$\theta = \frac{\alpha+2\pi k}{n}$$

for some integer k. Distinct values of k modulo n give distinct roots, and each number listed above indeed satisfies $z^n = z_0$. ☐

Example 1.3 (Roots of unity)**.** Taking $z_0 = 1 = \cos 0 + i\sin 0$, we obtain the nth roots of unity:

$$e^{2\pi ik/n} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \qquad k = 0, 1, \ldots, n-1.$$

These points are equally spaced on the unit circle.

CHAPTER 2

Topology of the Complex Plane

Complex analysis is built on a precise topological understanding of subsets of $\mathbb{C}$. This chapter develops the notions of neighborhood, closure, boundary, connectedness, and compactness that underlie the definitions of domain, continuity, path connectedness, and later existence theorems in the subject.

Learning objectives.

- Formulate openness, closedness, compactness, and connectedness in the complex plane with complete precision.
- Translate between geometric intuition and topological definitions.
- Understand why domains are the natural ambient spaces for complex analysis.

Chapter roadmap.

- The first sections develop the local language of neighborhoods, closures, and boundary points.
- The middle of the chapter studies connectedness, path connectedness, and the polygonal geometry of open subsets of $\mathbb{C}$.
- We end with continuity and compactness results that will later support maximum principles and contour arguments.

2.1 Neighborhoods, Open Sets, and Closed Sets

Complex analysis is a local theory, and its fundamental definitions are therefore phrased in topological language. Before differentiability and integration can be treated rigorously, we must understand how neighborhoods, openness, and closedness describe the local and global structure of subsets of $\mathbb{C}$.

We recall the basic topological notions on $\mathbb{C}$ induced by the Euclidean metric $d(z, w) = |z - w|$. For $z_0 \in \mathbb{C}$ and $r > 0$, the *open disk* of radius r centered at z_0 is

$$B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

The set $B(z_0, r) \setminus \{z_0\}$ is called a *deleted neighborhood* of z_0.

Definition 2.1 (Open and closed sets)**.** A set $U \subset \mathbb{C}$ is *open* if for every $z \in U$ there exists $r > 0$ such that $B(z, r) \subset U$. A set $F \subset \mathbb{C}$ is *closed* if its complement $F^c := \mathbb{C} \setminus F$ is open.

Definition 2.2 (Limit points and derived set)**.** Let $S \subset \mathbb{C}$. A point $z \in \mathbb{C}$ is a *limit point* of S if every deleted neighborhood of z meets S, that is,

$$(B(z,r) \setminus \{z\}) \cap S \neq \varnothing \qquad \text{for every } r > 0.$$

The set of all limit points of S is denoted by S' and is called the *derived set* of S.

Definition 2.3 (Interior, closure, and boundary)**.** For $S \subset \mathbb{C}$, the *interior* of S is

$$S^\circ := \{z \in S : \text{ there exists } r > 0 \text{ with } B(z,r) \subset S\}.$$

The *closure* of S is

$$\overline{S} := S \cup S'.$$

The *boundary* of S is

$$\partial S := \overline{S} \setminus S^\circ.$$

Proposition 2.4. *A set $S \subset \mathbb{C}$ is closed if and only if S^c is open.*

Proof. By definition, S is closed precisely when S^c is open. □

Definition 2.5 (Components)**.** A *component* of a topological space X is a maximal connected subset of X (with respect to inclusion).

Example 2.6. The set

$$X = \{z \in \mathbb{C} : \ |z| < \tfrac{1}{2}\} \ \cup \ \{z \in \mathbb{C} : \ \tfrac{1}{2} < |z| < 1\}$$

has two components, namely $\{\, |z| < \tfrac{1}{2} \,\}$ and $\{\, \tfrac{1}{2} < |z| < 1 \,\}$.

Remark 2.7. Components are always closed, pairwise disjoint, and their union is the whole space.

2.2 Examples of Limit Points and Derived Sets

Abstract definitions acquire real force only when tested on representative examples. The aim of this section is to develop geometric intuition for accumulation phenomena, since limit points govern closure, compactness, and the behavior of zeros of analytic functions.

Example 2.8 (A basic sequence accumulating at 0)**.** Let

$$S := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subset \mathbb{R} \subset \mathbb{C}.$$

Then the only limit point of S is 0; equivalently, $S' = \{0\}$. In particular, S is not closed and $\overline{S} = S \cup \{0\}$.

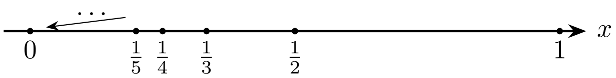

Figure 2.1: *The sequence* $(1/n)$ *accumulates at* 0.

Example 2.9 (A hyperbola has no new limit points at the origin)**.** Let

$$S := \left\{(x, \tfrac{1}{x}) : x \in \mathbb{R} \setminus \{0\}\right\} \subset \mathbb{R}^2 \cong \mathbb{C}.$$

Then every point of S is a limit point of S, and S has *no* additional limit points. More precisely,

$$S' = S \qquad \text{and hence} \qquad \overline{S} = S.$$

Indeed, the parametrization $x \mapsto (x, 1/x)$ is continuous on $(0, \infty)$ and on $(-\infty, 0)$, so points of S are non-isolated. On the other hand, near the origin the hyperbola stays outside a neighborhood of $(0, 0)$ (since $|1/x| \to \infty$ as $x \to 0$), so $(0, 0)$ is not a limit point.

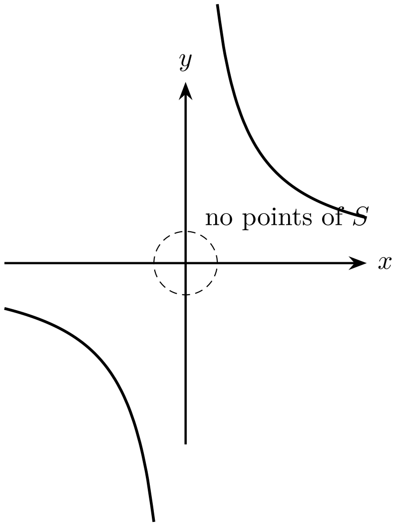

Figure 2.2: *The hyperbola* $y = 1/x$ *avoids a neighborhood of the origin.*

Example 2.10 (The topologist's sine curve (derived set))**.** Let

$$S := \left\{\left(x, \sin\frac{\pi}{x}\right) : x \in \mathbb{R},\ x \neq 0\right\} \subset \mathbb{R}^2.$$

Then

$$S' = S \cup \left(\{0\} \times [-1, 1]\right).$$

In particular, the closure $\overline{S}$ is obtained by adjoining the vertical segment $\{0\} \times [-1, 1]$ to the oscillating graph.

2.3 Closed Sets, Closure, and Boundary

> *Closure and boundary formalize the idea of adjoining the points that lie infinitesimally close to a set. These notions are indispensable in later arguments involving compactness, approximation, and the distinction between interior behavior and boundary behavior.*

Recall that a set $S \subset \mathbb{C}$ is *closed* if it contains all of its limit points; equivalently, $S' \subset S$. Since

$$\overline{S} = S \cup S',$$

this immediately yields the following characterization.

Proposition 2.11. *For a set $S \subset \mathbb{C}$, the following are equivalent:*

(i) *S is closed;*

(ii) *$S' \subset S$;*

(iii) *$S = \overline{S}$;*

(iv) *S^c is open.*

Proof. The equivalence of *(i)* and *(iv)* is exactly Theorem 2.4. If S is closed, then by definition $S' \subset S$, which is *(ii)*. Conversely, *(ii)* says precisely that S contains all of its limit points, so S is closed. Also, if S is closed then $\overline{S} = S \cup S' = S$, which yields *(iii)*. Conversely, if $S = \overline{S}$, then $S' \subset \overline{S} = S$, so S is closed. Hence all four conditions are equivalent. ☐

The boundary and interior describe how a set sits inside the ambient plane.

Proposition 2.12. *For every $S \subset \mathbb{C}$,*

$$\overline{S} = S^\circ \cup \partial S, \qquad S^\circ \cap \partial S = \varnothing.$$

In particular, the closure decomposes into the interior points of S together with its boundary points.

Proof. By definition, $\partial S = \overline{S} \setminus S^\circ$. Hence $\overline{S} = S^\circ \cup \partial S$, and the union is disjoint. ☐

Example 2.13. Let $S = \{z \in \mathbb{C} : |z| < 1\}$. Then $S^\circ = S$, $\overline{S} = \{z \in \mathbb{C} : |z| \leq 1\}$, and

$$\partial S = \{z \in \mathbb{C} : |z| = 1\}.$$

Thus the unit circle is precisely the boundary of the open unit disk.

2.4 Connected Sets

> *Many of the characteristic rigidity theorems of complex analysis require the underlying set to be in one piece. Connectedness is the topological condition that prevents decomposition into disjoint open parts and thereby underlies the notion of a domain.*

Definition 2.14 (Connectedness)**.** A set $E \subset \mathbb{C}$ is *connected* if it cannot be written as the union of two disjoint nonempty open subsets of E (with the relative topology). Equivalently, there do not exist disjoint nonempty open sets $U, V \subset \mathbb{C}$ such that

$$E \subset U \cup V, \qquad E \cap U \neq \varnothing, \qquad E \cap V \neq \varnothing.$$

Definition 2.15 (Path connectedness)**.** A set $E \subset \mathbb{C}$ is *path connected* if for every $a, b \in E$ there exists a continuous map $\gamma : [0,1] \to E$ with $\gamma(0) = a$ and $\gamma(1) = b$.

Proposition 2.16 (Continuous images of connected sets)**.** *If E is connected and $f : E \to Y$ is continuous, then $f(E)$ is connected.*

Proof. If $f(E) \subset U \cup V$ with U, V disjoint open in Y and both $f(E) \cap U$ and $f(E) \cap V$ nonempty, then $E = f^{-1}(U) \cup f^{-1}(V)$ is a separation of E into two disjoint nonempty open sets, contradicting connectedness. □

Remark 2.17. Every path connected set is connected (apply Theorem 2.16 to the path $[0,1] \to E$). The converse is false in general, but it becomes true for open subsets of $\mathbb{C}$; see the next section.

2.5 Polygons and Connectedness

> *Piecewise linear paths provide a concrete bridge between topology and integration. By understanding polygons and polygonal connectivity, we prepare the geometric language needed for contour integrals, homotopy arguments, and simply connected regions.*

A particularly useful class of paths in $\mathbb{C}$ is the class of *polygonal paths*, that is, continuous curves obtained by concatenating finitely many straight line segments.

Theorem 2.18. *Let $G \subset \mathbb{C}$ be open. Then G is connected if and only if G is path connected. Moreover, any two points of G can be joined by a polygonal path lying entirely in G.*

Proof strategy. *Fix one point and consider the set of points reachable from it by polygonal paths staying in the domain. Openness of the domain shows that this reachable set is open, while a small-ball argument around limit points shows that it is also closed in the relative topology. Connectedness then forces the reachable set to be the whole domain.*

Proof. If G is path connected, then it is connected by the remark following Theorem 2.16.

Conversely, assume that G is open and connected. Fix a point $z_0 \in G$ and let $A \subset G$ be the set of all points that can be joined to z_0 by a polygonal path contained in G. Then A is nonempty and satisfies $A \subset G$.

Step 1: A is open in G. Let $z \in A$. Since G is open, there exists $r > 0$ such that $B(z, r) \subset G$. For any $w \in B(z, r)$, the segment $[z, w] \subset B(z, r) \subset G$, and concatenating a polygonal path from z_0 to z with $[z, w]$ shows that $w \in A$. Hence $B(z, r) \subset A$, so A is open.

Step 2: A is closed in G. Let $(z_n) \subset A$ with $z_n \to z \in G$. Since G is open, choose $r > 0$ such that $B(z, r) \subset G$. For n large enough, $z_n \in B(z, r)$, and then the segment $[z_n, z] \subset B(z, r) \subset G$. Concatenating a polygonal path from z_0 to z_n with $[z_n, z]$ yields a polygonal path from z_0 to z in G. Thus $z \in A$, and A is closed.

Since G is connected and $A \subset G$ is nonempty, open, and closed in the relative topology, it follows that $A = G$. Therefore every point of G can be joined to z_0 by a polygonal path, and hence any two points of G can be joined by a polygonal path in G. □

Remark 2.19. The openness assumption is essential. A connected subset of $\mathbb{C}$ need not be path connected in general; the topologist's sine curve is the standard counterexample. What the theorem shows is that open connected sets are geometrically much better behaved than arbitrary connected sets.

2.6 Products of Connected Sets

> *It is important to know how connectedness behaves under natural constructions. Stability results of this kind allow us to manufacture new connected regions from familiar ones and sharpen our geometric control over subsets of the plane.*

In applications it is useful to pass between subsets of $\mathbb{C} \cong \mathbb{R}^2$ and Cartesian products of subsets of $\mathbb{R}$.

Proposition 2.20 (Products preserve connectedness)**.** *Let $A, B \subset \mathbb{R}$ be connected (equivalently, intervals). Then $A \times B \subset \mathbb{R}^2 \cong \mathbb{C}$ is connected. More generally, if $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are connected, then $A \times B \subset \mathbb{R}^{m+n}$ is connected.*

Proof. Fix $b_0 \in B$ and consider the "cross"

$$C := (A \times \{b_0\}) \cup (\{a_0\} \times B) \subset A \times B,$$

where $a_0 \in A$ is arbitrary. Both $A \times \{b_0\}$ and $\{a_0\} \times B$ are continuous images of connected sets, hence connected, and they intersect at (a_0, b_0). Therefore C is connected.

For each $a \in A$ set

$$C_a := (A \times \{b_0\}) \cup (\{a\} \times B).$$

As above, each C_a is connected and contains the common subset $A \times \{b_0\}$. Hence the union $\bigcup_{a \in A} C_a$ is connected. But $\bigcup_{a \in A} C_a = A \times B$, so $A \times B$ is connected. □

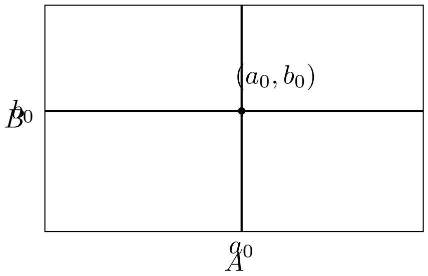

Figure 2.3: *A connected "cross" inside $A \times B$ (used in the proof of Theorem 2.20).*

Proposition 2.21 (Graphs of continuous functions)**.** *Let $S \subset \mathbb{C}$ be connected and let $f : S \to \mathbb{C}$ be continuous. Then the graph*

$$G_f := \{(z, f(z)) : \ z \in S\} \subset \mathbb{C}^2$$

is connected. Conversely, if G_f is connected, then S is connected.

Proof. The map $g : S \to \mathbb{C}^2$, $g(z) = (z, f(z))$, is continuous and satisfies $g(S) = G_f$. Hence G_f is connected by Theorem 2.16. Conversely, the projection $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$, $\pi_1(z, w) = z$, is continuous and $\pi_1(G_f) = S$; therefore S is connected whenever G_f is connected. ☐

Remark 2.22. A *component* of a topological space is a maximal connected subset. Components are always closed and pairwise disjoint, and their union is the whole space.

2.7 More Examples and Extensions

> *Further examples are useful not merely for illustration but for calibration: they show precisely how far the preceding definitions reach and where naive geometric intuition can fail. Such examples become increasingly valuable once analytic statements begin to depend sensitively on topological hypotheses.*

Proposition 2.23. *Let $A \subset \mathbb{C}$ be connected and let B satisfy $A \subseteq B \subseteq \overline{A}$. Then B is connected. In particular, $\overline{A}$ is connected.*

Proof. Assume for contradiction that B is disconnected. Then there exist disjoint nonempty open sets $U, V \subset \mathbb{C}$ such that $B \subset U \cup V$ and $B \cap U \neq \varnothing$, $B \cap V \neq \varnothing$. Since $A \subset B$ is connected, we must have $A \subset U$ or $A \subset V$; assume $A \subset U$. Because V is open and disjoint from U, we have $V \cap \overline{U} = \varnothing$, hence $V \cap \overline{A} = \varnothing$ (as $\overline{A} \subset \overline{U}$). But $B \subset \overline{A}$, so $B \cap V = \varnothing$, contradicting $B \cap V \neq \varnothing$. ☐

Example 2.24 (Topologist's sine curve)**.** Consider the subset $S \subset \mathbb{R}^2$ given by

$$S = \left\{ \left(x, \sin \tfrac{1}{x}\right) : 0 < x \leq 1 \right\} \cup \left(\{0\} \times [-1, 1]\right).$$

Then S is connected but not path connected. This example shows that connectedness and path connectedness may differ for sets that are not open.

2.8 Sequences, Limits, and Continuity

> *The language of limits and continuity provides the first rigorous description of local behavior. These notions are inherited from real analysis, but in the complex setting they serve as the immediate prelude to differentiability and later to uniform convergence of analytic functions.*

Let $S \subseteq \mathbb{C}$. A *complex-valued function* on S is a map

$$f : S \to \mathbb{C}.$$

When convenient, we write

$$f(z) = u(z) + iv(z),$$

where $u, v : S \to \mathbb{R}$ are the real and imaginary parts of f.

2.8.1 Sequences and Limits

A *sequence* in $\mathbb{C}$ is simply a map $\mathbb{N} \to \mathbb{C}$. We usually write it as $(z_n)_{n\geq 1}$, where

$$z_n = x_n + iy_n.$$

Definition 2.25 (Convergence of a sequence)**.** A sequence (z_n) converges to $z \in \mathbb{C}$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|z_n - z| < \epsilon \qquad \text{for all } n \geq N.$$

In that case we write

$$z_n \to z \qquad \text{or} \qquad \lim_{n\to\infty} z_n = z.$$

Proposition 2.26 (Componentwise convergence)**.** *Let $z_n = x_n + iy_n$ and $z = x + iy$. Then*

$$z_n \to z \quad \iff \quad x_n \to x \ \ \textit{and} \ \ y_n \to y.$$

Proof. If $z_n \to z$, then

$$|x_n - x| = |\Re(z_n - z)| \leq |z_n - z|, \qquad |y_n - y| = |\Im(z_n - z)| \leq |z_n - z|,$$

so $x_n \to x$ and $y_n \to y$.

Conversely, if $x_n \to x$ and $y_n \to y$, then

$$|z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} \leq |x_n - x| + |y_n - y|,$$

and the right-hand side tends to 0. Hence $z_n \to z$. ☐

Limits of functions. Let $z_0 \in \mathbb{C}$, let $r > 0$, and suppose

$$f : B(z_0, r) \setminus \{z_0\} \to \mathbb{C}.$$

We say that f has limit ℓ at z_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \Longrightarrow \quad |f(z) - \ell| < \epsilon.$$

In that case we write

$$\lim_{z \to z_0} f(z) = \ell.$$

If

$$f(z) = u(x, y) + iv(x, y), \qquad z = x + iy,\ z_0 = x_0 + iy_0,$$

then

$$\lim_{z \to z_0} f(z) = \alpha + i\beta$$

if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = \alpha \qquad \text{and} \qquad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = \beta.$$

Remark 2.27 (Uniqueness of limits)*.* If $\lim_{z \to z_0} f(z)$ exists, then it is unique.

Remark 2.28 (Path test)*.* A point z_0 may be approached along infinitely many curves. Therefore, if $\lim_{z \to z_0} f(z)$ exists, it must have the same value along every path tending to z_0. In practice, exhibiting two paths that yield different limiting values is a standard way to show that the limit does not exist.

Proposition 2.29. *Let* $f : B(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ *satisfy*

$$\lim_{z \to z_0} f(z) = \alpha \neq 0.$$

Then

$$\lim_{z \to z_0} \frac{1}{f(z)} = \frac{1}{\alpha}.$$

Proof. Since $\alpha \neq 0$, choose $\delta_1 > 0$ such that

$$0 < |z - z_0| < \delta_1 \quad \Longrightarrow \quad |f(z) - \alpha| < \frac{|\alpha|}{2}.$$

Then, for such z,

$$|f(z)| \geq |\alpha| - |f(z) - \alpha| > \frac{|\alpha|}{2},$$

so $f(z) \neq 0$ near z_0.

Now

$$\left| \frac{1}{f(z)} - \frac{1}{\alpha} \right| = \frac{|f(z) - \alpha|}{|f(z)|\,|\alpha|} \leq \frac{2}{|\alpha|^2}\,|f(z) - \alpha|.$$

Because $|f(z) - \alpha| \to 0$ as $z \to z_0$, the right-hand side tends to 0. This proves the claim. ☐

2.8.2 Continuity

Let $S \subseteq \mathbb{C}$ and let $f : S \to \mathbb{C}$. We say that f is *continuous at* $z_0 \in S$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$z \in S,\ |z - z_0| < \delta \quad \Longrightarrow \quad |f(z) - f(z_0)| < \epsilon.$$

Equivalently,

$$\lim_{z \to z_0} f(z) = f(z_0).$$

2.9 Continuity and Compactness in Complex Analysis

> *Compactness is the mechanism that converts local information into global conclusions. In complex analysis it appears constantly: in the attainment of extrema, in uniform continuity, and in many arguments where continuity on a small scale must be promoted to a global statement.*

Proposition 2.30 (Sequential criterion for continuity)**.** *Let $f : S \to \mathbb{C}$. Then f is continuous at $z_0 \in S$ if and only if for every sequence $(z_n) \subset S$ with $z_n \to z_0$, one has*

$$f(z_n) \to f(z_0).$$

Proof. Assume first that f is continuous at z_0. Let $\epsilon > 0$. Choose $\delta > 0$ such that

$$|z - z_0| < \delta \quad \Longrightarrow \quad |f(z) - f(z_0)| < \epsilon. \tag{2.1}$$

If $z_n \to z_0$, then $|z_n - z_0| < \delta$ for all sufficiently large n, and (2.1) implies $|f(z_n) - f(z_0)| < \epsilon$ eventually. Hence $f(z_n) \to f(z_0)$.

Conversely, suppose f is not continuous at z_0. Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ one can find $z \in S$ with

$$|z - z_0| < \delta \qquad \text{but} \qquad |f(z) - f(z_0)| \geq \epsilon_0.$$

Choosing $\delta = 1/n$ for each n, we obtain a sequence $(z_n) \subset S$ such that

$$|z_n - z_0| < \frac{1}{n} \qquad \text{and} \qquad |f(z_n) - f(z_0)| \geq \epsilon_0.$$

Then $z_n \to z_0$, but $f(z_n) \not\to f(z_0)$, a contradiction. ☐

Proposition 2.31 (Basic continuity properties)**.** *Let $S \subseteq \mathbb{C}$ and let $z_0 \in S$.*

(i) *If $f : S \to \mathbb{C}$ is continuous at z_0 and $f(z_0) \neq 0$, then $1/f$ is continuous at z_0.*

(ii) *If $f = u + iv$, then f is continuous at z_0 if and only if both u and v are continuous at z_0.*

(iii) *If $f : S \to \mathbb{C}$ is continuous at z_0 and $f(z_0) \neq 0$, then there exists $\delta > 0$ such that*

$$z \in S \cap B(z_0, \delta) \quad \Longrightarrow \quad f(z) \neq 0.$$

Proof. Part *(i)* follows from Theorem 2.29. Part *(ii)* is immediate from the componentwise characterization of limits. For *(iii)*, continuity at z_0 gives $\delta > 0$ such that

$$|f(z) - f(z_0)| < \frac{1}{2}|f(z_0)| \qquad \text{whenever } z \in S,\ |z - z_0| < \delta.$$

Then

$$|f(z)| \geq |f(z_0)| - |f(z) - f(z_0)| > \frac{1}{2}|f(z_0)| > 0,$$

so $f(z) \neq 0$ on $S \cap B(z_0, \delta)$. □

2.9.1 Uniform Continuity

A function $f : S \to \mathbb{C}$ is said to be *uniformly continuous* on S if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $z, w \in S$,

$$|z - w| < \delta \quad \Longrightarrow \quad |f(z) - f(w)| < \epsilon.$$

Remark 2.32. Uniform continuity is stronger than pointwise continuity because the same δ must work simultaneously for all points of S. If f and g are uniformly continuous on S, then $f + g$ is uniformly continuous on S.

2.9.2 Compactness

Definition 2.33 (Compactness)**.** A set $K \subset \mathbb{C}^n$ is called *compact* if it is closed and bounded.

Remark 2.34. In $\mathbb{C}^n \cong \mathbb{R}^{2n}$, this is exactly the Heine–Borel characterization of compactness.

Theorem 2.35 (Continuous images of compact sets)**.** *Let $K \subset \mathbb{C}$ be compact and let $f : K \to \mathbb{C}$ be continuous. Then $f(K)$ is compact.*

Proof. Because compactness in $\mathbb{C}$ is equivalent to sequential compactness, it suffices to show that every sequence in $f(K)$ has a convergent subsequence whose limit still lies in $f(K)$.

Let $(w_n) \subset f(K)$. For each n choose $z_n \in K$ such that $w_n = f(z_n)$. Since K is compact, there exists a subsequence (z_{n_k}) and a point $z \in K$ with

$$z_{n_k} \to z.$$

By continuity of f,

$$w_{n_k} = f(z_{n_k}) \to f(z) \in f(K).$$

Thus every sequence in $f(K)$ has a convergent subsequence with limit in $f(K)$, so $f(K)$ is compact. $\square$

Chapter 3

Holomorphic Functions

Here we enter the central theme of the course: complex differentiability. We define holomorphic functions, derive the Cauchy–Riemann equations, interpret the derivative as a complex-linear differential, and isolate the decisive distinction between real differentiability and genuine holomorphicity.

Learning objectives.

- Distinguish real differentiability from complex differentiability and understand the rigidity introduced by complex linearity.
- Derive and interpret the Cauchy–Riemann equations.
- Connect local differential criteria with the global analytic viewpoint.

Chapter roadmap.

- We begin by defining complex differentiability and comparing it with ordinary differentiability on $\mathbb{R}^2$.
- The Cauchy–Riemann equations and the complex-linearity of the Jacobian reveal the algebraic content of holomorphy.
- The chapter culminates in sufficient criteria, permanence properties, and the first constancy principle.

3.1 Complex Differentiation

Complex differentiability is the decisive new notion of the subject. Although it is defined by the same formal limit as in real calculus, its consequences are vastly stronger, and the remainder of the theory may be viewed as an unfolding of that single definition.

Let $D \subset \mathbb{C}$ be open. A function $f : D \to \mathbb{C}$ is said to be *complex differentiable* at $z_0 \in D$ if the limit

$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{3.1}$$

exists. If f is complex differentiable at every point of D, then f is called *holomorphic* on D.

The crucial feature of (3.1) is that the increment $h \in \mathbb{C}$ may approach 0 from *every* complex direction. This makes complex differentiability much more restrictive than ordinary differentiability for functions of two real variables.

Example 3.1. For $f(z) = z^2$,

$$\frac{(z+h)^2 - z^2}{h} = 2z + h \longrightarrow 2z,$$

so $f'(z) = 2z$. More generally, if $f(z) = z^n$ with $n \in \mathbb{N}$, then $f'(z) = nz^{n-1}$.

Example 3.2. The function $f(z) = \overline{z}$ is nowhere complex differentiable. Indeed,

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{h}}{h}.$$

If $h = t \in \mathbb{R} \setminus \{0\}$, this quotient equals 1, whereas for $h = it$ it equals -1. Hence the limit does not exist.

Example 3.3. The function $f(z) = |z|^2 = z\overline{z}$ is complex differentiable only at $z = 0$. Indeed,

$$\frac{|z+h|^2 - |z|^2}{h} = \overline{z} + z\frac{\overline{h}}{h} + \overline{h}.$$

If $z \neq 0$, the term $z\,\overline{h}/h$ depends on the direction of approach, so the limit fails to exist. At $z = 0$ the quotient reduces to $\overline{h} \to 0$, so $f'(0) = 0$.

3.2 Algebra of Differentiation

> *Once differentiability has been defined, one must verify that it interacts well with the usual algebraic operations. These rules make the class of holomorphic functions manageable and permit later constructions by composition, quotient formation, and iterative differentiation.*

The familiar algebraic rules of differential calculus remain valid in the complex setting.

Proposition 3.4 (Linearity, product rule, quotient rule)**.** *Suppose f and g are complex differentiable at z_0 and let $\alpha, \beta \in \mathbb{C}$. Then:*

(i) $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$*;*

(ii) $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$*;*

(iii) *if $g(z_0) \neq 0$, then f/g is complex differentiable at z_0 and*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof. Part *(i)* follows immediately from the linearity of limits. For the product rule, write

$$\frac{f(z_0+h)g(z_0+h) - f(z_0)g(z_0)}{h} = g(z_0+h)\frac{f(z_0+h) - f(z_0)}{h} + f(z_0)\frac{g(z_0+h) - g(z_0)}{h}.$$

Since differentiability implies continuity, $g(z_0+h) \to g(z_0)$ as $h \to 0$; passing to the limit therefore gives

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

For *(iii)*, set $q = 1/g$ near z_0. The identity $gq \equiv 1$ and the product rule yield

$$0 = (gq)'(z_0) = g'(z_0)q(z_0) + g(z_0)q'(z_0),$$

so

$$q'(z_0) = -\frac{g'(z_0)}{g(z_0)^2}.$$

Applying the product rule to $f/g = fq$ gives the stated quotient formula. □

3.3 The Cauchy–Riemann Equations

> *The Cauchy–Riemann equations provide the first visible manifestation of the rigidity of complex differentiability. They translate the single complex derivative into a coupled system of real partial differential equations and thereby reveal that holomorphic functions are highly constrained objects.*

Write $z = x + iy$ and let

$$f(z) = u(x, y) + iv(x, y),$$

where u and v are real-valued functions on $D \subset \mathbb{C} \cong \mathbb{R}^2$. A necessary condition for complex differentiability is obtained by taking the limit in (3.1) along the real and imaginary axes.

Theorem 3.5 (Necessary form of the derivative)**.** *Assume that* $f = u+iv$ *is complex differentiable at* $z_0 = x_0 + iy_0$*. Then the first-order partial derivatives of* u *and* v *exist at* (x_0, y_0) *and satisfy*

$$u_x(x_0, y_0) = v_y(x_0, y_0), \qquad u_y(x_0, y_0) = -v_x(x_0, y_0). \tag{3.2}$$

Moreover,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0). \tag{3.3}$$

Proof strategy. *Evaluate the difference quotient along two distinguished families of increments: first along the real axis and then along the imaginary axis. Because a complex derivative, if it exists, is independent of the direction of approach, the two limiting expressions must agree. Comparing real and imaginary parts produces the Cauchy–Riemann equations.*

Proof. Take $h = t \in \mathbb{R}$. Then

$$\frac{f(z_0 + t) - f(z_0)}{t} = \frac{u(x_0 + t, y_0) - u(x_0, y_0)}{t} + i\,\frac{v(x_0 + t, y_0) - v(x_0, y_0)}{t}.$$

As $t \to 0$, the left-hand side tends to $f'(z_0)$. Hence its real and imaginary parts converge, which shows that $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ exist and that

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0).$$

Now take $h = it$ with $t \in \mathbb{R}$. Since $1/(it) = -i/t$,

$$\frac{f(z_0 + it) - f(z_0)}{it} = -i \, \frac{u(x_0, y_0 + t) - u(x_0, y_0)}{t} + \frac{v(x_0, y_0 + t) - v(x_0, y_0)}{t}.$$

Letting $t \to 0$, we conclude that $u_y(x_0, y_0)$ and $v_y(x_0, y_0)$ exist and that

$$f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, y_0).$$

Comparing the two expressions for $f'(z_0)$ and equating real and imaginary parts yields (3.2) and (3.3). □

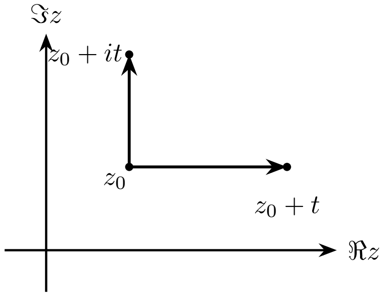

Figure 3.1: *Real and imaginary directions used in the derivation of the Cauchy–Riemann equations.*

Remark 3.6. The Cauchy–Riemann equations are *necessary* for holomorphy, but by themselves they are not sufficient at a single point. A counterexample is given in Section 3.6.

Example 3.7 (Consistency check)**.** If $f(z) = z^2$, then $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. One computes

$$u_x = 2x, \qquad u_y = -2y, \qquad v_x = 2y, \qquad v_y = 2x,$$

so the Cauchy–Riemann equations hold everywhere and

$$f'(z) = u_x + i v_x = 2x + 2iy = 2z.$$

Although elementary, this example illustrates exactly how the real Jacobian becomes multiplication by a complex scalar.

3.4 Complex-Linearity of the Differential

> *The derivative of a holomorphic function is more than a real linear map on $\mathbb{R}^2$: it is given by multiplication by a complex number. This observation gives a conceptual explanation of angle preservation and clarifies why complex differentiability is fundamentally stronger than ordinary real differentiability.*

The Jacobian matrix of $f = u + iv$ at (x, y) is the real 2×2 matrix

$$Df(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

If f is complex differentiable, this real linear map must in fact be multiplication by a complex number.

Proposition 3.8. *Suppose that the first-order partial derivatives of u and v exist at (x_0, y_0). Then the following are equivalent:*

(i) *the Jacobian matrix $Df(x_0, y_0)$ is of the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for some $a, b \in \mathbb{R}$;

(ii) *the Cauchy–Riemann equations hold at (x_0, y_0);*

(iii) *the real differential acts as multiplication by the complex number $a + ib$.*

In this case one has $a = u_x(x_0, y_0)$ and $b = v_x(x_0, y_0)$.

Proof. The equivalence of *(i)* and *(ii)* is immediate by comparing matrix entries. The matrix in *(i)* acts on $(\xi, \eta) \in \mathbb{R}^2$ exactly as multiplication by $a + ib$ acts on $\xi + i\eta \in \mathbb{C}$, so *(i)* and *(iii)* are equivalent as well. ☐

Thus complex differentiability can be interpreted as a very rigid compatibility condition between real differentiation and multiplication by i.

3.5 Polar Form of the Cauchy–Riemann Equations

> *Many important functions in complex analysis are most naturally expressed in polar coordinates. Rewriting the Cauchy–Riemann equations in this form is therefore not merely a change of variables; it is a practical tool for studying radial and angular behavior, especially near the origin or on punctured domains.*

Assume $f = u + iv$ is holomorphic on a punctured neighborhood of the origin, and write $z = re^{i\theta}$ with $r > 0$. Using the chain rule together with

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

one obtains the polar form of the Cauchy–Riemann equations.

Proposition 3.9 (Polar Cauchy–Riemann equations)**.** *Whenever the derivatives exist,*

$$u_r = \frac{1}{r} v_\theta, \qquad v_r = -\frac{1}{r} u_\theta. \tag{3.4}$$

Proof. By the chain rule,

$$u_r = u_x \cos\theta + u_y \sin\theta, \qquad v_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta).$$

Using $u_x = v_y$ and $u_y = -v_x$, we get

$$\frac{1}{r} v_\theta = -v_x \sin\theta + v_y \cos\theta = u_y \sin\theta + u_x \cos\theta = u_r.$$

The second identity follows similarly. ☐

3.6 A Cautionary Example: Cauchy–Riemann at a Point

> *A single example can be as instructive as a theorem itself. The purpose of this section is to show that satisfying the Cauchy–Riemann equations at one point, without suitable neighborhood hypotheses, does not by itself guarantee complex differentiability.*

The following example shows that the Cauchy–Riemann equations at a *single point* do not guarantee complex differentiability there.

Example 3.10 (Cauchy–Riemann at 0 without differentiability)**.** Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$ and, for $z = x + iy \neq 0$,

$$f(z) = \frac{x^3}{x^2 + y^2} + i\,\frac{y^3}{x^2 + y^2}.$$

Then u and v satisfy the Cauchy–Riemann equations at $(0, 0)$, but f is not complex differentiable at 0.

Proof. Since $u(x, 0) = x$ and $u(0, y) = 0$, we obtain

$$u_x(0,0) = 1, \qquad u_y(0,0) = 0.$$

Similarly, $v(0, y) = y$ and $v(x, 0) = 0$, so

$$v_x(0,0) = 0, \qquad v_y(0,0) = 1.$$

Hence the Cauchy–Riemann equations hold at the origin. However,

$$\frac{f(z) - f(0)}{z} = \frac{f(z)}{z}$$

has different limits along different paths. Along the real axis $z = x \to 0$, one has $f(x) = x$, so the quotient tends to 1. Along the diagonal $z = x(1+i)$, one computes

$$f(x+ix) = \frac{x}{2}(1+i), \qquad \frac{f(x+ix)}{x+ix} = \frac{1}{2}.$$

Since the limits disagree, $f'(0)$ does not exist. □

3.7 Chain Rule for Complex Differentiation

> *Holomorphic functions acquire much of their strength from their stability under composition. The chain rule is therefore indispensable: it allows complicated maps to be built from simpler ones and lies behind the behavior of elementary transcendental functions and conformal changes of variable.*

Theorem 3.11 (Chain rule). *Let $D_1, D_2 \subset \mathbb{C}$ be open. If $f : D_1 \to D_2$ is complex differentiable at $z \in D_1$ and $g : D_2 \to \mathbb{C}$ is complex differentiable at $f(z)$, then $g \circ f$ is complex differentiable at z and*

$$(g \circ f)'(z) = g'(f(z))\, f'(z).$$

Proof strategy. *Write the increment of g at $f(z)$ as its linear part plus a remainder that is small relative to the increment in the intermediate variable. Substituting the increment produced by f turns this into a first-order expansion for $g \circ f$. The small remainder remains negligible because $f(z+h) - f(z) \to 0$ as $h \to 0$.*

Proof. Set $w = f(z)$. Since f is differentiable at z, we may write

$$f(z+h) = w + f'(z)h + \varepsilon(h)h, \qquad \varepsilon(h) \to 0.$$

Likewise, differentiability of g at w gives

$$g(w+k) = g(w) + g'(w)k + \delta(k)k, \qquad \delta(k) \to 0 \quad (k \to 0).$$

Apply this with $k = f(z+h) - f(z)$. Then

$$\frac{g(f(z+h)) - g(f(z))}{h} = g'(w)\frac{f(z+h) - f(z)}{h} + \delta(k)\frac{k}{h}.$$

As $h \to 0$, the first factor tends to $g'(w)f'(z)$, while $\delta(k) \to 0$ and $k/h \to f'(z)$. Hence the second term tends to 0. This proves the formula. □

3.8 A Sufficient Condition for Complex Differentiability

> *In practice one often knows more about the real and imaginary parts of a function than about the complex difference quotient itself. This section records a useful criterion showing that appropriate differentiability, together with the Cauchy–Riemann equations, is enough to recover holomorphy.*

The next theorem is the standard converse to the necessary condition in Theorem 3.5 under a mild regularity hypothesis.

Theorem 3.12 (Cauchy–Riemann equations with continuous partial derivatives)**.** *Let $D \subset \mathbb{C}$ be open and let $f = u + iv : D \to \mathbb{C}$. Assume that the first-order partial derivatives of u and v exist on D, are continuous on D, and satisfy*

$$u_x = v_y, \qquad u_y = -v_x$$

throughout D. Then f is holomorphic on D; moreover, for every $z = x + iy \in D$,

$$f'(z) = u_x(x, y) + iv_x(x, y) = v_y(x, y) - iu_y(x, y).$$

Proof strategy. *Use ordinary differentiability of the map $(u, v) : \mathbb{R}^2 \to \mathbb{R}^2$, which follows from the continuity of the first-order partial derivatives. The Cauchy–Riemann equations force the Jacobian matrix to be complex-linear, so the real first-order approximation is already the correct complex-linear approximation.*

Proof. Fix $z_0 = x_0 + iy_0 \in D$. Since the partial derivatives are continuous near (x_0, y_0), the map $(u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ is real differentiable there. Hence for small real increments h, k we have

$$u(x_0 + h, y_0 + k) = u(x_0, y_0) + u_x h + u_y k + \rho\, \varepsilon_1(h, k),$$

$$v(x_0 + h, y_0 + k) = v(x_0, y_0) + v_x h + v_y k + \rho\, \varepsilon_2(h, k),$$

where $\rho = (h^2 + k^2)^{1/2}$ and $\varepsilon_1(h, k), \varepsilon_2(h, k) \to 0$ as $(h, k) \to (0, 0)$. Setting $\varepsilon = \varepsilon_1 + i\varepsilon_2$ and using the Cauchy–Riemann equations at (x_0, y_0), we obtain

$$\begin{aligned} f(z_0 + h + ik) - f(z_0) &= (u_x + iv_x)h + (u_y + iv_y)k + \rho\, \varepsilon(h, k) \\ &= (u_x + iv_x)(h + ik) + \rho\, \varepsilon(h, k). \end{aligned}$$

Dividing by $h + ik \neq 0$ gives

$$\frac{f(z_0 + h + ik) - f(z_0)}{h + ik} = u_x + iv_x + \frac{\rho\, \varepsilon(h, k)}{h + ik}.$$

Since $|h + ik| = \rho$, the last term tends to 0 as $(h, k) \to (0, 0)$. Therefore the difference quotient converges to $u_x + iv_x$, proving that f is holomorphic at z_0. As z_0 was arbitrary, f is holomorphic on D. □

Remark 3.13. The theorem above is one of the first places where complex analysis separates decisively from real analysis. Over $\mathbb{R}$, differentiability does not force any local power-series expansion; over $\mathbb{C}$, the compatibility encoded in the Cauchy–Riemann equations and continuity of the first derivatives already pushes the function into the holomorphic regime.

3.9 Holomorphic Versus Analytic

> *One of the central surprises of the subject is that complex differentiability automatically yields power-series expansion. The equivalence between being holomorphic and being analytic has no analogue in ordinary real-variable calculus and marks the beginning of the deep rigidity of complex function theory.*

Definition 3.14 (Holomorphic and analytic)**.** Let $\Omega \subset \mathbb{C}$ be open.

(i) A function $f : \Omega \to \mathbb{C}$ is *holomorphic* if it is complex differentiable at every point of Ω.

(ii) A function f is *analytic at* $z_0 \in \Omega$ if there exist $r > 0$ and coefficients $(a_n)_{n\geq 0}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < r).$$

(iii) A function is *analytic on* Ω if it is analytic at each point of Ω.

At this stage, *holomorphic* is the primary notion and *analytic* is a stronger-looking one. One of the deepest basic theorems of the subject, proved later from Cauchy's integral formula, says that these notions are in fact equivalent on open sets.

Example 3.15. A holomorphic function on all of $\mathbb{C}$ is called *entire*. Polynomials and the exponential function are entire, whereas $1/z$ is holomorphic only on $\mathbb{C} \setminus \{0\}$.

3.10 Basic Examples and Permanence Properties

> *After the foundational theory has been established, it is essential to build a flexible stock of examples. Closure properties show that holomorphic functions form a robust class, while explicit examples give concrete models to which the abstract theory can be applied.*

Proposition 3.16. *Let* $\Omega \subset \mathbb{C}$ *be open.*

(i) *If* $f, g \in \mathrm{Hol}(\Omega)$ *and* $\alpha, \beta \in \mathbb{C}$, *then* $\alpha f + \beta g \in \mathrm{Hol}(\Omega)$ *and* $fg \in \mathrm{Hol}(\Omega)$.

(ii) *If* $f, g \in \mathrm{Hol}(\Omega)$ *and* g *has no zeros on* Ω, *then* $f/g \in \mathrm{Hol}(\Omega)$.

(iii) *If* $f : \Omega \to \Omega_1$ *and* $g : \Omega_1 \to \mathbb{C}$ *are holomorphic, then* $g \circ f$ *is holomorphic on* Ω.

Proof. Parts *(i)* and *(ii)* follow pointwise from Theorem 3.4. Part *(iii)* follows from Theorem 3.11. □

Example 3.17. For a fixed branch of the logarithm, the function

$$2^z = e^{z \log 2}$$

is entire, and compositions such as 3^{2^z} are holomorphic because they arise from repeated composition of holomorphic maps.

3.11 Domains and a First Constancy Principle

Connectedness begins to interact seriously with holomorphy in this section. The main point is that an analytic relation that holds on a set with accumulation can force a global conclusion on an entire domain, illustrating the first major rigidity principle of the theory.

Definition 3.18. A nonempty open connected subset of $\mathbb{C}$ is called a *domain.*

Connectedness already has strong consequences in complex analysis.

Proposition 3.19. *Let $D \subset \mathbb{C}$ be a domain and let $f = u + iv \in \mathrm{Hol}(D)$. If either u or v is constant on D, then f is constant on D.*

Proof. Assume first that u is constant. Then $u_x = u_y = 0$ throughout D. By the Cauchy–Riemann equations,

$$v_x = -u_y = 0, \qquad v_y = u_x = 0$$

on D. Hence the gradient of v vanishes identically. Since D is connected, v is constant on D, and therefore $f = u + iv$ is constant. The case where v is constant is identical. □

The proposition is an early indication of the rigidity of holomorphic functions: the real and imaginary parts cannot vary independently.

Chapter 4

Elementary Functions

The transcendental functions of one complex variable already display the essential richness of the subject. We construct the exponential and trigonometric functions, analyze the multivalued character of the logarithm, and explain how branch choices encode the topology of the underlying domain.

Learning objectives.

- Develop the complex exponential, trigonometric functions, and logarithm in a way compatible with holomorphy.
- Understand multivaluedness and the role of branches.
- Use these examples to test the global topological constraints of the theory.

Chapter roadmap.

- We first establish the basic algebraic and analytic properties of the exponential and trigonometric functions.
- We then study the logarithm as the first genuinely multivalued elementary function.
- The chapter explains how branch choices encode the topology of the underlying domain.

4.1 Elementary Functions: Exponential and Trigonometric Functions

The exponential, trigonometric, and related functions provide the basic transcendental examples of holomorphic maps. Their complex-analytic behavior is richer than their real-variable counterpart and will later illuminate periodicity, logarithms, conformal maps, and residue computations.

The exponential function is defined on $\mathbb{C}$ by the absolutely convergent power series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (z \in \mathbb{C}). \tag{4.1}$$

Termwise differentiation of the series (4.1) shows that e^z is entire and satisfies $(e^z)' = e^z$. The trigonometric functions are defined by

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z := \frac{e^{iz} + e^{-iz}}{2}. \tag{4.2}$$

They are entire and satisfy the familiar identities $\sin^2 z + \cos^2 z = 1$ and $(\sin z)' = \cos z$, $(\cos z)' = -\sin z$.

Proposition 4.1 (Periodicity and Euler's formula)**.** *For all $z \in \mathbb{C}$ one has $e^{z+2\pi i} = e^z$. Moreover,*

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (\theta \in \mathbb{R}). \tag{4.3}$$

Proof. The identity $e^{z+2\pi i} = e^z\, e^{2\pi i}$ follows from the exponential series and the Cauchy product. Since $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$ by (4.2), periodicity follows. The formula (4.3) is immediate from (4.2) with $z = \theta \in \mathbb{R}$. □

Remark 4.2. The periodicity in Proposition 4.1 is the source of the multivalued nature of the complex logarithm and, more generally, of complex powers $z^\alpha = e^{\alpha \log z}$.

4.2 The Complex Logarithm and Branches

> *Unlike the exponential function, the logarithm cannot be defined globally as a single-valued holomorphic function on $\mathbb{C} \setminus \{0\}$. The need to choose branches is one of the first places where topology enters analytic function theory in an essential way.*

The exponential map $\exp : \mathbb{C} \to \mathbb{C}$ is entire and $2\pi i$–periodic:

$$e^{z+2\pi i k} = e^z \qquad (k \in \mathbb{Z}).$$

Moreover exp is surjective onto $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Thus for each $z \in \mathbb{C}^*$ the equation

$$e^w = z \tag{4.4}$$

has infinitely many solutions, differing by integer multiples of $2\pi i$.

4.2.1 The Multivalued Logarithm

Write $z = re^{i\theta}$ with $r = |z| > 0$ and $\theta \in \arg(z)$. Then any solution w of (4.4) has the form

$$w = \ln r + i(\theta + 2\pi k), \qquad k \in \mathbb{Z}.$$

This motivates the *multivalued logarithm*

$$\log z := \ln|z| + i\,\arg(z) = \{\ln|z| + i(\operatorname{Arg} z + 2\pi k) : \ k \in \mathbb{Z}\},$$

and the branch relation

$$\log z = \operatorname{Log} z + 2\pi i k, \qquad k \in \mathbb{Z}. \tag{4.5}$$

4.2.2 The Principal Branch

The *principal argument* $\operatorname{Arg} z \in (-\pi, \pi]$ defines the *principal logarithm*

$$\operatorname{Log} z := \ln |z| + i \operatorname{Arg} z.$$

It is single valued and holomorphic on the slit plane $\mathbb{C} \setminus (-\infty, 0]$, and satisfies

$$(\operatorname{Log} z)' = \frac{1}{z} \qquad \text{on } \mathbb{C} \setminus (-\infty, 0].$$

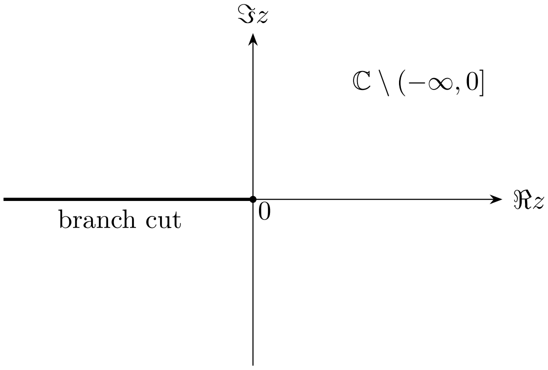

Figure 4.1: *Domain of the principal branch* $\operatorname{Log} z$.

4.2.3 Branches on Simply Connected Domains

Theorem 4.3 (Existence of a logarithm branch)**.** *Let $\Omega \subset \mathbb{C}$ be simply connected and assume $0 \notin \Omega$. Then there exists $L \in \operatorname{Hol}(\Omega)$ such that $e^{L(z)} = z$ for all $z \in \Omega$. Any two such branches differ by an additive constant $2\pi i k$, $k \in \mathbb{Z}$.*

Proof. Since $0 \notin \Omega$, the function $1/z$ is holomorphic on Ω. Because Ω is simply connected, $1/z$ admits a primitive $L \in \operatorname{Hol}(\Omega)$. Consider

$$H(z) := e^{-L(z)} z.$$

Then

$$H'(z) = e^{-L(z)} - L'(z) e^{-L(z)} z = e^{-L(z)} - \frac{1}{z} e^{-L(z)} z = 0,$$

so H is constant on Ω. Write $H \equiv c$ with $c \neq 0$, choose $\lambda \in \mathbb{C}$ such that $e^{\lambda} = c$, and replace L by $L - \lambda$. For the new primitive, still denoted by L, we have $e^{-L(z)} z \equiv 1$, hence $e^{L(z)} = z$ on Ω. If L_1 and L_2 are two such branches, then $e^{L_1 - L_2} \equiv 1$. Therefore $L_1 - L_2$ takes values in the discrete set $2\pi i \mathbb{Z}$; since it is continuous on the connected set Ω, it must be constant, and that constant belongs to $2\pi i \mathbb{Z}$. ☐

Proposition 4.4 (Logarithms of zero-free holomorphic functions)**.** *Let* $\Omega \subset \mathbb{C}$ *be simply connected and let* $f \in \mathrm{Hol}(\Omega)$ *satisfy* $f(z) \neq 0$ *for all* $z \in \Omega$*. Then there exists* $L \in \mathrm{Hol}(\Omega)$ *such that*

$$e^{L(z)} = f(z) \qquad (z \in \Omega),$$

and every such L *satisfies*

$$L'(z) = \frac{f'(z)}{f(z)} \qquad (z \in \Omega).$$

Any two such branches differ by an additive constant $2\pi ik$ *with* $k \in \mathbb{Z}$*.*

Proof. Since f never vanishes, the quotient f'/f is holomorphic on Ω. Because Ω is simply connected, the holomorphic function f'/f admits a primitive $L \in \mathrm{Hol}(\Omega)$. Consider

$$H(z) := f(z)e^{-L(z)}.$$

Then

$$H'(z) = f'(z)e^{-L(z)} - f(z)L'(z)e^{-L(z)} = 0,$$

so H is constant on Ω. Choose $z_0 \in \Omega$ and let $c := H(z_0) \neq 0$. Fix any logarithm λ of c and replace L by $L - \lambda$. Then the new function, still denoted by L, satisfies

$$f(z)e^{-L(z)} \equiv 1,$$

that is, $e^{L(z)} = f(z)$ on Ω. Differentiating $e^L = f$ gives $L'e^L = f'$, hence $L' = f'/f$. If L_1 and L_2 are two such branches, then $e^{L_1 - L_2} \equiv 1$, so $L_1 - L_2$ is locally constant and therefore constant on the connected set Ω. Its value must lie in $2\pi i\,\mathbb{Z}$. □

Corollary 4.5 (Nonexistence of a logarithm on a punctured disk)**.** *There is no holomorphic branch of* $\log z$ *on the punctured disk* $0 < |z| < 1$*. Consequently there is no continuous single-valued branch of* $\arg z$ *there.*

Proof. Assume that L is a holomorphic branch of $\log z$ on $0 < |z| < 1$. Fix $r \in (0, 1)$ and consider the circle $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$. Set

$$u(t) := L(\gamma(t)).$$

Then u is continuous on $[0, 2\pi]$ and

$$e^{u(t)} = re^{it} \qquad (0 \leq t \leq 2\pi).$$

Hence $\Re u(t) = \log r$, while

$$e^{i\Im u(t)} = e^{it}.$$

Therefore $\Im u(t) - t \in 2\pi\mathbb{Z}$ for every t. Since $t \mapsto \Im u(t) - t$ is continuous and takes values in the discrete set $2\pi\mathbb{Z}$, it must be constant:

$$\Im u(t) = t + 2\pi k \qquad \text{for some } k \in \mathbb{Z} \text{ and all } t.$$

But $\gamma(0) = \gamma(2\pi) = r$, so single-valuedness of L gives $u(0) = u(2\pi)$. The displayed formula yields instead

$$\Im u(2\pi) = 2\pi + 2\pi k \neq 2\pi k = \Im u(0),$$

a contradiction.

The same argument shows that no continuous single-valued branch of $\arg z$ can exist on $0 < |z| < 1$: if θ were such a branch, then $\theta(re^{it}) - t$ would be a continuous $2\pi\mathbb{Z}$-valued function on $[0, 2\pi]$, hence constant, again contradicting the equality of the endpoint values at $t = 0$ and $t = 2\pi$. □

Example 4.6 (Principal square root and general powers)**.** On the slit plane $\mathbb{C} \setminus (-\infty, 0]$, the principal logarithm determines the *principal square root*

$$\sqrt{z} := \exp\left(\frac{1}{2}\operatorname{Log} z\right),$$

which is holomorphic and satisfies $(\sqrt{z})^2 = z$. More generally, for $\alpha \in \mathbb{C}$ the principal branch of the power function is

$$z^\alpha := \exp(\alpha \operatorname{Log} z), \qquad z \in \mathbb{C} \setminus (-\infty, 0].$$

Different branch choices for the logarithm lead to different branches of z^α, which is why branch cuts are an essential part of the theory of multivalued elementary functions.

4.2.4 Complex Powers and Roots

Once a branch L of log is fixed on a domain Ω (for instance $L = \operatorname{Log}$ on $\mathbb{C} \setminus (-\infty, 0]$), complex powers are defined by

$$z^\alpha := \exp(\alpha L(z)), \qquad z \in \Omega,\ \alpha \in \mathbb{C}.$$

Different choices of L may lead to different values, reflecting the intrinsic multivaluedness of the logarithm; branch cuts provide a convenient mechanism for enforcing single-valuedness. In particular, the nth roots of $z \neq 0$ are given by

$$z^{1/n} = \exp\left(\frac{1}{n}\log z\right) = \left\{|z|^{1/n}\exp\left(i\frac{\operatorname{Arg} z + 2\pi k}{n}\right) : k = 0, 1, \ldots, n-1\right\}.$$

Remark 4.7 (Riemann surface viewpoint (optional)). The multivalued logarithm becomes single valued on a suitable Riemann surface obtained by gluing countably many copies of the slit plane along the branch cut. This viewpoint clarifies analytic continuation around the origin and explains the appearance of monodromy.

Chapter 5

Complex Integration

Contour integration provides the first global mechanism in complex analysis. Starting from parametrized curves and contour integrals, we develop Cauchy's theorem and Cauchy's integral formula, and we derive from them a chain of classical consequences including Liouville's theorem, the maximum-modulus principle, and the fundamental theorem of algebra.

Learning objectives.

- Define contour integrals carefully and compute them in basic situations.
- Understand the progression from local antiderivatives to global integral theorems.
- Use Cauchy's theorem and Cauchy's integral formula as the main engines of classical complex analysis.

Chapter roadmap.

- We start from parameterized curves and the algebra of contour integrals.
- The middle of the chapter establishes Cauchy–Goursat, simply connected versions of Cauchy's theorem, and contour deformation.
- The chapter culminates in Cauchy's integral formula and its immediate consequences, including Morera's theorem, Liouville's theorem, and the maximum modulus principle.

5.1 Contour Integrals

Integration along curves is the bridge between the geometric and analytic aspects of the subject. Once contour integrals are in place, the theory gains access to Cauchy's theorem, integral formulas, and the profound principle that interior information can be recovered from boundary data.

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 curve and let f be continuous on $\gamma([a, b])$. The *complex line integral* of f along γ is defined by

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t. \tag{5.1}$$

When γ is closed we also write $\oint_\gamma f(z)\,\mathrm{d}z$.

For a continuous curve $\gamma : [a, b] \to \mathbb{C}$, its *length* is

$$L(\gamma) := \sup_P \sum_{j=1}^{m} |\gamma(t_j) - \gamma(t_{j-1})|,$$

where the supremum is taken over all partitions $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a, b]$. If $L(\gamma) < \infty$, the curve is called *rectifiable.*

Proposition 5.1 (Piecewise C^1 curves are rectifiable)**.** *Every piecewise C^1 curve $\gamma : [a, b] \to \mathbb{C}$ is rectifiable, and*

$$L(\gamma) = \int_a^b |\gamma'(t)| \,\mathrm{d}t.$$

Proof. It is enough to prove the statement on a C^1 subinterval, since lengths add over concatenation. If $u < v$, then by the fundamental theorem of calculus applied to the real and imaginary parts of γ,

$$\gamma(v) - \gamma(u) = \int_u^v \gamma'(t) \,\mathrm{d}t,$$

hence

$$|\gamma(v) - \gamma(u)| \leq \int_u^v |\gamma'(t)| \,\mathrm{d}t.$$

Summing over a partition gives

$$\sum_{j=1}^{m} |\gamma(t_j) - \gamma(t_{j-1})| \leq \int_a^b |\gamma'(t)| \,\mathrm{d}t,$$

so γ is rectifiable and $L(\gamma) \leq \int_a^b |\gamma'(t)| \,\mathrm{d}t$.

For the reverse inequality, use uniform continuity of γ' on each smooth piece: choose a partition so fine that γ' varies little on every subinterval, and compare the increment $\gamma(t_j) - \gamma(t_{j-1})$ with $\gamma'(\xi_j)(t_j - t_{j-1})$. Passing to the limit yields

$$\int_a^b |\gamma'(t)| \,\mathrm{d}t \leq L(\gamma).$$

Therefore equality holds. ☐

Proposition 5.2 (Basic properties)**.** *If f, g are continuous on $\gamma([a, b])$ and $\alpha, \beta \in \mathbb{C}$, then*

$$\int_\gamma (\alpha f + \beta g) \,\mathrm{d}z = \alpha \int_\gamma f \,\mathrm{d}z + \beta \int_\gamma g \,\mathrm{d}z.$$

If γ is reparametrized by an orientation-preserving C^1-diffeomorphism, the integral (5.1) *is unchanged. Reversing orientation changes the sign.*

Proof. Linearity follows immediately from the linearity of the ordinary integral in the definition

$$\int_\gamma f(z) \,\mathrm{d}z = \int_a^b f(\gamma(t))\, \gamma'(t) \,\mathrm{d}t.$$

Now let $\phi : [c, d] \to [a, b]$ be an orientation-preserving C^1-diffeomorphism, and set $\widetilde{\gamma} := \gamma \circ \phi$. Then

$$\int_{\widetilde{\gamma}} f(z)\,\mathrm{d}z = \int_c^d f(\gamma(\phi(s)))\,\gamma'(\phi(s))\,\phi'(s)\,\mathrm{d}s.$$

By the real-variable change-of-variables formula $t = \phi(s)$, this equals

$$\int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t = \int_\gamma f(z)\,\mathrm{d}z.$$

If instead ϕ is orientation reversing, the same computation introduces a minus sign, so reversing orientation changes the sign of the contour integral. □

Lemma 5.3 (Estimation lemma (ML-inequality))**.** *Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise C^1 and set $L(\gamma) := \int_a^b |\gamma'(t)|\,\mathrm{d}t$. If $|f(z)| \le M$ on $\gamma([a, b])$, then*

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le M\,L(\gamma).$$

Proof. By (5.1) and the triangle inequality,

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t \le M \int_a^b |\gamma'(t)|\,\mathrm{d}t = M\,L(\gamma).$$

□

Theorem 5.4 (Integral of a derivative)**.** *Let $\Omega \subset \mathbb{C}$ be open, let $f \in \mathrm{Hol}(\Omega)$, and suppose $F \in \mathrm{Hol}(\Omega)$ satisfies $F' = f$ on Ω. Then for every piecewise C^1 curve $\gamma : [a, b] \to \Omega$,*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Define $h : [a, b] \to \mathbb{C}$ by $h(t) := F(\gamma(t))$. By the chain rule,

$$h'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t).$$

Integrating over $[a, b]$ and using the real-variable fundamental theorem of calculus gives

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b h'(t)\,\mathrm{d}t = h(b) - h(a) = F(\gamma(b)) - F(\gamma(a)).$$

□

Corollary 5.5 (Path independence)**.** *If f has a primitive on Ω, then the integral $\int_\gamma f(z)\,\mathrm{d}z$ depends only on the endpoints of γ. In particular, every closed contour integral of f vanishes.*

Proof. If $\gamma : [a, b] \to \Omega$ is any piecewise C^1 curve, Theorem 5.4 gives

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)),$$

which depends only on the endpoints. If γ is closed, then $\gamma(a) = \gamma(b)$ and therefore

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

□

5.1.1 Cauchy's Integral Theorem

The first fundamental theorem in the subject asserts that holomorphic functions have vanishing integrals over closed curves under appropriate geometric hypotheses.

Theorem 5.6 (Cauchy–Goursat theorem for triangles)**.** *Let $\Omega \subset \mathbb{C}$ be open and let $f \in \mathrm{Hol}(\Omega)$. Then, for every triangle T whose closure is contained in Ω,*

$$\oint_{\partial T} f(z)\,\mathrm{d}z = 0. \tag{5.2}$$

Proof strategy. *Subdivide the triangle repeatedly into four congruent subtriangles and retain one on which the boundary integral stays proportionally large. Nested compactness yields a limiting point where holomorphy gives a linear approximation to f. The integral of the linear part vanishes, while the error term becomes arbitrarily small, forcing the original integral to be zero.*

Proof. Assume, for contradiction, that (5.2) fails for some triangle $T \subset \Omega$ with $\overline{T} \subset \Omega$. Write

$$I(S) := \oint_{\partial S} f(z)\,\mathrm{d}z$$

for a triangle S. Subdivide T into four congruent subtriangles $T^{(1)}, \ldots, T^{(4)}$ by joining the midpoints of the sides. The integrals over the interior edges cancel in pairs, so

$$I(T) = I(T^{(1)}) + I(T^{(2)}) + I(T^{(3)}) + I(T^{(4)}).$$

Hence at least one subtriangle, call it T_1, satisfies

$$|I(T_1)| \geq \frac{1}{4}|I(T)|.$$

Repeating this construction inductively, we obtain nested triangles

$$T \supset T_1 \supset T_2 \supset \cdots$$

with

$$|I(T_n)| \geq 4^{-n}|I(T)|, \qquad \operatorname{diam}(T_n) = 2^{-n}\operatorname{diam}(T).$$

Because the closures $\overline{T_n}$ are nested nonempty compact sets with diameters tending to 0, their intersection consists of a single point $z_0 \in \overline{T} \subset \Omega$.

Since f is complex differentiable at z_0, there exists a function η with $\eta(z) \to 0$ as $z \to z_0$ such that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0). \tag{5.3}$$

Integrating (5.3) over ∂T_n and using

$$\oint_{\partial T_n} 1 \,\mathrm{d}z = 0, \qquad \oint_{\partial T_n} (z - z_0)\,\mathrm{d}z = 0,$$

we obtain

$$I(T_n) = \oint_{\partial T_n} \eta(z)(z - z_0)\,\mathrm{d}z.$$

Let $\varepsilon > 0$. Since $\eta(z) \to 0$ as $z \to z_0$, there exists N such that $|\eta(z)| \leq \varepsilon$ for all $z \in T_n$ whenever $n \geq N$. Therefore, by the ML-estimate,

$$|I(T_n)| \leq \varepsilon \sup_{z \in T_n} |z - z_0|\, L(\partial T_n) \leq C\varepsilon\,(\operatorname{diam} T_n)^2 = C\varepsilon\, 4^{-n}(\operatorname{diam} T)^2,$$

where $C > 0$ is an absolute constant (for example, one may take $C = 3$). Combining this with the lower bound gives, for $n \geq N$,

$$4^{-n}|I(T)| \leq |I(T_n)| \leq C\varepsilon\, 4^{-n}(\operatorname{diam} T)^2.$$

Thus

$$|I(T)| \leq C\varepsilon\,(\operatorname{diam} T)^2.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $I(T) = 0$, a contradiction. Hence (5.2) holds for every such triangle T. □

Theorem 5.7 (Cauchy's integral theorem on simply connected domains)**.** *Let $\Omega \subset \mathbb{C}$ be simply connected and let $f \in \operatorname{Hol}(\Omega)$. Then, for every piecewise C^1 closed curve γ in Ω,*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0. \tag{5.4}$$

Proof. We first treat a positively oriented simple closed polygon Γ whose interior is contained in Ω. Triangulate $\operatorname{int}(\Gamma)$ into finitely many triangles $T_1, \dots, T_N$ whose closures lie in Ω. By Theorem 5.6,

$$\oint_{\partial T_j} f(z)\,\mathrm{d}z = 0 \qquad (j = 1, \dots, N).$$

Summing these identities and observing that every interior edge is traversed twice with opposite orientations, we obtain

$$\oint_{\Gamma} f(z)\,\mathrm{d}z = 0.$$

A general closed polygonal loop in Ω that is null-homotopic can be decomposed into finitely many such simple polygons, so its integral also vanishes.

Now let γ be an arbitrary closed piecewise C^1 curve in Ω. Since Ω is simply connected, γ is null-homotopic in Ω. Approximating a null-homotopy of γ by a sufficiently fine polygonal mesh and applying the preceding polygonal case cell-by-cell (each cell split into two triangles) shows that the integral of f is invariant under such homotopies. Because a constant loop has integral 0, it follows that

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 0.$$

This proves (5.4). ☐

Theorem 5.8 (Deformation of contours)**.** *Let γ_1 and γ_2 be positively oriented simple closed piecewise C^1 curves such that*

$$\overline{\mathrm{int}(\gamma_2)} \subset \mathrm{int}(\gamma_1).$$

Assume that f is holomorphic on an open neighborhood of the compact annular region

$$K := \overline{\mathrm{int}(\gamma_1)} \setminus \mathrm{int}(\gamma_2).$$

Then

$$\oint_{\gamma_1} f(z)\,\mathrm{d}z = \oint_{\gamma_2} f(z)\,\mathrm{d}z.$$

Proof. Choose points $z_j \in \gamma_j$ and connect them by a smooth arc σ contained in K, meeting γ_1 and γ_2 only at its endpoints. Cutting K along σ produces a simply connected region U whose positively oriented boundary consists of γ_1, the reverse of γ_2, and the two sides of the slit σ, which are traversed in opposite directions. By Theorem 5.7,

$$0 = \oint_{\partial U} f(z)\,\mathrm{d}z = \oint_{\gamma_1} f(z)\,\mathrm{d}z - \oint_{\gamma_2} f(z)\,\mathrm{d}z + \int_{\sigma} f(z)\,\mathrm{d}z - \int_{\sigma} f(z)\,\mathrm{d}z.$$

The slit integrals cancel, leaving the desired identity. ☐

5.1.2 Cauchy's Integral Formula and Consequences

Theorem 5.9 (Cauchy's integral formula)**.** *Let $\Omega \subset \mathbb{C}$ be open, $f \in \mathrm{Hol}(\Omega)$, and let γ be a positively oriented simple closed piecewise C^1 curve such that its interior* $\mathrm{int}(\gamma)$ *satisfies* $\overline{\mathrm{int}(\gamma)} \subset \Omega$*. Then for every z in the interior,*

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{5.5}$$

Proof strategy. *Subtract the singular part carrying the value $f(z)$ and study $\frac{f(\zeta)-f(z)}{\zeta-z}$. The numerator cancels the pole, leaving a holomorphic function to which Cauchy's theorem applies. What remains is exactly the contribution of the simple pole at $\zeta = z$.*

Proof. Fix z inside γ. The function $\zeta \mapsto \frac{f(\zeta)-f(z)}{\zeta-z}$ is holomorphic on a neighborhood of $\overline{\operatorname{int}(\gamma)}$, hence its integral over γ vanishes by Theorem 5.7. Expanding,

$$0 = \oint_\gamma \frac{f(\zeta)-f(z)}{\zeta-z}\,\mathrm{d}\zeta = \oint_\gamma \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta - f(z)\oint_\gamma \frac{1}{\zeta-z}\,\mathrm{d}\zeta.$$

The last integral equals $2\pi i$ (it is the integral of $(\log(\zeta-z))'$ along γ), yielding (5.5). □

Corollary 5.10 (Cauchy estimates)**.** *Let $f \in \operatorname{Hol}(\Omega)$ and assume $\overline{D(z_0,R)} \subset \Omega$. If $M := \max_{|\zeta-z_0|=R}|f(\zeta)|$, then for each $n \geq 0$,*

$$|f^{(n)}(z_0)| \leq \frac{n!\,M}{R^n}. \tag{5.6}$$

Proof. Differentiate (5.5) under the integral sign to obtain

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{|\zeta-z_0|=R} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,\mathrm{d}\zeta$$

and apply Lemma 5.3. □

Example 5.11 (A model application of Cauchy's integral formula)**.** If $f(\zeta) = e^\zeta$ and $|z| < 2$, then

$$\frac{1}{2\pi i}\int_{|\zeta|=2} \frac{e^\zeta}{\zeta-z}\,\mathrm{d}\zeta = e^z.$$

Thus the integral formula should be read not merely as an evaluation device, but as a representation theorem: the values of a holomorphic function inside a contour are completely encoded by its boundary values.

Theorem 5.12 (Morera's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be continuous. If*

$$\oint_{\partial T} f(z)\,\mathrm{d}z = 0 \tag{5.7}$$

for every triangle T with $\overline{T} \subset \Omega$, then f is holomorphic on Ω.

Proof strategy. *The vanishing of integrals over triangles implies local path-independence for the integral of f. On a small disk one can therefore define a primitive by integrating from a fixed base point. Once a primitive exists, differentiation shows that the primitive has derivative f, so f must be holomorphic.*

Proof. Fix $z_0 \in \Omega$ and choose $r > 0$ such that $\overline{D(z_0, r)} \subset \Omega$. For $z \in D(z_0, r)$ define

$$F(z) := \int_{\gamma_{z_0,z}} f(\zeta)\,\mathrm{d}\zeta,$$

where $\gamma_{z_0,z}$ is any polygonal path in $D(z_0, r)$ joining z_0 to z. Because integrals over boundaries of triangles vanish, the same is true for polygonal loops obtained by triangulating their interiors; hence F is well defined.

To compute the derivative, fix $z \in D(z_0, r)$ and choose h so small that the segment $[z, z+h]$ lies in $D(z_0, r)$. Then

$$F(z+h) - F(z) = \int_{[z,z+h]} f(\zeta)\,\mathrm{d}\zeta.$$

Parametrizing the segment by $\zeta = z + th$ $(0 \le t \le 1)$, we obtain

$$\frac{F(z+h) - F(z)}{h} = \int_0^1 f(z+th)\,\mathrm{d}t.$$

Therefore

$$\frac{F(z+h) - F(z)}{h} - f(z) = \int_0^1 \big(f(z+th) - f(z)\big)\,\mathrm{d}t,$$

and the right-hand side tends to 0 as $h \to 0$ by continuity of f. Thus F is holomorphic on $D(z_0, r)$ and

$$F'(z) = f(z) \qquad (z \in D(z_0, r)).$$

Since F is holomorphic, Cauchy's integral formula implies that F' is holomorphic on $D(z_0, r)$. Hence $f = F'$ is holomorphic on $D(z_0, r)$. As z_0 was arbitrary, f is holomorphic on all of Ω. ☐

Theorem 5.13 (Locally uniform limits of holomorphic functions)**.** *Let* $\Omega \subset \mathbb{C}$ *be a domain and let* $\{f_n\} \subset \mathrm{Hol}(\Omega)$*. If* $f_n \to f$ *uniformly on compact subsets of* Ω*, then* $f \in \mathrm{Hol}(\Omega)$*.*

Proof. Since locally uniform limits of continuous functions are continuous, f is continuous on Ω. Let T be any triangle with $\overline{T} \subset \Omega$. Then ∂T is compact, so $f_n \to f$ uniformly on ∂T. Hence

$$\oint_{\partial T} f(z)\,\mathrm{d}z = \lim_{n\to\infty} \oint_{\partial T} f_n(z)\,\mathrm{d}z = 0,$$

because each f_n is holomorphic. Morera's theorem now implies that f is holomorphic on Ω. ☐

Proposition 5.14 (Holomorphic dependence under integration)**.** *Let* $\Omega \subset \mathbb{C}$ *be open and let* $h : [a, b] \times \Omega \to \mathbb{C}$ *be continuous. Suppose that for each fixed* $t \in [a, b]$*, the function* $z \mapsto h(t, z)$ *is holomorphic on* Ω*. Define*

$$H(z) := \int_a^b h(t, z)\,\mathrm{d}t, \qquad z \in \Omega.$$

Then $H \in \mathrm{Hol}(\Omega)$*.*

Proof. The continuity of h implies that H is continuous on Ω. Let T be a triangle with $\overline{T} \subset \Omega$. Since h is continuous on the compact set $[a,b] \times \partial T$, Fubini's theorem applies and yields

$$\oint_{\partial T} H(z)\,\mathrm{d}z = \oint_{\partial T} \left(\int_a^b h(t,z)\,\mathrm{d}t \right) \mathrm{d}z = \int_a^b \left(\oint_{\partial T} h(t,z)\,\mathrm{d}z \right) \mathrm{d}t = 0,$$

because for each fixed t the function $z \mapsto h(t,z)$ is holomorphic. Morera's theorem shows that H is holomorphic. ☐

Theorem 5.15 (Liouville's theorem)**.** *If f is entire and bounded, then f is constant.*

Proof. Fix $z_0 \in \mathbb{C}$ and apply the Cauchy estimate (5.6) with $n = 1$ on disks of radius $R \to \infty$. If $|f| \le M$ on $\mathbb{C}$, then $|f'(z_0)| \le M/R$ for all $R > 0$, hence $f'(z_0) = 0$. Since z_0 was arbitrary, $f' \equiv 0$ and f is constant. ☐

Theorem 5.16 (Maximum modulus principle)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in \mathrm{Hol}(\Omega)$ be nonconstant. Then $|f|$ cannot attain a local maximum inside Ω.*

Proof. Assume $|f|$ attains a local maximum at $z_0 \in \Omega$. Choose $r > 0$ with $\overline{D(z_0, r)} \subset \Omega$. By Cauchy's formula (5.5),

$$f(z_0) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0}\,\mathrm{d}\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t.$$

Taking absolute values and using the maximality of $|f(z_0)|$ on the circle gives equality in the triangle inequality. Hence $f(z_0 + re^{it})$ has constant argument for all t, which forces f to be constant on the circle. By the identity theorem (proved in Section 6.3), f is constant on Ω, a contradiction. ☐

Corollary 5.17 (Minimum modulus principle)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in \mathrm{Hol}(\Omega)$ be nonconstant. If f has no zeros in Ω, then $|f|$ cannot attain a local minimum in Ω.*

Proof. If $|f|$ had a local minimum at z_0, then $1/f$ would be holomorphic near z_0 and

$$\left| \frac{1}{f(z)} \right|$$

would attain a local maximum at z_0. By the maximum modulus principle, $1/f$ would be constant, hence f would be constant as well. ☐

CHAPTER 6

Power Series and Analytic Continuation

Power series furnish the local normal form of holomorphic functions. We study their convergence, algebraic operations, termwise differentiation and integration, and Taylor expansions, and we use analytic continuation to explain how local data propagate through connected domains.

Learning objectives.

- Use power series as the local normal form of holomorphic functions.
- Understand radius of convergence, termwise operations, and analytic continuation from the power-series viewpoint.
- Derive fundamental mapping theorems from Taylor expansions and local zero structure.

Chapter roadmap.

- The first sections establish the analytic technology of power series and the geometry of convergence disks.
- Taylor expansion then turns holomorphicity into a genuinely local algebraic object.
- We conclude with identity, open mapping, and zero-set theorems that display the rigidity of analytic continuation.

6.1 Series and Power Series

Power series are the local normal form of holomorphic functions. This section begins with a brief review of convergence for numerical series and then develops the coefficient tests and geometric examples that prepare the way for Taylor expansions, analytic continuation, and Laurent theory.

Let (a_n) be a sequence in $\mathbb{C}$. The series

$$\sum_{n=0}^{\infty} a_n$$

is said to *converge* to $I \in \mathbb{C}$ if its partial sums converge to I; equivalently, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{n=0}^{m} a_n - I \right| < \varepsilon \qquad (m \geq N).$$

An equivalent Cauchy formulation is that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left|\sum_{n=m'}^{m} a_n\right| < \varepsilon \qquad (m, m' \geq N).$$

6.1.1 Series and Convergence

A series $\sum a_n$ is said to be *absolutely convergent* if the numerical series $\sum |a_n|$ converges.

Every absolutely convergent series is convergent. Indeed, if $\sum |a_n|$ converges, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{n=m'}^{m} |a_n| < \varepsilon \qquad (m, m' \geq N).$$

Consequently, whenever $m \geq k \geq N$,

$$\left|\sum_{n=k}^{m} a_n\right| \leq \sum_{n=k}^{m} |a_n| < \varepsilon.$$

Thus the sequence of partial sums is Cauchy in $\mathbb{C}$, and since $\mathbb{C}$ is complete, the series $\sum a_n$ converges.

Lim sup and lim inf. We define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k, \quad \liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k.$$

The following elementary properties will be used repeatedly:

(i) $\liminf a_n \leq \limsup a_n$.

(ii) $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

(iii) $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$.

6.1.2 Power Series

An infinite series of the form $\sum a_n(z-a)^n$ is called a **power series about** a.

The simplest power series is $\sum_{n=0}^{\infty} z^n$.

We can show that:

$$1 - z^{n+1} = (1-z)(1 + z + z^2 + \cdots + z^n)$$

So

$$1 + z + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

If $|z| < 1$, then $\lim_{n\to\infty} z^n = 0$, so

$$|1 + z + \cdots + z^n| \leq 1 + |z| + |z|^2 + \cdots + |z|^n \leq \frac{1}{1 - |z|}.$$

Thus, the series $\sum z^n$ is absolutely convergent, and hence convergent, for $|z| < 1$.

Therefore,

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

and as $n \to \infty$,

$$\left|1 + z + \cdots + z^n - \frac{1}{1-z}\right| = \left|\frac{z^{n+1}}{1-z}\right| \to 0.$$

Theorem 6.1 (Radius of convergence)**.** *Let* $0 \le R \le \infty$ *be defined by*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n},$$

with the conventions $1/\infty = 0$ *and* $1/0 = \infty$*. Then the power series*

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

has the following properties:

(i) it converges absolutely for $|z - a| < R$*;*

(ii) it diverges for $|z - a| > R$*;*

(iii) for every $0 < r < R$*, it converges uniformly on the closed disk* $\overline{D(a, r)}$*.*

Moreover, R *is the unique number with properties* (i) *and* (ii)*.*

Proof. By translating the center, we may assume $a = 0$. Set

$$\rho := \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}.$$

Suppose first that $|z| < R$. Choose r so that $|z| < r < R$. Then $1/r > \rho$. By the definition of the limit superior, there exists N such that

$$|a_n|^{1/n} < \frac{1}{r} \qquad (n \ge N).$$

Hence, for $n \ge N$,

$$|a_n z^n| \le \left(\frac{|z|}{r}\right)^n.$$

Since $|z|/r < 1$, the tail is dominated by a convergent geometric series, so $\sum a_n z^n$ converges absolutely. The same estimate shows more: if $|z| \le s < r < R$, then

$$|a_n z^n| \le \left(\frac{s}{r}\right)^n \qquad (n \ge N),$$

and the Weierstrass M-test yields uniform convergence on $\overline{D(0, s)}$. This proves *(i)* and *(iii)*.

Now suppose $|z| > R$. Choose r with $R < r < |z|$. Then $1/r < \rho$, so by the definition of $\limsup$ there are infinitely many n for which

$$|a_n|^{1/n} > \frac{1}{r}.$$

For those n,

$$|a_n z^n| > \left(\frac{|z|}{r}\right)^n.$$

Because $|z|/r > 1$, these terms do not tend to 0. Therefore the series cannot converge. This proves *(ii)*.

The uniqueness of R is immediate from *(i)* and *(ii)*. □

Remark 6.2 (Behavior on the boundary). The theorem says nothing definitive on the circle $|z - a| = R$; every behavior is possible there. For example,

$$\sum_{n=1}^{\infty} z^n, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

all have radius of convergence 1, but their boundary behavior is quite different.

Proposition 6.3 (Ratio criterion for coefficients)**.** *Assume that* $a_n \neq 0$ *for all sufficiently large* n *and that the limit*

$$L := \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

exists in $[0, \infty]$*. Then the radius of convergence of* $\sum a_n(z-a)^n$ *is* L*.*

Proof. Again assume $a = 0$. If $0 < L < \infty$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{L}.$$

Fix $\varepsilon > 0$. For all sufficiently large n we have

$$\frac{1}{L+\varepsilon} < \left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{L-\varepsilon}.$$

Iterating these inequalities shows that, for large n,

$$C_1(L+\varepsilon)^{-n} \le |a_n| \le C_2(L-\varepsilon)^{-n}$$

for suitable constants $C_1, C_2 > 0$. Taking nth roots and letting $n \to \infty$ gives

$$\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{L},$$

so Theorem 6.1 yields radius L.

If $L = \infty$, then $|a_{n+1}/a_n| \to 0$, hence $|a_n|^{1/n} \to 0$ and the radius is ∞. If $L = 0$, then $|a_{n+1}/a_n| \to \infty$, hence $|a_n|^{1/n} \to \infty$ and the radius is 0. ☐

Proposition 6.4 (Cauchy product of absolutely convergent series)**.** *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be absolutely convergent, and define*

$$c_n := \sum_{k=0}^{n} a_k b_{n-k} \qquad (n \geq 0).$$

Then $\sum_{n=0}^{\infty} c_n$ *converges absolutely and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} |c_n| &\leq \sum_{n=0}^{\infty}\sum_{k=0}^{n} |a_k|\,|b_{n-k}| \\ &= \sum_{k=0}^{\infty} |a_k| \sum_{m=0}^{\infty} |b_m| \\ &= \left(\sum_{k=0}^{\infty} |a_k|\right)\left(\sum_{m=0}^{\infty} |b_m|\right) < \infty.\end{aligned}$$

Thus the Cauchy product is absolutely convergent. Applying the same computation to the partial sums and then passing to the limit gives the identity for the sums. ☐

Proposition 6.5 (Product of power series)**.** *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n(z-a)^n$$

have radii of convergence R_1 *and* R_2*, respectively. Define*

$$c_n := \sum_{k=0}^{n} a_k b_{n-k}.$$

Then the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ *has radius of convergence at least* $\min\{R_1, R_2\}$*, and for every* $|z-a| < \min\{R_1, R_2\}$ *one has*

$$\left(\sum_{n=0}^{\infty} a_n(z-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(z-a)^n\right) = \sum_{n=0}^{\infty} c_n(z-a)^n.$$

Proof. Fix z with $|z-a| < \min\{R_1, R_2\}$. Then both numerical series

$$\sum_{n=0}^{\infty} a_n(z-a)^n, \qquad \sum_{n=0}^{\infty} b_n(z-a)^n$$

are absolutely convergent. The previous proposition therefore applies and yields the claimed product formula. Since this is true for every such z, the Cauchy-product series converges absolutely on that disk, so its radius of convergence is at least $\min\{R_1, R_2\}$. □

Theorem 6.6 (Termwise differentiation and integration of power series)**.** *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

have radius of convergence $R > 0$. Then:

(i) the derived series

$$\sum_{n=1}^{\infty} na_n(z-a)^{n-1}$$

has the same radius of convergence R;

(ii) on every closed disk $\overline{D(a,r)}$ with $0 < r < R$, one may differentiate term by term, and

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1};$$

(iii) for every $k \in \mathbb{N}$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k},$$

and this series also has radius of convergence R;

(iv) the integrated series

$$F(z) := \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}$$

has radius of convergence R and satisfies $F'(z) = f(z)$ on $D(a,R)$.

Proof. Let $b_n := (n+1)a_{n+1}$. Since $(n+1)^{1/n} \to 1$, we have

$$\limsup_{n\to\infty} |b_n|^{1/n} = \limsup_{n\to\infty} (n+1)^{1/n}|a_{n+1}|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Hence the derived series has the same radius of convergence R. The same argument applies after each further differentiation, proving the radius statement in *(iii)*.

Fix r with $0 < r < R$. Then the derivative series converges uniformly on $\overline{D(a,r)}$. Let

$$g(z) := \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}.$$

For the partial sums

$$p_N(z) := \sum_{n=0}^{N} a_n (z-a)^n,$$

we have $p_N'(z) = \sum_{n=1}^{N} n a_n (z-a)^{n-1}$. Since $p_N \to f$ and $p_N' \to g$ uniformly on $\overline{D(a,r)}$, we may integrate along the line segment from a to z and pass to the limit:

$$p_N(z) - p_N(a) = \int_{[a,z]} p_N'(\zeta)\,\mathrm{d}\zeta \quad \Longrightarrow \quad f(z) - f(a) = \int_{[a,z]} g(\zeta)\,\mathrm{d}\zeta.$$

The right-hand side has derivative $g(z)$ by the fundamental theorem for primitives, so $f'(z) = g(z)$ on $D(a,r)$. Since $r < R$ was arbitrary, *(ii)* follows on all of $D(a,R)$. Iterating gives the formula in *(iii)*.

Finally, the coefficients $a_n/(n+1)$ have the same nth-root growth as a_n, so F has radius of convergence R. Differentiating term by term gives

$$F'(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = f(z),$$

which proves *(iv)*. ☐

Corollary 6.7 (Coefficients from derivatives)**.** *If*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

on $D(a,R)$*, then*

$$a_n = \frac{f^{(n)}(a)}{n!} \qquad (n \geq 0).$$

Proof. Set $z = a$ in the formula for $f^{(n)}$ from Theorem 6.6. All terms vanish except the one with index n. ☐

6.1.3 Standard Examples from Power Series

Proposition 6.8 (The exponential series)**.** *The series*

$$\exp z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has radius of convergence ∞ *and therefore defines an entire function satisfying*

$$(\exp z)' = \exp z, \qquad \exp(z+w) = \exp z\, \exp w \qquad (z, w \in \mathbb{C}).$$

In particular, $\exp z \neq 0$ *for all* $z \in \mathbb{C}$ *and* $\exp(-z) = 1/\exp z$*.*

Proof. Since

$$\left|\frac{1/n!}{1/(n+1)!}\right| = n+1 \to \infty,$$

the radius of convergence is ∞. Termwise differentiation therefore gives $(\exp z)' = \exp z$.

For the multiplicative identity, Proposition 6.5 shows that

$$\exp z\, \exp w = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \right).$$

Using the binomial formula,

$$\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \frac{(z+w)^n}{n!},$$

so

$$\exp z\, \exp w = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \exp(z+w).$$

Taking $w = -z$ yields $\exp z\, \exp(-z) = 1$, so $\exp z \neq 0$ for every $z \in \mathbb{C}$. □

Proposition 6.9 (Sine and cosine from power series)**.** *Define*

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Then sin *and* cos *are entire and satisfy*

$$(\sin z)' = \cos z, \qquad (\cos z)' = -\sin z, \qquad \exp(iz) = \cos z + i \sin z.$$

Consequently,

$$\cos^2 z + \sin^2 z = 1 \qquad (z \in \mathbb{C}).$$

Proof. Both series have infinite radius of convergence by the ratio criterion. Termwise differentiation gives the derivative identities. Separating the even and odd powers in the series for $\exp(iz)$ gives Euler's formula

$$\exp(iz) = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \cos z + i \sin z.$$

Replacing z by $-z$ and multiplying the resulting formulas for $\exp(iz)$ and $\exp(-iz)$ gives

$$1 = \exp(iz)\exp(-iz) = (\cos z + i\sin z)(\cos z - i\sin z) = \cos^2 z + \sin^2 z.$$

☐

6.2 Taylor's Theorem and Analytic Continuation

Taylor's theorem shows that holomorphic functions are controlled locally by their derivatives at a single point. Analytic continuation then asks how far such local data can be propagated, thereby introducing one of the main themes of complex analysis: the passage from local expansion to global structure.

Power series are the local normal form of holomorphic functions.

Theorem 6.10 (Taylor expansion)**.** *Let $f \in \mathrm{Hol}(\Omega)$ and assume $\overline{D(z_0, R)} \subset \Omega$. Then for every z with $|z - z_0| < R$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \tag{6.1}$$

where the series converges absolutely and uniformly on compact subsets of $D(z_0, R)$.

Proof strategy. *Start from Cauchy's integral formula on a circle centered at z_0 and expand the kernel $(\zeta - z)^{-1}$ as a geometric series in $(z - z_0)/(\zeta - z_0)$. Uniform convergence on smaller disks justifies termwise integration and reveals the Taylor coefficients as Cauchy integrals of the derivatives.*

Proof. By Cauchy's formula (5.5) applied to the circle $|\zeta - z_0| = R$,

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z_0}\,\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}\,\mathrm{d}\zeta.$$

For $|z - z_0| < R$ we expand $(1 - w)^{-1} = \sum_{n \geq 0} w^n$ with $w = \frac{z - z_0}{\zeta - z_0}$ and interchange sum and integral (uniform convergence on the circle). This yields (6.1), with coefficients given by Cauchy's integral formula for derivatives. ☐

Example 6.11 (Classical Taylor expansions)**.** For $|z| < 1$ one has

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n, \qquad \log(1 + z) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{z^n}{n}.$$

The first is geometric; the second is obtained by integrating the geometric series termwise on compact subsets of the disk. Both expansions illustrate how local analytic information is encoded in the coefficients.

6.2.1 Analytic Continuation

A holomorphic function is determined by its values on any set with an accumulation point.

Theorem 6.12 (Identity theorem). *Let $\Omega \subset \mathbb{C}$ be a domain and let $f, g \in \operatorname{Hol}(\Omega)$. If $f = g$ on a set $E \subset \Omega$ with an accumulation point in Ω, then $f \equiv g$ on Ω.*

Proof. Set $h := f - g \in \operatorname{Hol}(\Omega)$. Assume that h is not identically zero. Let $a \in \Omega$ be an accumulation point of the zero set of h. By Taylor's theorem, there exist an integer $m \geq 0$ and a holomorphic function φ in a neighborhood of a such that

$$h(z) = (z - a)^m \varphi(z), \qquad \varphi(a) \neq 0.$$

If $m = 0$, then $h(a) \neq 0$, contradicting that a is a limit point of zeros of h. If $m \geq 1$, continuity and $\varphi(a) \neq 0$ imply that $\varphi(z) \neq 0$ for all z sufficiently close to a. Hence the only zero of h near a is a itself, again contradicting that a is an accumulation point of distinct zeros. Therefore $h \equiv 0$, so $f \equiv g$ on Ω. □

Remark 6.13. Analytic continuation is the process of extending a holomorphic function along paths by repeatedly applying the uniqueness guaranteed by Theorem 6.12. Branch cuts and Riemann surfaces arise when continuation depends on the chosen path.

6.3 Zeros and Mapping Theorems

Zeros of holomorphic functions are never arbitrary; they reflect the underlying rigidity of the analytic structure. The mapping theorems developed here show that local information about derivatives and zeros has strong global consequences for the image of a domain.

Proposition 6.14 (Zeros are isolated). *Let $\Omega \subset \mathbb{C}$ be a domain and $f \in \operatorname{Hol}(\Omega)$. If f is not identically zero, then every zero of f is isolated. More precisely, if $f(z_0) = 0$, then there exists an integer $m \geq 1$ and a holomorphic function g with $g(z_0) \neq 0$ such that*

$$f(z) = (z - z_0)^m g(z) \quad \text{in a neighborhood of } z_0. \tag{6.2}$$

Proof. Apply the Taylor expansion (6.1) to f at z_0. Let m be the least index for which $f^{(m)}(z_0) \neq 0$. Then (6.2) follows by factoring $(z - z_0)^m$ from the Taylor series, and $g(z_0) = f^{(m)}(z_0)/m! \neq 0$. Consequently, $f(z) \neq 0$ for $0 < |z - z_0|$ sufficiently small. □

Theorem 6.15 (Fundamental theorem of algebra). *Every nonconstant complex polynomial has a zero in $\mathbb{C}$.*

Proof. Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \qquad a_n \neq 0, \quad n \geq 1.$$

Assume, for contradiction, that p has no zero in $\mathbb{C}$. Then

$$g(z) := \frac{1}{p(z)}$$

is entire. Since

$$p(z) = a_n z^n \left(1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_0}{a_n z^n}\right),$$

the bracket tends to 1 as $|z| \to \infty$. Hence there exists $R > 0$ such that

$$|p(z)| \geq \frac{|a_n|}{2}|z|^n, \qquad |z| \geq R.$$

Therefore

$$|g(z)| \leq \frac{2}{|a_n|\,|z|^n}, \qquad |z| \geq R,$$

so g is bounded outside $D(0, R)$. On the compact disk $\overline{D(0,R)}$, the continuous function g is also bounded. Thus g is bounded on all of $\mathbb{C}$, and Liouville's theorem implies that g is constant. Consequently p is constant, a contradiction. ☐

Corollary 6.16 (Polynomials have exactly as many zeros as their degree)**.** *A polynomial of degree $n \geq 1$ has exactly n zeros in $\mathbb{C}$, counted with multiplicity.*

Proof. Proceed by induction on n. The case $n = 1$ is immediate. For $n \geq 2$, Theorem 6.15 yields a zero $a \in \mathbb{C}$, so

$$p(z) = (z - a)q(z)$$

for some polynomial q of degree $n - 1$. By the induction hypothesis, q has exactly $n - 1$ zeros counted with multiplicity. Adding the zero a (with its multiplicity) gives a total of n zeros for p. ☐

Theorem 6.17 (Open mapping theorem)**.** *If $\Omega \subset \mathbb{C}$ is a domain and $f \in \mathrm{Hol}(\Omega)$ is nonconstant, then $f(\Omega)$ is open in $\mathbb{C}$.*

Proof. Fix $z_0 \in \Omega$ and set $w_0 := f(z_0)$. Since f is nonconstant, the function $f - w_0$ is not identically zero. By Theorem 6.14, z_0 is an isolated zero of $f - w_0$. Choose $r > 0$ such that

$$\overline{D(z_0, r)} \subset \Omega \qquad \text{and} \qquad f(z) \neq w_0 \ \text{ for } |z - z_0| = r.$$

Set

$$m := \min_{|z - z_0| = r} |f(z) - w_0| > 0.$$

We claim that every w with $|w - w_0| < m/2$ belongs to $f(D(z_0, r))$. Suppose, to the contrary, that such a w is omitted by f on $D(z_0, r)$. Then

$$g(z) := \frac{1}{f(z) - w}$$

is holomorphic on $D(z_0, r)$ and continuous on $\overline{D(z_0, r)}$. By the maximum modulus principle applied on the disk,

$$|g(z_0)| \leq \max_{|z-z_0|=r} |g(z)|.$$

That is,

$$\frac{1}{|w_0 - w|} \leq \max_{|z-z_0|=r} \frac{1}{|f(z) - w|}.$$

But for $|z - z_0| = r$ we have

$$|f(z) - w| \geq |f(z) - w_0| - |w - w_0| \geq m - \frac{m}{2} = \frac{m}{2},$$

so

$$\max_{|z-z_0|=r} \frac{1}{|f(z) - w|} \leq \frac{2}{m}.$$

Since $|w - w_0| < m/2$, we also have

$$\frac{1}{|w_0 - w|} > \frac{2}{m},$$

a contradiction.

Therefore $D(w_0, m/2) \subset f(D(z_0, r))$. So every point $w_0 = f(z_0)$ is an interior point of $f(\Omega)$, and hence $f(\Omega)$ is open. □

Corollary 6.18 (Inverse function theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open and let $f \in \mathrm{Hol}(\Omega)$ be injective. Then $f(\Omega)$ is open, $f'(z) \neq 0$ for every $z \in \Omega$, and the inverse map*

$$f^{-1} : f(\Omega) \to \Omega$$

is holomorphic. Moreover,

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \qquad (w \in f(\Omega)).$$

Proof. The openness of $f(\Omega)$ is exactly Theorem 6.17. To prove that $f'(z) \neq 0$, assume to the contrary that $f'(z_0) = 0$ for some $z_0 \in \Omega$. By Theorem 6.14, applied to $f - f(z_0)$, there exist an integer $m \geq 2$ and a holomorphic function g with $g(z_0) \neq 0$ such that

$$f(z) - f(z_0) = (z - z_0)^m g(z)$$

in a neighborhood of z_0. After shrinking the neighborhood, choose a disk U centered at z_0 on which g never vanishes. Since U is simply connected, Theorem 4.4 yields a holomorphic function h on U such that

$$h(z)^m = g(z) \qquad (z \in U).$$

Define

$$u(z) := (z - z_0)h(z), \qquad z \in U.$$

Then

$$f(z) - f(z_0) = u(z)^m \qquad (z \in U).$$

Moreover u is nonconstant because $u'(z_0) = h(z_0) \neq 0$. By the open mapping theorem, $u(U)$ contains a disk $D(0, \rho)$ for some $\rho > 0$. Choose $r \in (0, \rho)$ and let

$$\omega_1 = r, \qquad \omega_2 = re^{2\pi i/m}.$$

These are distinct points of $u(U)$, so there exist $z_1, z_2 \in U$ with

$$u(z_1) = \omega_1, \qquad u(z_2) = \omega_2.$$

Since $\omega_1 \neq \omega_2$ we have $z_1 \neq z_2$, but

$$f(z_1) - f(z_0) = \omega_1^m = r^m = \omega_2^m = f(z_2) - f(z_0),$$

contradicting injectivity. Hence $f'(z) \neq 0$ everywhere on Ω.

Now let $w_0 = f(z_0) \in f(\Omega)$ and consider

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} \qquad (w \neq w_0).$$

Writing $z = f^{-1}(w)$ and using injectivity,

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}.$$

As $w \to w_0$ we have $z \to z_0$, because f^{-1} is continuous between the open sets $f(\Omega)$ and Ω. Since $f'(z_0) \neq 0$,

$$\lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)}.$$

Therefore $(f^{-1})'(w_0)$ exists and equals $1/f'(z_0)$, proving holomorphy of the inverse and the derivative formula. □

Remark 6.19. The inverse function theorem is the analytic counterpart of conformality: an injective holomorphic map is locally a biholomorphism, and its derivative never vanishes. This explains why conformal maps preserve angles and local orientation away from critical points.

Chapter 7

Harmonic Functions and Conformal Mappings

This chapter develops the geometric side of holomorphic function theory. Using the analytic machinery built earlier, we study harmonic functions, harmonic conjugates on simply connected domains, mean value phenomena, and the conformal viewpoint, culminating in a systematic discussion of Möbius transformations.

Learning objectives.

- Relate holomorphic functions to harmonic functions through real and imaginary parts.
- Use harmonic conjugates, the mean value property, and the maximum principle as structural tools.
- Interpret nonvanishing derivatives geometrically through conformality.

Chapter roadmap.

- The first half of the chapter studies harmonicity as the real-variable shadow of holomorphy.
- The second half turns to conformal mappings and Möbius transformations as the basic rigid motions of the holomorphic category.
- Throughout, the emphasis is on how local differential information leads to strong geometric conclusions.

7.1 Harmonic Functions and Harmonic Conjugates

Holomorphic functions may be decomposed into real and imaginary parts, and these satisfy the Laplace equation. This section explains that relation and thereby connects complex analysis with potential theory, boundary-value problems, and the geometry of conformal maps.

7.1.1 Holomorphic Functions as Harmonic Pairs

Write $z = x + iy$ and let $f = u + iv$ on a domain $\Omega \subset \mathbb{C}$, with $u, v : \Omega \to \mathbb{R}$. Recall that f is holomorphic on Ω if and only if u, v have continuous first partial derivatives and satisfy the Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x. \tag{7.1}$$

Proposition 7.1 (Harmonicity of real and imaginary parts)**.** *If $f = u + iv$ is holomorphic on Ω and $u, v \in C^2(\Omega)$, then both u and v are harmonic on Ω, i.e.*

$$\Delta u = u_{xx} + u_{yy} = 0, \qquad \Delta v = v_{xx} + v_{yy} = 0.$$

Proof. Differentiate (7.1): from $u_x = v_y$ we obtain $u_{xx} = v_{yx}$, and from $u_y = -v_x$ we obtain $u_{yy} = -v_{xy}$. Since mixed partials agree, we find

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

Similarly, differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x gives

$$v_{yy} = u_{xy}, \qquad v_{xx} = -u_{yx},$$

and therefore

$$\Delta v = v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.$$

Thus both u and v are harmonic on Ω. □

7.1.2 A Rigidity Lemma

Proposition 7.2 (Constant argument forces constancy)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in \mathrm{Hol}(\Omega)$ be nonvanishing. If $\operatorname{Arg} f$ is constant on Ω (equivalently, $f(\Omega)$ is contained in a ray $\{re^{i\alpha} : r > 0\}$), then f is constant.*

Proof. Multiply by $e^{-i\alpha}$ to reduce to the case $f(\Omega) \subset (0, \infty) \subset \mathbb{R}$. Then $\Im f \equiv 0$, hence by the Cauchy–Riemann equations $\Re f$ has zero gradient and is constant on each connected component of Ω, i.e. on Ω. □

Theorem 7.3 (Zero derivative implies constancy)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and $f \in \mathrm{Hol}(\Omega)$. If $f'(z) = 0$ for all $z \in \Omega$, then f is constant on Ω.*

Proof. Fix $z_0 \in \Omega$ and let $z \in \Omega$ be arbitrary. Since Ω is path connected, there exists a piecewise C^1 curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = z_0$ and $\gamma(1) = z$. Choose a partition

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

such that γ is C^1 on each open subinterval (t_{j-1}, t_j). For each j define $g_j := f \circ \gamma$ on $[t_{j-1}, t_j]$. Then, for $t \in (t_{j-1}, t_j)$, the chain rule gives

$$g_j'(t) = f'(\gamma(t))\, \gamma'(t) = 0.$$

Hence g_j is constant on $[t_{j-1}, t_j]$. Because consecutive subintervals meet at the partition points and $f \circ \gamma$ is continuous, these constants agree from one subinterval to the next. Therefore $f \circ \gamma$ is constant on all of $[0, 1]$, and

$$f(z) = f(\gamma(1)) = f(\gamma(0)) = f(z_0).$$

Since $z \in \Omega$ was arbitrary, f is constant on Ω. □

7.1.3 Harmonic Functions

Definition 7.4 (Harmonic function)**.** A function $u : \Omega \to \mathbb{R}$ on an open set $\Omega \subset \mathbb{C} \cong \mathbb{R}^2$ is *harmonic* if $u \in C^2(\Omega)$ and

$$\Delta u := u_{xx} + u_{yy} = 0 \quad \text{on } \Omega.$$

Definition 7.5 (Harmonic conjugate)**.** Let u be harmonic on Ω. A function $v : \Omega \to \mathbb{R}$ is called a *harmonic conjugate* of u if $f := u + iv$ is holomorphic on Ω.

Example 7.6. For $u(x, y) = x^2 - y^2$ one may take $v(x, y) = 2xy$ (up to an additive constant), since $u_x = 2x = v_y$ and $u_y = -2y = -v_x$.

Proposition 7.7 (Uniqueness up to a constant)**.** *If v_1 and v_2 are harmonic conjugates of the same harmonic function u on a domain Ω, then $v_1 - v_2$ is constant on Ω.*

Proof. The difference $(u + iv_1) - (u + iv_2) = i(v_1 - v_2)$ is holomorphic and has zero real part. By the Cauchy–Riemann equations its derivative is 0, hence it is constant by Theorem 7.3. □

7.1.4 Existence of Harmonic Conjugates on Simply Connected Domains

The obstruction to the existence of a global harmonic conjugate is *topological*: on non-simply connected domains the would-be conjugate may become multivalued (the logarithm provides the basic example).

Theorem 7.8 (Existence of harmonic conjugates)**.** *Let $\Omega \subset \mathbb{C}$ be simply connected and let $u \in C^2(\Omega)$ be harmonic. Then there exists $v : \Omega \to \mathbb{R}$ such that $f = u + iv \in \mathrm{Hol}(\Omega)$. Moreover, v is unique up to an additive constant.*

Proof strategy. *The Cauchy–Riemann equations suggest the differential form $\omega = -u_y\, dx + u_x\, dy$. Harmonicity gives the integrability condition $d\omega = 0$, and simple connectivity then upgrades this closed form to an exact one. Integrating ω produces the desired conjugate.*

Proof. Consider the 1-form on Ω

$$\omega := -u_y \, \mathrm{d}x + u_x \, \mathrm{d}y.$$

A direct computation gives

$$\mathrm{d}\omega = (u_{xx} + u_{yy}) \, \mathrm{d}x \wedge \mathrm{d}y = (\Delta u) \, \mathrm{d}x \wedge \mathrm{d}y = 0,$$

so ω is closed. Since Ω is simply connected, every closed 1-form is exact; hence there exists a C^1 function v with $\mathrm{d}v = \omega$, that is,

$$v_x = -u_y, \qquad v_y = u_x.$$

Thus u and v satisfy the Cauchy–Riemann equations, so $f = u + iv$ is holomorphic. Uniqueness up to a constant follows from Theorem 7.7. □

Example 7.9 (Why $\log|z|$ has no global conjugate on $\mathbb{C} \setminus \{0\}$)**.** The function $u(z) = \log|z|$ is harmonic on $\mathbb{C}\setminus\{0\}$. Locally it has a harmonic conjugate, but globally no continuous single-valued conjugate exists on $\mathbb{C} \setminus \{0\}$ because this domain is not simply connected: any conjugate would produce a single-valued holomorphic function $u + iv$ whose exponential equals z on $\mathbb{C} \setminus \{0\}$, contradicting the multivalued nature of the argument. Equivalently, $\oint_{|z|=1} \omega = 2\pi \neq 0$, so ω is not exact on $\mathbb{C} \setminus \{0\}$.

7.1.5 Mean Value Property and Maximum Principle

Theorem 7.10 (Mean value property)**.** *Let u be harmonic on a domain Ω and let $\overline{D(z_0, r)} \subset \Omega$. Then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, \mathrm{d}t.$$

Proof. On the disk $D(z_0, r)$, which is simply connected, Theorem 7.8 yields v such that $f = u+iv$ is holomorphic on $D(z_0, r)$. Applying Cauchy's integral formula to f at z_0 and parametrizing $\zeta = z_0 + re^{it}$ gives

$$f(z_0) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0}\, \mathrm{d}\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, \mathrm{d}t.$$

Taking real parts yields the stated identity. □

Theorem 7.11 (Maximum principle for harmonic functions)**.** *Let Ω be a bounded domain and $u \in C(\overline{\Omega})$ be harmonic on Ω. Then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$. In particular, if u attains its maximum at an interior point, then u is constant.*

Proof. Because $\overline{\Omega}$ is compact and u is continuous on $\overline{\Omega}$, the maximum

$$M := \max_{\overline{\Omega}} u$$

is attained at some point $z_0 \in \overline{\Omega}$. If $z_0 \in \partial\Omega$, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$ and we are done.

Assume now that $z_0 \in \Omega$. Choose $r > 0$ such that $\overline{D(z_0, r)} \subset \Omega$. By Theorem 7.10,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, \mathrm{d}t.$$

Since $u(z_0) = M$ and every value on the circle is $\leq M$, the average can equal M only if

$$u(z_0 + re^{it}) = M \qquad \text{for all } t \in [0, 2\pi].$$

Hence every point of that circle is also a maximum point.

Let

$$E := \{z \in \Omega : \ u(z) = M\}.$$

The set E is closed in Ω by continuity of u. The argument above shows that if $z \in E$, then a small disk around z is contained in E; thus E is also open in Ω. Since $z_0 \in E$ and Ω is connected, we obtain $E = \Omega$. Therefore $u \equiv M$ on Ω, so u is constant.

In particular, unless u is constant, the maximum of u on $\overline{\Omega}$ cannot be attained in the interior. Hence

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

□

Remark 7.12 (Poisson kernel (optional)). If $u \in C(\partial\mathbb{D})$ and u is harmonic on $\mathbb{D}$ with these boundary values, then for $0 \leq r < 1$ and $\theta \in \mathbb{R}$,

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, u(e^{it})\, \mathrm{d}t, \qquad P_r(\phi) := \frac{1 - r^2}{1 - 2r\cos\phi + r^2}.$$

This is the Poisson integral formula, which also implies the mean value property and the harmonic maximum principle.

Remark 7.13. The simply connected hypothesis cannot be dropped. On $\mathbb{C} \setminus \{0\}$ the harmonic function $\log|z|$ is locally the real part of a holomorphic function, but there is no single-valued global conjugate because the candidate argument function winds by 2π around the origin.

7.2 Conformal Mappings

> *One of the most beautiful aspects of complex analysis is its geometric interpretation in terms of angle-preserving maps. Conformal mappings translate analytic information into geometry and provide the natural equivalences between planar domains.*

Complex differentiability imposes strong geometric rigidity. One of the most important manifestations of this rigidity is the fact that holomorphic functions preserve angles whenever the derivative does not vanish.

Definition 7.14 (Conformality at a point). Let $\Omega \subset \mathbb{C}$ be open, and let $f : \Omega \to \mathbb{C}$ be differentiable at $z_0 \in \Omega$ in the complex sense. We say that f is *conformal at* z_0 if it preserves oriented angles between C^1-curves through z_0.

Proposition 7.15 (Holomorphic maps are conformal away from critical points). *Let* $\Omega \subset \mathbb{C}$ *be open and* $f : \Omega \to \mathbb{C}$ *be holomorphic. If* $f'(z_0) \neq 0$*, then* f *is conformal at* z_0*.*

Proof. Fix $z_0 \in \Omega$ with $f'(z_0) \neq 0$. By complex differentiability,

$$f(z_0 + h) = f(z_0) + f'(z_0)\,h + o(h) \qquad (h \to 0). \tag{7.2}$$

Let $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \Omega$ be C^1-curves with $\gamma_j(0) = z_0$ and $\gamma_j'(0) \neq 0$. Applying (7.2) with $h = \gamma_j(t) - z_0$ and dividing by t yields

$$(f \circ \gamma_j)'(0) = f'(z_0)\,\gamma_j'(0).$$

Multiplication by a nonzero complex number is the composition of a rotation and a dilation; hence it preserves oriented angles. Therefore the angle between $(f \circ \gamma_1)'(0)$ and $(f \circ \gamma_2)'(0)$ equals the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$, proving conformality at z_0. □

7.3 Möbius Transformations

> *Möbius transformations form the most explicit and most flexible family of biholomorphic self-maps of the Riemann sphere. They serve simultaneously as examples, as computational tools, and as the basic building blocks for many later normal forms.*

A *Möbius transformation* is a map of the form

$$T(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

It is holomorphic on $\mathbb{C} \setminus \{-d/c\}$ (and extends meromorphically to the Riemann sphere $\widehat{\mathbb{C}}$). A direct calculation gives

$$T'(z) = \frac{ad - bc}{(cz + d)^2}.$$

In particular, $T'(z) \neq 0$ on its domain of holomorphy, so every Möbius transformation is conformal wherever it is holomorphic.

Example 7.16 (Half-plane to disk)**.** The map

$$\phi(z) = \frac{z - i}{z + i}$$

is a Möbius transformation sending the upper half-plane $\{z : \Im z > 0\}$ biholomorphically onto the unit disk $\mathbb{D} = \{w : |w| < 1\}$.

Remark 7.17. Conformal maps are the natural isomorphisms in complex analysis: a bijective holomorphic map with holomorphic inverse is called a *biholomorphism*. By Proposition 7.15 such maps are conformal and, in particular, preserve harmonicity via pullback.

Chapter 8

Schwarz Lemma and Automorphisms of the Disk

The unit disk is the natural testing ground for rigidity in holomorphic mapping theory. In this chapter we prove the Schwarz lemma, classify the automorphisms of the disk, and derive the Schwarz–Pick lemma, thereby obtaining strong metric and geometric control on holomorphic self-maps.

Learning objectives.

- Understand the rigidity of holomorphic self-maps of the unit disk.
- Use normalization by disk automorphisms to convert geometric questions into differential estimates at the origin.
- Recognize Schwarz–Pick as a prototype of negative-curvature behavior in complex analysis.

Chapter roadmap.

- The chapter opens with Schwarz's lemma as a sharp normalization statement for disk self-maps fixing the origin.
- We then transport the result to arbitrary points of the disk using automorphisms.
- The outcome is the Schwarz–Pick lemma, which controls both values and derivatives of disk maps.

8.1 Schwarz Lemma and Disk Self-Maps

> *The unit disk is the canonical laboratory for rigidity phenomena in holomorphic mapping theory. Schwarz's lemma shows that a surprisingly mild normalization at one point imposes strong metric restrictions everywhere else.*

The maximum modulus principle admits a remarkably sharp refinement on the unit disk. Besides being a basic tool in itself, it is the starting point for the intrinsic geometry of the disk (the hyperbolic metric) and for the description of all holomorphic automorphisms of $\mathbb{D}$.

Theorem 8.1 (Schwarz lemma)**.** *Let* $f \in \operatorname{Hol}(\mathbb{D})$ *satisfy* $f(0) = 0$ *and* $|f(z)| \le 1$ *for all* $z \in \mathbb{D}$. *Then*

$$|f(z)| \le |z|, \qquad z \in \mathbb{D},$$

and moreover $|f'(0)| \le 1$*. If equality holds at some point* $z_0 \in \mathbb{D} \setminus \{0\}$*, or if* $|f'(0)| = 1$*, then there exists* $\theta \in \mathbb{R}$ *such that*

$$f(z) = e^{i\theta}\, z, \qquad z \in \mathbb{D}.$$

Proof. Define

$$g(z) = \begin{cases} \dfrac{f(z)}{z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

Since $f(0) = 0$, the function g is holomorphic on $\mathbb{D}$ (removable singularity at 0). Fix $0 < r < 1$ and consider the closed disk $\overline{D(0,r)}$. For $|z| = r$ we have $|g(z)| = |f(z)|/r \le 1/r$, hence by the maximum modulus principle (Theorem 5.16) applied to g on $D(0,r)$,

$$|g(z)| \le \frac{1}{r}, \qquad |z| < r.$$

Letting $r \to 1^-$ gives $|g(z)| \le 1$ for all $z \in \mathbb{D}$, and therefore $|f(z)| = |z|\,|g(z)| \le |z|$. Evaluating at $z = 0$ yields $|f'(0)| = |g(0)| \le 1$.

If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|g(z_0)| = 1$. Since $|g| \le 1$ on $\mathbb{D}$, the maximum modulus principle forces g to be constant, say $g \equiv e^{i\theta}$ with $|e^{i\theta}| = 1$, and hence $f(z) = e^{i\theta} z$. The same conclusion follows if $|f'(0)| = |g(0)| = 1$. ☐

8.1.1 Automorphisms of the Disk

For $a \in \mathbb{D}$ define the Möbius map

$$\phi_a(z) := \frac{z - a}{1 - \overline{a}\, z}, \qquad z \in \mathbb{D}. \tag{8.1}$$

Lemma 8.2. *For each* $a \in \mathbb{D}$*, the map* ϕ_a *is a biholomorphism of* $\mathbb{D}$ *onto itself, with inverse* $\phi_a^{-1} = \phi_{-a}$*. Moreover,*

$$1 - |\phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}\, z|^2}, \qquad z \in \mathbb{D}. \tag{8.2}$$

Proof. A direct computation gives (8.2). In particular, the right-hand side is positive for $z \in \mathbb{D}$, hence $|\phi_a(z)| < 1$ and $\phi_a(\mathbb{D}) \subset \mathbb{D}$. Since the same identity holds with a replaced by $-a$, the map ϕ_{-a} also sends $\mathbb{D}$ into itself. One checks algebraically that $\phi_{-a} \circ \phi_a = \phi_a \circ \phi_{-a} = \mathrm{id}$, so ϕ_a is bijective with holomorphic inverse ϕ_{-a}. ☐

8.1.2 Schwarz–Pick Lemma

Theorem 8.3 (Schwarz–Pick lemma)**.** *Let* $f \in \mathrm{Hol}(\mathbb{D})$ *satisfy* $f(\mathbb{D}) \subset \mathbb{D}$*. Then for all* $z, w \in \mathbb{D}$*,*

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)}\, f(z)} \right| \le \left| \frac{z - w}{1 - \overline{w}\, z} \right|. \tag{8.3}$$

Equivalently,

$$|\phi_{f(w)}(f(z))| \le |\phi_w(z)|, \qquad z, w \in \mathbb{D},$$

where ϕ_a *is defined by* (8.1)*. In particular, for each* $w \in \mathbb{D}$ *we have the differential estimate*

$$|f'(w)| \le \frac{1 - |f(w)|^2}{1 - |w|^2}. \tag{8.4}$$

If equality holds in (8.3) *for one pair* $z \ne w$*, or if equality holds in* (8.4) *at some point* w*, then* f *is an automorphism of* $\mathbb{D}$*.*

Proof strategy. *Normalize both the source and target by disk automorphisms sending* w *and* $f(w)$ *to the origin. The conjugated map fixes* 0*, so Schwarz's lemma applies immediately. Undoing the normalization recovers the invariant estimate.*

Proof. Fix $w \in \mathbb{D}$ and consider the conjugate map

$$F := \phi_{f(w)} \circ f \circ \phi_w^{-1}.$$

By Lemma 8.2, both ϕ_w and $\phi_{f(w)}$ are biholomorphisms of $\mathbb{D}$. Thus $F \in \mathrm{Hol}(\mathbb{D})$ and $F(\mathbb{D}) \subset \mathbb{D}$. Moreover, $F(0) = 0$ because $\phi_w^{-1}(0) = w$ and $\phi_{f(w)}(f(w)) = 0$. Applying Schwarz lemma (Theorem 8.1) gives $|F(\zeta)| \le |\zeta|$ for all $\zeta \in \mathbb{D}$. Taking $\zeta = \phi_w(z)$ yields

$$|\phi_{f(w)}(f(z))| = |F(\phi_w(z))| \le |\phi_w(z)|, \qquad z \in \mathbb{D},$$

which is equivalent to (8.3).

To obtain (8.4), differentiate $F = \phi_{f(w)} \circ f \circ \phi_w^{-1}$ at 0 and use $|F'(0)| \le 1$ from Schwarz lemma. Using $\phi_a'(z) = \dfrac{1 - |a|^2}{(1 - \overline{a}\, z)^2}$, we have $\phi_{f(w)}'(f(w)) = \dfrac{1}{1 - |f(w)|^2}$ and $(\phi_w^{-1})'(0) = 1 - |w|^2$. The chain rule gives

$$|F'(0)| = \frac{|f'(w)|(1 - |w|^2)}{1 - |f(w)|^2} \le 1,$$

which is exactly (8.4).

Finally, if equality occurs in (8.3) for some $z \ne w$, then equality occurs in Schwarz lemma for the map F at the point $\zeta = \phi_w(z) \ne 0$, forcing $F(\zeta) = e^{i\theta}\zeta$ for some $\theta \in \mathbb{R}$. Equivalently, f is a disk automorphism. The same conclusion follows from equality in (8.4) since that implies $|F'(0)| = 1$. ☐

Corollary 8.4 (Automorphisms of the disk)**.** *A map* $f : \mathbb{D} \to \mathbb{D}$ *is biholomorphic if and only if it has the form*

$$f(z) = e^{i\theta}\, \phi_a(z) = e^{i\theta}\, \frac{z - a}{1 - \overline{a}\, z}$$

for some $a \in \mathbb{D}$ *and* $\theta \in \mathbb{R}$*.*

Proof. Suppose first that f is a biholomorphism of $\mathbb{D}$. Let $a := f^{-1}(0) \in \mathbb{D}$ and set $g := f \circ \phi_a^{-1}$. Then g is a biholomorphism of $\mathbb{D}$ and $g(0) = 0$. Applying Schwarz lemma (Theorem 8.1) to g gives $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Applying the same lemma to g^{-1} yields $|z| \leq |g(z)|$. Hence equality holds, and the rigidity statement in Theorem 8.1 implies $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. Therefore $f(z) = g(\phi_a(z)) = e^{i\theta}\phi_a(z)$. Conversely, each map $z \mapsto e^{i\theta}\phi_a(z)$ is a composition of biholomorphisms of $\mathbb{D}$, hence a biholomorphism. □

Remark 8.5 (Hyperbolic contraction)*.* Inequality (8.4) can be rewritten as

$$(1 - |z|^2)\,|f'(z)| \leq 1 - |f(z)|^2, \qquad z \in \mathbb{D},$$

which expresses that holomorphic self-maps of $\mathbb{D}$ are contractions for the Poincaré metric on the disk.

Chapter 9

Laurent Series and Isolated Singularities

Near an isolated singularity, power series are no longer sufficient; Laurent series provide the correct local model. We develop Laurent expansions, classify isolated singularities, and interpret poles and essential singularities as the first genuinely meromorphic phenomena, including behavior at infinity.

Learning objectives.

- Extend the power-series viewpoint from disks to annuli.
- Classify isolated singularities in terms of the principal part of a Laurent expansion.
- Understand how local expansions detect removable singularities, poles, and essential singularities.

Chapter roadmap.

- The chapter begins with Laurent's theorem, which is the natural annular analogue of Taylor's theorem.
- We then interpret the coefficients of the negative powers as precise local obstruction data.
- The singularity classification culminates in the first strong picture of meromorphic behavior.

9.1 Laurent Series and Singularities

Laurent series extend the power-series point of view to punctured domains, where negative powers record the behavior near singular points. This expansion is the natural language for classifying isolated singularities and for preparing residue calculus.

9.1.1 Laurent Expansion

Let $A = \{z : r < |z - z_0| < R\}$ be an annulus. If f is holomorphic on A, then it admits a convergent expansion in powers of $(z - z_0)$ and $(z - z_0)^{-1}$.

Theorem 9.1 (Laurent's theorem). *Let f be holomorphic on $A = \{z : r < |z - z_0| < R\}$. Then for each $z \in A$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{9.1}$$

where the coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta, \qquad r < \rho < R. \tag{9.2}$$

The series (9.1) *converges absolutely and uniformly on compact subsets of A.*

Proof strategy. *Apply Cauchy's integral formula on a region bounded by two circles and decompose the kernel into geometric series valid on the two sides of the point z. The outer boundary contributes nonnegative powers and the inner boundary contributes negative powers, reflecting the two-sided geometry of an annulus.*

Proof. Fix $z \in A$ and choose radii ρ_1, ρ_2 such that

$$r < \rho_1 < |z - z_0| < \rho_2 < R.$$

Apply Cauchy's integral formula to f on the annulus bounded by the circles $|\zeta - z_0| = \rho_1$ and $|\zeta - z_0| = \rho_2$. With the positive orientation on the outer boundary and the negative orientation on the inner boundary, we obtain

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho_2} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho_1} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

For $|\zeta - z_0| = \rho_2$ we have $|z - z_0| < |\zeta - z_0|$, so

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

For $|\zeta - z_0| = \rho_1$ we have $|\zeta - z_0| < |z - z_0|$, so

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0} \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}}.$$

Substituting these series into the two contour integrals and integrating term by term yields

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n},$$

where

$$a_n = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta \qquad (n \in \mathbb{Z},\ r < \rho < R).$$

This is exactly (9.1) with coefficients (9.2). The convergence is absolute and uniform on compact subannuli because the geometric-series expansions above are uniform there. ☐

9.1.2 Classification of Isolated Singularities

Let Ω be open and let $z_0 \in \Omega$. Suppose f is holomorphic on $\Omega \setminus \{z_0\}$. The Laurent expansion (9.1) at z_0 allows one to classify the behavior of f near z_0:

- z_0 is *removable* if $a_n = 0$ for all $n < 0$.
- z_0 is a *pole of order* m if $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$.
- z_0 is *essential* if infinitely many negative-index coefficients are nonzero.

Example 9.2 (A basic Laurent expansion)**.** On the annulus $0 < |z| < 1$,

$$\frac{1}{z(1-z)} = \frac{1}{z} + \sum_{n=0}^{\infty} z^n.$$

The principal part consists of the single term z^{-1}, which records the simple pole at the origin, while the holomorphic part is the ordinary geometric series. This is the local model behind many residue computations.

Theorem 9.3 (Riemann's removable singularity theorem)**.** *Let f be holomorphic on $\Omega \setminus \{z_0\}$, where Ω is a neighborhood of z_0. If f is bounded in a punctured neighborhood of z_0, then z_0 is a removable singularity of f, i.e. f extends holomorphically to Ω.*

Proof. Let $\overline{D(z_0, r)} \subset \Omega$ and assume $|f(z)| \leq M$ for $0 < |z - z_0| < r$. For $0 < \rho < r$, Cauchy's integral formula on $|z - z_0| = \rho$ gives the Laurent coefficients

$$a_{-n} = \frac{1}{2\pi i} \oint_{|z-z_0|=\rho} f(z)(z-z_0)^{n-1}\,\mathrm{d}z \qquad (n \geq 1).$$

Estimating the integral yields

$$|a_{-n}| \leq \frac{1}{2\pi}\,(2\pi\rho)\,M\,\rho^{n-1} = M\rho^n,$$

hence $a_{-n} = 0$ after letting $\rho \downarrow 0$. Thus the principal part vanishes and the singularity is removable. ☐

Corollary 9.4 (A useful limit criterion for removability)**.** *Let f be holomorphic on $\Omega \setminus \{z_0\}$. If*

$$\lim_{z \to z_0} (z - z_0) f(z) = 0,$$

then z_0 is a removable singularity of f. In particular, if $\lim_{z \to z_0} f(z)$ exists in $\mathbb{C}$, then the singularity is removable.

Proof. Choose $r > 0$ so that $\overline{D(z_0, r)} \subset \Omega$. By hypothesis there exists $\delta \in (0, r)$ such that

$$|(z - z_0) f(z)| \leq 1 \qquad (0 < |z - z_0| < \delta).$$

Hence on the smaller punctured disk $0 < |z - z_0| < \delta/2$ we have

$$|f(z)| \le \frac{1}{|z - z_0|} \le \frac{2}{\delta}.$$

Thus f is bounded in a punctured neighborhood of z_0, and Theorem 9.3 applies. If $\lim_{z \to z_0} f(z)$ exists, then multiplying by $z - z_0 \to 0$ gives the displayed limit automatically. □

Theorem 9.5 (Characterization of poles)**.** *Let f be holomorphic on $\Omega \setminus \{z_0\}$. Then z_0 is a pole of order m if and only if $(z - z_0)^m f(z)$ extends holomorphically to Ω and is nonzero at z_0. Equivalently, z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. If z_0 is a pole of order m, then the Laurent expansion of f at z_0 has the form

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + a_0 + a_1(z - z_0) + \cdots, \qquad a_{-m} \neq 0.$$

Multiplying by $(z - z_0)^m$ gives a holomorphic function near z_0 whose value at z_0 is $a_{-m} \neq 0$.

Conversely, assume that

$$h(z) := (z - z_0)^m f(z)$$

extends holomorphically to Ω and that $h(z_0) \neq 0$. Then h is nonvanishing in some neighborhood of z_0, so

$$f(z) = \frac{h(z)}{(z - z_0)^m}$$

has a pole of order exactly m at z_0.

Now suppose that z_0 is a pole. Then $h(z) = (z - z_0)^m f(z) \to h(z_0) \neq 0$, so for z near z_0 we have $|h(z)| \ge c > 0$. Hence

$$|f(z)| = \frac{|h(z)|}{|z - z_0|^m} \ge \frac{c}{|z - z_0|^m} \to \infty \qquad (z \to z_0).$$

Conversely, if $|f(z)| \to \infty$ as $z \to z_0$, then

$$g(z) := \frac{1}{f(z)}$$

satisfies $g(z) \to 0$ as $z \to z_0$. Thus g is bounded near z_0, so by Theorem 9.3 it extends holomorphically across z_0 with $g(z_0) = 0$. If the extension were identically zero near z_0, then by the identity theorem g would vanish on a punctured neighborhood, impossible. Therefore g has a zero of some finite order $m \ge 1$ at z_0:

$$g(z) = (z - z_0)^m \psi(z), \qquad \psi(z_0) \neq 0.$$

Hence

$$f(z) = \frac{1}{(z - z_0)^m \psi(z)},$$

so f has a pole of order m at z_0. ☐

Remark 9.6 (Meromorphic functions). A function is called *meromorphic* on a domain if it is holomorphic except at isolated poles. Meromorphic functions behave like rational functions locally and are the natural class for the argument principle and residue calculus.

9.1.3 Singularities at Infinity (Optional)

The point at infinity is often treated exactly like an isolated singularity after the substitution $\zeta = 1/z$.

Definition 9.7. Let f be holomorphic for $|z| > R$ for some $R > 0$. We say that f has an *isolated singularity at infinity* if the function

$$g(\zeta) := f(1/\zeta)$$

is holomorphic for $0 < |\zeta| < 1/R$. The singularity of f at ∞ is called removable, a pole, or essential according as the singularity of g at 0 has the corresponding type.

Proposition 9.8 (Entire functions and the point at infinity)**.** *Let f be entire. Then the singularity of f at ∞ is removable if and only if f is constant, it is a pole if and only if f is a polynomial, and it is essential if and only if f is a non-polynomial entire function.*

Proof. Write the Taylor expansion of f at the origin as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then for $\zeta \neq 0$,

$$g(\zeta) = f(1/\zeta) = \sum_{n=0}^{\infty} a_n \zeta^{-n},$$

which is the Laurent expansion of g about 0. The singularity at 0 is removable exactly when all negative-power coefficients vanish, that is, when $a_n = 0$ for every $n \geq 1$, so f is constant. It is a pole exactly when only finitely many negative-power coefficients are nonzero, which means precisely that only finitely many a_n are nonzero, i.e. f is a polynomial. The remaining case is therefore the essential one. ☐

Remark 9.9. This viewpoint packages several classical facts elegantly: a nonconstant polynomial maps $\mathbb{C}$ onto $\mathbb{C}$, while a non-polynomial entire function has an essential singularity at infinity and is therefore governed by the dramatic value-distribution behavior associated with essential singularities.

Theorem 9.10 (Casorati–Weierstrass)**.** *If z_0 is an essential singularity of f, then f maps every punctured neighborhood of z_0 densely into $\mathbb{C}$.*

Proof. Let U be a punctured disk about z_0. If $f(U)$ were not dense, there would exist $w_0 \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - w_0| \geq \delta$ on U. Then $g(z) := \frac{1}{f(z)-w_0}$ is holomorphic and bounded on U. The singularity of g at z_0 is removable, so g extends holomorphically across z_0, which implies that f has at most a pole at z_0, contradicting essentiality. $\square$

Chapter 10

Argument Principle and Residue Calculus

This chapter passes from local singular behavior to global counting principles. The argument principle and Rouché's theorem relate zeros and poles to winding numbers, while residues and the residue theorem provide the decisive computational instrument for meromorphic contour integrals.

Learning objectives.

- Compute contour integrals by extracting local coefficient data.
- Understand the argument principle as a bridge between integration and zero-counting.
- Use Rouché's theorem to obtain robust perturbative information about zeros.

Chapter roadmap.

- Residues compress the local behavior of a meromorphic function into a single coefficient.
- The residue theorem converts global contour integrals into finite local data.
- This framework then yields zero-counting tools such as the argument principle and Rouché's theorem.

10.1 Argument Principle and Rouché's Theorem

A recurring theme in complex analysis is that information on the boundary controls what happens inside. The argument principle and Rouché's theorem turn contour integrals into effective tools for counting zeros and comparing analytic functions.

10.1.1 Residues

If f has a Laurent expansion (9.1) about z_0, the coefficient a_{-1} is called the *residue* of f at z_0 and is denoted by $\mathrm{Res}(f; z_0)$.

10.1.2 Computing Residues

In practice one rarely computes a full Laurent series. The following formulas cover the most common situations.

Proposition 10.1 (Simple poles)**.** *Let f have a simple pole at a. Then*

$$\mathrm{Res}(f; a) = \lim_{z \to a} (z - a) f(z).$$

If $f = g/h$ with g, h holomorphic near a, $h(a) = 0$, and $h'(a) \neq 0$, then a is a simple pole of f and

$$\operatorname{Res}\left(\frac{g}{h}; a\right) = \frac{g(a)}{h'(a)}.$$

Proof. For a simple pole, $f(z) = \frac{c_{-1}}{z-a} + h_0(z)$ with h_0 holomorphic, hence $(z-a)f(z) \to c_{-1} = \operatorname{Res}(f; a)$. For g/h, write $h(z) = (z-a)h_1(z)$ with $h_1(a) = h'(a) \neq 0$, so $\frac{g}{h} = \frac{g(z)}{(z-a)h_1(z)}$ and the residue equals $g(a)/h_1(a) = g(a)/h'(a)$. ☐

Proposition 10.2 (Poles of order m)**.** *If f has a pole of order $m \geq 1$ at a, then*

$$\operatorname{Res}(f; a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\Big((z-a)^m f(z)\Big).$$

Proof. Set

$$h(z) := (z-a)^m f(z).$$

Then h is holomorphic near a, and

$$f(z) = \frac{h(z)}{(z-a)^m}.$$

Expand h in its Taylor series at a:

$$h(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!}(z-a)^n.$$

Dividing by $(z-a)^m$ gives the Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!}(z-a)^{n-m}.$$

The residue is therefore the coefficient of $(z-a)^{-1}$, namely the term with $n = m-1$:

$$\operatorname{Res}(f; a) = \frac{h^{(m-1)}(a)}{(m-1)!}.$$

Since h is holomorphic, this is exactly

$$\operatorname{Res}(f; a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\Big((z-a)^m f(z)\Big).$$

☐

Example 10.3. Let $f(z) = \dfrac{e^z}{z^2(z-1)}$. Then $z = 0$ is a pole of order 2 and

$$\operatorname{Res}(f; 0) = \lim_{z \to 0} \frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{e^z}{z-1}\Big) = \frac{e^z(z-1) - e^z}{(z-1)^2}\bigg|_{z=0} = -2.$$

Also $z = 1$ is a simple pole, and $\operatorname{Res}(f; 1) = e$.

Remark 10.4 (Residue at infinity (optional)). For meromorphic f on the Riemann sphere, the residue at infinity is defined by

$$\operatorname{Res}(f; \infty) := -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right),$$

and satisfies $\sum_{a \in \widehat{\mathbb{C}}} \operatorname{Res}(f; a) = 0$ (sum over all poles, including ∞). This is convenient for rational integrals.

Theorem 10.5 (Residue theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let γ be a positively oriented simple closed piecewise C^1 curve with $\overline{\operatorname{int}(\gamma)} \subset \Omega$. If f is holomorphic on Ω except for finitely many isolated singularities $a_1, \dots, a_N$ in the interior of γ, then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f; a_k). \tag{10.1}$$

Proof strategy. *Remove small circles around the singularities and apply Cauchy's theorem on the punctured region. The outer contour is then homologous to the sum of the small inner circles. Each local integral is read off from the Laurent expansion, and only the coefficient of $(z-a)^{-1}$ survives.*

Proof. For each k, let

$$P_k(z) = \sum_{m=1}^{M_k} c_{k,m} (z - a_k)^{-m}$$

be the principal part of the Laurent expansion of f at a_k. Then

$$g(z) := f(z) - \sum_{k=1}^{N} P_k(z)$$

is holomorphic on a neighborhood of $\overline{\operatorname{int}(\gamma)}$, so Cauchy's theorem gives

$$\oint_\gamma g(z)\,\mathrm{d}z = 0.$$

Hence

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{N} \oint_\gamma P_k(z)\,\mathrm{d}z.$$

Now each term of P_k except $c_{k,1}(z-a_k)^{-1}$ has primitive on the punctured neighborhood of a_k and therefore contributes 0 to the contour integral. The term $c_{k,1}(z-a_k)^{-1}$ contributes $2\pi i\, c_{k,1}$. Since $c_{k,1} = \operatorname{Res}(f; a_k)$, summing over k yields

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f; a_k),$$

which is (10.1). □

Remark 10.6. The residue theorem should be viewed as a localization principle: once the contour is fixed, only the coefficients of $(z-a)^{-1}$ at the enclosed singularities survive. All higher-order local data become invisible to the contour integral.

Theorem 10.7 (Argument principle)**.** *Let Ω be a domain and γ as in Theorem 10.5. Suppose f is meromorphic on a neighborhood of $\overline{\operatorname{int}(\gamma)}$ and has no zeros or poles on γ. Then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = N_0 - N_\infty, \tag{10.2}$$

where N_0 and N_∞ are the numbers of zeros and poles of f in $\operatorname{int}(\gamma)$, counted with multiplicity.

Proof strategy. *Factor f near each zero or pole as a power of $(z-a)$ times a holomorphic non-vanishing factor. The logarithmic derivative therefore splits into a simple pole plus a holomorphic term. The residue theorem converts those local multiplicities into the global contour integral.*

Proof. The function f'/f is meromorphic on a neighborhood of $\overline{\operatorname{int}(\gamma)}$, and its singularities occur precisely at the zeros and poles of f.

If a is a zero of order m, then

$$f(z) = (z-a)^m g(z)$$

with g holomorphic and $g(a) \neq 0$. Therefore

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)},$$

and the residue at a is m. If a is a pole of order m, then

$$f(z) = (z-a)^{-m} g(z)$$

with g holomorphic and nonvanishing at a, so

$$\frac{f'(z)}{f(z)} = -\frac{m}{z-a} + \frac{g'(z)}{g(z)},$$

and the residue at a is $-m$.

Applying the residue theorem to f'/f, we obtain

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{a\in\operatorname{int}(\gamma)} \operatorname{Res}\Big(\frac{f'}{f};a\Big) = N_0 - N_\infty,$$

where zeros and poles are counted with multiplicity. This proves (10.2). □

Theorem 10.8 (Rouché's theorem)**.** *Let γ be a positively oriented simple closed curve and let f, g be holomorphic on a neighborhood of $\overline{\operatorname{int}(\gamma)}$. If*

$$|g(z)| < |f(z)| \quad \textit{for all } z \in \gamma, \tag{10.3}$$

then f and $f+g$ have the same number of zeros in $\operatorname{int}(\gamma)$, counted with multiplicity.

Proof strategy. *Study the homotopy $f+tg$ for $0 \le t \le 1$. The boundary inequality prevents any member of the family from vanishing on the contour, so the logarithmic derivative varies continuously throughout the deformation. The argument principle then shows that the number of zeros inside cannot change with t.*

Proof. For $z \in \gamma$ we have $|g(z)| < |f(z)|$, so in particular $f(z) \neq 0$ and

$$\left|\frac{g(z)}{f(z)}\right| < 1.$$

Hence

$$\frac{f(z)+g(z)}{f(z)} = 1 + \frac{g(z)}{f(z)}$$

never vanishes on γ. Therefore $f+g$ also has no zeros on γ, and the quotient $(f+g)/f$ admits a holomorphic logarithm on a neighborhood of γ.

Applying the argument principle to f and to $f+g$, and subtracting, we find that the difference in the numbers of zeros inside γ is

$$\frac{1}{2\pi i}\oint_\gamma \left(\frac{(f+g)'(z)}{f(z)+g(z)} - \frac{f'(z)}{f(z)}\right)\mathrm{d}z.$$

Since

$$\left(\log\frac{f+g}{f}\right)' = \frac{(f+g)'}{f+g} - \frac{f'}{f},$$

this difference equals

$$\frac{1}{2\pi i}\oint_\gamma \left(\log\frac{f+g}{f}\right)'\mathrm{d}z = 0.$$

Thus f and $f+g$ have the same number of zeros in $\operatorname{int}(\gamma)$, counted with multiplicity. □

Example 10.9 (A standard use of Rouché's theorem)**.** Consider the polynomial

$$p(z) = z^5 + 3z + 1.$$

On the circle $|z| = 2$ we obtain

$$|3z + 1| \leq 7 < 32 = |z^5|.$$

Hence p and z^5 have the same number of zeros in $|z| < 2$, namely five. Similarly, on $|z| = \frac{1}{2}$ one has

$$|z^5 + 3z| \leq \frac{1}{32} + \frac{3}{2} < 1 = |1|,$$

so p and the constant function 1 have the same number of zeros in $|z| < \frac{1}{2}$, namely none. Therefore all five zeros of p lie in the annulus

$$\frac{1}{2} < |z| < 2.$$

This example illustrates the typical use of Rouché's theorem: one chooses different comparison functions on different contours to localize zeros efficiently.

Chapter 11

Applications of Residues to Definite and Principal Value Integrals

The final chapter shows how the abstract machinery of residues translates into concrete evaluation methods for real-variable problems. We treat rational and trigonometric integrals, keyhole-contour arguments, Fourier-type integrals, Jordan's lemma, and Cauchy principal values in a unified framework.

Learning objectives.

- Translate real-variable integrals into contour integrals suited to residue methods.
- Choose contours and growth estimates adapted to the integrand at hand.
- See residue calculus functioning as a computational method rather than only as a structural theorem.

Chapter roadmap.

- We review the contour templates that arise most often in practice.
- The main examples illustrate how symmetry, exponential damping, and pole structure guide the choice of contour.
- The goal is to make residue calculus a flexible computational tool for oscillatory and rational integrals.

11.1 Applications of the Residue Theorem to Real Integrals

Residues are not only a structural theorem of complex analysis but also a remarkably efficient computational device. The purpose of this section is to show how carefully chosen contours convert difficult real integrals into manageable questions about poles and residues.

We record a few standard templates illustrating how Theorem 10.5 yields real integral evaluations.

11.1.1 Rational Integrals on the Real Line

Let P, Q be polynomials with $\deg Q \geq \deg P + 2$ and assume Q has no real zeros. Consider

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x.$$

One applies the residue theorem to $f(z) = \frac{P(z)}{Q(z)}$ on a semicircular contour in the upper half-plane. The decay condition implies that the arc contribution tends to 0 as the radius tends to ∞, giving

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x = 2\pi i \sum_{\substack{a:\ Q(a)=0 \\ \Im a>0}} \operatorname{Res}\!\left(\frac{P}{Q}; a\right). \tag{11.1}$$

Example 11.1. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1}$.

Proof. Here $f(z) = \frac{1}{z^2+1}$ has a simple pole at $z = i$ in the upper half-plane with residue

$$\operatorname{Res}(f; i) = \lim_{z\to i} \frac{z-i}{(z-i)(z+i)} = \frac{1}{2i}.$$

By (11.1), the integral equals $2\pi i \cdot \frac{1}{2i} = \pi$. ☐

11.1.2 Trigonometric Integrals

Many integrals over $[0, 2\pi]$ can be reduced to contour integrals on the unit circle via the substitution $z = e^{it}$, so that $\mathrm{d}t = \frac{1}{iz}\mathrm{d}z$ and $\cos t = \frac{1}{2}(z + z^{-1})$, $\sin t = \frac{1}{2i}(z - z^{-1})$.

Example 11.2. Evaluate $\displaystyle\int_{0}^{2\pi} \frac{\mathrm{d}t}{2+\cos t}$.

Proof. With $z = e^{it}$, one obtains

$$\int_0^{2\pi} \frac{\mathrm{d}t}{2+\cos t} = \oint_{|z|=1} \frac{1}{2+\frac{1}{2}(z+z^{-1})}\,\frac{1}{iz}\,\mathrm{d}z = \oint_{|z|=1} \frac{2}{i(z^2+4z+1)}\,\mathrm{d}z.$$

The poles are at $z = -2 \pm \sqrt{3}$; only $z_0 = -2 + \sqrt{3}$ lies inside $|z| < 1$. The residue at z_0 is

$$\operatorname{Res}\!\left(\frac{2}{i(z^2+4z+1)}; z_0\right) = \frac{2}{i(2z_0+4)} = \frac{1}{i\sqrt{3}}.$$

Hence the integral equals $2\pi i \cdot \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$. ☐

11.1.3 Keyhole Contour and Branch Cuts

Integrals involving powers x^α (non-integer α) naturally lead to branch cuts and *keyhole contours*. A standard prototype is Euler's beta integral.

Theorem 11.3 (Euler integral via residues)**.** *Let* $0 < \alpha < 1$. *Then*

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin(\pi\alpha)}.$$

Proof. Consider $f(z) = \dfrac{z^{\alpha-1}}{1+z}$, where $z^{\alpha-1} = e^{(\alpha-1)\operatorname{Log} z}$ uses a branch of Log with a cut along the positive real axis, so that crossing the cut multiplies $z^{\alpha-1}$ by $e^{2\pi i(\alpha-1)}$. Integrate f over a keyhole contour encircling $(0,\infty)$ once counterclockwise. The only pole inside is at $z=-1$, with residue $\operatorname{Res}(f;-1)=(-1)^{\alpha-1}=-e^{i\pi(\alpha-1)}$. The large and small circular arcs vanish in the limit. The contributions from the two sides of the cut differ by the factor $e^{2\pi i(\alpha-1)}$, yielding

$$\left(1-e^{2\pi i(\alpha-1)}\right)\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x = 2\pi i\ \operatorname{Res}(f;-1) = 2\pi i\bigl(-e^{i\pi(\alpha-1)}\bigr).$$

A short simplification gives the stated value. □

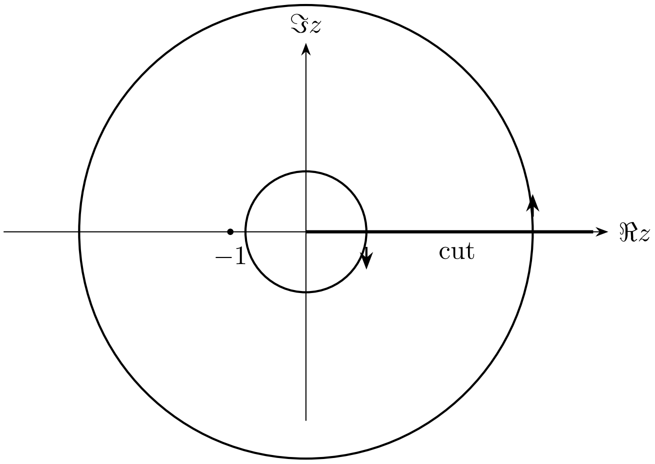

Figure 11.1: *A keyhole contour around the branch cut* $(0,\infty)$.

11.1.4 Jordan's Lemma and Fourier-Type Integrals

When integrands contain e^{iaz} with $a>0$, semicircular contours in the upper half-plane often work because e^{iaz} decays on the upper semicircle. The following estimate packages this observation.

Lemma 11.4 (Jordan's lemma)**.** *Let $a>0$ and let g be continuous on $\{Re^{it}: 0\le t\le\pi\}$ for all large R, with $|g(z)|\le M/R$ on the semicircle $|z|=R$, $\Im z\ge 0$. Then*

$$\lim_{R\to\infty}\int_{|z|=R,\ \Im z\ge 0} e^{iaz} g(z)\,\mathrm{d}z = 0.$$

Proof. Parametrize the upper semicircle by

$$z = Re^{it}, \qquad 0\le t\le\pi,$$

so that $\mathrm{d}z = iRe^{it}\,\mathrm{d}t$. Then

$$\left|\int_{|z|=R,\ \Im z\ge 0} e^{iaz} g(z)\,\mathrm{d}z\right| \le \int_0^\pi \bigl|e^{iaRe^{it}}\bigr|\,|g(Re^{it})|\,R\,\mathrm{d}t.$$

Now

$$\bigl|e^{iaRe^{it}}\bigr| = e^{-aR\sin t},$$

and by hypothesis $|g(Re^{it})| \le M/R$. Hence

$$\left|\int_{|z|=R,\ \Im z\ge 0} e^{iaz} g(z)\,\mathrm{d}z\right| \le M\int_0^{\pi} e^{-aR\sin t}\,\mathrm{d}t.$$

Using symmetry about $\pi/2$,

$$\int_0^{\pi} e^{-aR\sin t}\,\mathrm{d}t = 2\int_0^{\pi/2} e^{-aR\sin t}\,\mathrm{d}t.$$

For $0 \le t \le \pi/2$ we have $\sin t \ge \frac{2}{\pi}t$, so

$$2\int_0^{\pi/2} e^{-aR\sin t}\,\mathrm{d}t \le 2\int_0^{\pi/2} e^{-(2aR/\pi)t}\,\mathrm{d}t = \frac{\pi}{aR}\Big(1-e^{-aR}\Big).$$

Therefore

$$\left|\int_{|z|=R,\ \Im z\ge 0} e^{iaz} g(z)\,\mathrm{d}z\right| \le \frac{\pi M}{aR}\Big(1-e^{-aR}\Big) \xrightarrow[R\to\infty]{} 0.$$

This proves the lemma. □

Example 11.5 (A basic Fourier integral)**.** For $a \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,\mathrm{d}x = \pi e^{-|a|}.$$

Consequently, for $a \ge 0$,

$$\int_0^{\infty} \frac{\cos(ax)}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2}e^{-a}.$$

Proof. Assume $a > 0$ (the case $a < 0$ is analogous with the lower half-plane). Apply the residue theorem to $f(z) = \dfrac{e^{iaz}}{z^2+1}$ on the upper semicircle. By Jordan's lemma, the arc contribution vanishes as $R \to \infty$. The only pole in the upper half-plane is $z = i$, with residue

$$\mathrm{Res}\left(\frac{e^{iaz}}{z^2+1}; i\right) = \frac{e^{iai}}{2i} = \frac{e^{-a}}{2i}.$$

Hence $\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,\mathrm{d}x = 2\pi i \cdot \frac{e^{-a}}{2i} = \pi e^{-a}$. Taking real parts yields the cosine integral. □

11.1.5 Principal Value Integrals

Residues also evaluate Cauchy principal values when the integrand has poles on the contour. A standard example is

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

This can be obtained by integrating $\frac{e^{iz}}{z}$ along an indented contour that avoids the pole at 0 by a small semicircle, and keeping track of the half-residue contribution.

Problem Sets

These problem sets are intended to function as a second line of exposition rather than as a detached exercise bank. In each set, the early problems reinforce definitions and standard estimates, the middle layer develops the central techniques of the chapter, and the final layer is designed for longer written arguments and seminar-style discussion.

Conventions for the problem sets. Problems marked by $(\star)$ are typically longer or conceptually more synthetic. A good working rhythm is to solve most of Part A before lecture, use Part B to consolidate the theory after class, and reserve Part C for written assignments, seminar discussion, or examination-style revision.

Problem Set 1: Complex Numbers, Geometry, and Fractional Linear Maps

Guide to the set.

> These problems reinforce the algebra and geometry of the complex plane and also preview the fractional linear transformations that reappear later in the course as Möbius maps and disk automorphisms. The problems are arranged progressively: begin with Part A, use Part B for course-level mastery, and treat Part C as extended written work.

Part A. Foundations and preliminary checks

1. Show that $|z| \leq |\Re(z)| + |\Im(z)| \leq \sqrt{2}|z|$ for all $z \in \mathbb{C}$.
2. If $z_1, z_2 \in \mathbb{C}$, then $|z_1 + z_2| \leq |z_1| + |z_2|$. Show that equality holds if and only if one of them is a nonnegative scalar multiple of the other.
3. If either $|z_1| = 1$ or $|z_2| = 1$, but not both, then prove that $\left|\dfrac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right| = 1$. What exception must be made for the validity of the above equality when $|z_1| = |z_2| = 1$?
4. Show that the equation $z^4 + z + 5 = 0$ has no solution in the set $\{z \in \mathbb{C} \colon\ |z| < 1\}$.
5. If z and w are in $\mathbb{C}$ such that $\Im(z) > 0$ and $\Im(w) > 0$, show that $|\frac{z-w}{z-\overline{w}}| < 1$.
6. When does $az + b\overline{z} + c = 0$ have exactly one solution?
7. Let $a, b, c \in \mathbb{C}$ such that $a \neq 0$ and $|a| \neq |c|$. Show that a root (in $\mathbb{C}$) of the equation $az^2 + bz + c = 0$ has modulus 1 if and only if $|\overline{a}b - \overline{b}c| = |a\overline{a} - c\overline{c}|$.
8. If $1 = z_0, z_1, \ldots, z_{n-1}$ are distinct n^{th} roots of unity, prove that

$$\prod_{j=1}^{n-1} (z - z_j) = \sum_{j=0}^{n-1} z^j.$$

9. Let $z, w \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$. Show that $|z+w|^2 \leq (1+\lambda)|z|^2 + (1+\frac{1}{\lambda})|w|^2$.
10. Let $z, w \in \mathbb{C}$ such that $(1+|z|^2)w = (1+|w|^2)z$. Show that $z = w$ or $z\overline{w} = 1$.
11. Let $z \in \mathbb{C} \setminus \mathbb{R}$ such that $\dfrac{1+z+z^2}{1-z+z^2} \in \mathbb{R}$. Show that $|z| = 1$.
12. If $z, w \in \mathbb{D}$, then show that $|(1-|z|^2)w + (1-|w|^2)z| < |1-z^2w^2|$.

Part B. Core theory and proof-based exercises

13. If $z, w \in \mathbb{C}$, then show that $|1+z| + |1+w| + |1+zw| \geq 2$.
14. Let $z \in \mathbb{C} \setminus \{1\}$ such that $z^n = 1$, where $n \in \mathbb{N}$. Show that $1 + 2z + \cdots + nz^{n-1} = \dfrac{n}{z-1}$.
15. Let $n \in \mathbb{N}$ and let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that $a_0 \geq a_1 \geq \cdots \geq a_{n-1} \geq a_n > 0$. Show that $|a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n| > 0$ for all $z \in \mathbb{D}$.
16. Let $z \in \mathbb{C} \setminus \{0\}$ such that $\left|z^3 + \dfrac{1}{z^3}\right| \leq 2$. Show that $\left|z + \dfrac{1}{z}\right| \leq 2$.
17. Show that all the roots of the equation $(z+1)^3 + z^3 = 0$ lie on the line $\Re(z) + \frac{1}{2} = 0$.
18. Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that all the roots $z \in \mathbb{C}$ of the equation $\left(\dfrac{1+iz}{1-iz}\right)^n = a + ib$ are real if and only if $a^2 + b^2 = 1$.
19. Let $f(x) = \dfrac{1+ix}{1-ix}$ for all $x \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{C}$ is one-to-one. Also, determine the range of f.
20. Let $a \in \mathbb{R}$ such that $|a| < 1$ and let $f(z) = \dfrac{z-a}{1-\overline{a}z}$ for all $z \in \mathbb{D}$. Show that $f : \mathbb{D} \to \mathbb{D}$ is bijective.
21. Let $a, b \in \mathbb{C}$ and let $T(z) = az + b\overline{z}$ for all $z \in \mathbb{C}$. Show that $T : \mathbb{C} \to \mathbb{C}$ is bijective if and only if $|a| \neq |b|$.
22. Let $z_1, z_2 \in \mathbb{C}$ such that $\Re(z_1) > 0$ and $\Re(z_2) > 0$. Show that $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$.
23. Let $z \in \mathbb{C}$ such that $\Re(z^n) \geq 0$ for all $n \in \mathbb{N}$. Show that z is a non-negative real number.
24. If $d(z, w) = \dfrac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z, w \in \mathbb{C}$, then show that d is a metric on $\mathbb{C}$.

Part C. Extensions, synthesis, and challenge problems

25. If $z \in \mathbb{C}$ such that $|z| = 2$, then show that $\left|\dfrac{1}{z^4 - 4z^2 + 3}\right| \leq \dfrac{1}{3}$.
26. If $z \in \mathbb{C}$ such that $|z| = 3$, then show that $\dfrac{5}{13} \leq \left|\dfrac{2z-1}{z^2+4}\right| \leq \dfrac{7}{5}$.
27. Let $z, w \in \mathbb{C}$ such that $\Re(z) > 0$ and $\Re(w) > 0$. Show that $|z - w| < |\overline{z} + w|$.
28. Let $z_1, z_2, z_3 \in \mathbb{C}$ such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$. Show that $z_1^2 + z_2^2 + z_3^2 = 0$.
29. Show that $|z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1| \leq |z_1| + |z_2| + |z_3| + |z_1 + z_2 + z_3|$ for all $z_1, z_2, z_3 \in \mathbb{C}$.
30. Let z_1, z_2, z_3 be distinct nonzero complex numbers such that $|z_1| = |z_2| = |z_3|$. If $z_1 + z_2 z_3$, $z_2 + z_1 z_3$, $z_3 + z_1 z_2 \in \mathbb{R}$, then show that $z_1 z_2 z_3 = 1$.

31. Let $n \in \mathbb{N}$ and let $z_1, \ldots, z_n \in \mathbb{C}$ such that $\left|\sum\limits_{j=1}^{n} z_j w_j\right| \leq 1$ for all $w_1, \ldots, w_n \in \mathbb{C}$ with $\sum\limits_{j=1}^{n} |w_j|^2 \leq 1$. Is it necessary that $\sum\limits_{j=1}^{n} |z_j|^2 \leq 1$? Justify.
32. Let $n \in \mathbb{N}$ and let $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$. Show that

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) - \sum_{1 \leq j < k \leq n} |z_j \overline{w}_k - z_k \overline{w}_j|^2.$$

Hence deduce the Cauchy–Schwarz inequality:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right).$$

33. Show that

$$\frac{(1-i)^{49}\left(\cos\frac{\pi}{40} + i\sin\frac{\pi}{40}\right)^{10}}{(8i - 8\sqrt{3})^6} = -\sqrt{2}.$$

34. Show that three distinct given points $z_1, z_2, z_3 \in \mathbb{C}$ represent the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

35. Let $z_1, z_2, z_3 \in \mathbb{C}$ such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$. Show that z_1, z_2, and z_3 represent the vertices of an equilateral triangle inscribed in the circle $|z| = 1$.
36. Let $z_1, z_2, z_3 \in \mathbb{C}$ represent the vertices of an isosceles triangle, with a right angle at the vertex z_2. Show that

$$z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3).$$

Problem Set 2: Topology of $\mathbb{C}$ and Continuity

Guide to the set.

> These exercises consolidate the topological language used throughout the course: closure, compactness, continuity, connectedness, and the geometry of domains. The problems are arranged progressively: begin with Part A, use Part B for course-level mastery, and treat Part C as extended written work.

Part A. Foundations and preliminary checks

1. For each subset of $\mathbb{C}$, determine whether it is open, closed, or neither, with justification:
   (a) $\{z \in \mathbb{C} : \Re(z) = 1 \text{ and } \Im(z) \neq 4\}$
   (b) $B(1,1) \cup B(2, \frac{1}{2}) \cup B(3, \frac{1}{3})$
   (c) $\left\{z \in \mathbb{C} : \left|\frac{z-1}{z+1}\right| = 2\right\}$
   (d) $\{z \in \mathbb{C} : \sin(\Re(z)) < \Im(z) < 1\}$.

2. For each of the following subsets of $\mathbb{C}$, determine their interior, exterior, and boundary:
   (a) $\{z \in \mathbb{C} : |z| < 1 \text{ and } \Im(z) \neq 0\} \cup \{z \in \mathbb{C} : |z| > 1 \text{ and } \Im(z) = 0\}$
   (b) $\left\{r\left(\cos\left(\frac{1}{n}\right) + i\sin\left(\frac{1}{n}\right)\right) \in \mathbb{C} : r > 0, n \in \mathbb{N}\right\} \cup \{z \in \mathbb{C} : \Re(z) < 0\}$.
3. Determine whether $\bigcup\limits_{n=1}^{\infty}\{z \in \mathbb{C} : z^n = 1\}$ is open or closed in (a) $\mathbb{C}$ and (b) $\partial\mathbb{D}$.
4. Determine all limit points in $\mathbb{C}$ of the following sets:
   (a) $\left\{\dfrac{1}{m} + \dfrac{i}{n} : m, n \in \mathbb{N}\right\}$,
   (b) $\left\{x + \dfrac{i}{x} : x \in \mathbb{R},\ x > 0\right\}$.
5. State TRUE or FALSE with justification: Every uncountable subset of $\mathbb{C}$ has a limit point in $\mathbb{C}$.
6. Show that $\{\cos n + i\sin n : n \in \mathbb{N}\}$ is dense in $\partial\mathbb{D}$.
7. State TRUE or FALSE with justification: If (z_n) is a sequence in $\mathbb{C}$ such that (z_n) has no convergent subsequence, then it is necessary that $\lim\limits_{n\to\infty}|z_n| = \infty$.
8. Let Ω be an open set in $\mathbb{C}$. Show that for each $n \in \mathbb{N}$, there exists a compact set K_n in $\mathbb{C}$ such that $\Omega = \bigcup\limits_{n=1}^{\infty} K_n$.
9. Let (z_n) be a sequence in $\mathbb{C}$ and let $z \in \mathbb{C}$. Show that the series $\sum\limits_{n=1}^{\infty} z_n$ converges with sum z if and only if both the series $\sum\limits_{n=1}^{\infty}\Re(z_n)$ and $\sum\limits_{n=1}^{\infty}\Im(z_n)$ converge with sums $\Re(z)$ and $\Im(z)$ respectively.
10. If $z \in \mathbb{C}$ such that $1 < |z| < 2$, then show that the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2 + z^2}$ is convergent.

Part B. Core theory and proof-based exercises

11. If $z \in \mathbb{D}$, then show that the series $\sum\limits_{n=1}^{\infty}\dfrac{z^{2n}}{2 + z^n + z^{5n}}$ is convergent.
12. Let (z_n) be a sequence in $\mathbb{C}$ such that $\Re(z_n) \geq 0$ for all $n \in \mathbb{N}$. If both the series $\sum\limits_{n=1}^{\infty} z_n$ and $\sum\limits_{n=1}^{\infty} z_n^2$ are convergent, then show that the series $\sum\limits_{n=1}^{\infty}|z_n|^2$ is also convergent.
13. Let (z_n) be a sequence in $\mathbb{C}$ such that $\sup\limits_{n\in\mathbb{N}}|\operatorname{Arg}(z_n)| < \frac{\pi}{2}$. If the series $\sum\limits_{n=1}^{\infty} z_n$ is convergent, then show that the series $\sum\limits_{n=1}^{\infty} z_n$ is absolutely convergent.
14. If $z \in \mathbb{C}\setminus\{0\}$ such that $|\operatorname{Arg}(z)| \leq \dfrac{\pi}{4}$, then show that the series $\sum\limits_{n=1}^{\infty}\dfrac{z}{(1+z^2)^n}$ is convergent.
15. If $z \in \mathbb{C}\setminus\mathbb{N}$, then show that the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{z-n} + \dfrac{1}{n}\right)$ is convergent.

16. Let $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ be absolutely convergent series in $\mathbb{C}$. If $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N} \cup \{0\}$, then show that the series $\sum\limits_{n=0}^{\infty} c_n$ is absolutely convergent.
17. Determine whether the following subsets of $\mathbb{C}$ are connected.
    (a) $\{z \in \mathbb{C} : \Re(z),\ \Im(z) \in \mathbb{Q}\}$
    (b) $\{z \in \mathbb{C} : \Re(z)\Im(z) > 0\}$
    (c) $\{z \in \mathbb{C} : |z-1| \leq 1\} \cup \{z \in \mathbb{C} : |z-1+2i| < 1\}$
    (d) $\{z \in \mathbb{C} : \Re(z)^2 + \Im(z)^3 \in \mathbb{R} \setminus \mathbb{Q}\}$
    (e) $\{z \in \mathbb{C} : |z| \leq \sqrt{2}\} \cup \{z \in \mathbb{C} : |z-2-2i| \leq \sqrt{2}\}$
    (f) $\{z \in \mathbb{C} : |\mathrm{Re}(z)| < |\mathrm{Im}(z)|\}$
    (g) $\mathbb{C} \setminus \{z \in \mathbb{C} : \Re(z) \in \mathbb{Q},\ \Im(z) \in \mathbb{Q}\}$
    (h) $\{x + i \sin \frac{1}{x} : x \in (0,1]\} \cup \{\frac{i}{3}\} \cup \{-\frac{i}{4}\}$
18. Show that the function $f : \mathbb{C} \to \mathbb{C}$, defined by $f(z) = \begin{cases} z \sin \dfrac{1}{z} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$ is not continuous at 0.
19. If $f : \mathbb{C} \to \mathbb{C}$ is continuous such that $f(2z) = f(z)$ for all $z \in \mathbb{C}$, then show that f is a constant function.
20. State TRUE or FALSE with justification:
    (a) There exists a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(\sin n) = \frac{n\pi}{2}$ for all $n \in \mathbb{N}$.
    (b) If $f : \mathbb{C} \to \mathbb{C}$ is continuous and Ω is a bounded subset of $\mathbb{C}$, then $f(\Omega)$ must be a bounded subset of $\mathbb{C}$.
21. Let $f : K \to \mathbb{C}$ be a continuous function, where K is a compact set in $\mathbb{C}$. If $f(z) \neq 0$ for all $z \in K$, then show that there exists $r > 0$ such that $f(K) \subset \mathbb{C} \setminus B_r(0)$.

Part C. Extensions, synthesis, and challenge problems

22. Let $f : \mathbb{C} \to \mathbb{C}$ be continuous and $\lim\limits_{|z| \to \infty} f(z) = 0$. Show that f is bounded and that there exists $z_0 \in \mathbb{C}$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \mathbb{C}$.
23. Let $f : \Omega \to \mathbb{C}$ be continuous, where Ω is a domain in $\mathbb{C}$. If $|f(z)^2 - 1| < 1$ for all $z \in \Omega$, then show that either $|f(z) - 1| < 1$ for all $z \in \Omega$ or $|f(z) + 1| < 1$ for all $z \in \Omega$.
24. Let Ω be a domain in $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be continuous such that $\{z \in \Omega : |f(z)| < |g(z)|\} \neq \emptyset$ and $\{z \in \Omega : |f(z)| > |g(z)|\} \neq \emptyset$. Show that $\{z \in \Omega : |f(z)| = |g(z)|\} \neq \emptyset$.
25. State TRUE or FALSE with justification: For every continuous function $f : \partial\mathbb{D} \to \mathbb{R}$, there exists $z \in \partial\mathbb{D}$ such that $f(z) = f(-z)$.
26. Show that there is no continuous function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $f(z)^2 = z$ for all $z \in \mathbb{C} \setminus \{0\}$.

27. State TRUE or FALSE with proper justification: An unbounded function $f : \mathbb{D} \to \mathbb{C}$ cannot be uniformly continuous (on $\mathbb{D}$).
28. Let $f(z) = \dfrac{z}{1+|z|}$ for all $z \in \mathbb{C}$. Show that $f : \mathbb{C} \to \mathbb{D}$ is a homeomorphism.
29. State TRUE or FALSE with justification:
$$\bigcup_{n=1}^{\infty} \{z \in \mathbb{C} : z^n = 1\} = \{z \in \mathbb{C} : |z| = 1\}.$$
30. Let $z_1, z_2 \in \mathbb{C}$ such that $z_1 \neq z_2$ and let $\lambda > 0$. Describe the locus of z in the complex plane satisfying the equation $|z - z_1| = \lambda|z - z_2|$.
31. Let $z_0 \in \mathbb{C}$ and $r > 0$. If Ω is an open set in $\mathbb{C}$ such that $B_r[z_0] \subset \Omega$, then show that there exists $s > r$ such that $B_s(z_0) \subset \Omega$.
32. Examine whether
$$\lim_{z\to 0} \frac{(\Re z)^5(\Im z)^2}{(\Re z)^6 + (\Im z)^6}$$
exists (in $\mathbb{C}$).
33. If $f : \mathbb{C} \to \mathbb{C}$ is continuous at $z_0 \in \mathbb{C}$ and $f(z_0) \neq 0$, then show that there exists $\delta > 0$ such that $f(z) \neq 0$ for all $z \in B_\delta(z_0)$.

Problem Set 3: Differentiability, Holomorphy, and Power Series

Guide to the set.

This set moves from computational criteria for differentiability to structural properties of holomorphic functions and their power-series expansions. The later problems are best approached in written proof format. The problems are arranged progressively: begin with Part A, use Part B for course-level mastery, and treat Part C as extended written work.

Part A. Foundations and preliminary checks

1. Show that $f : \mathbb{C} \to \mathbb{C}$ is nowhere differentiable on $\mathbb{C}$, where for $z = x + iy \in \mathbb{C}$ one has:
   (a) $f(z) = \Re(z)$
   (b) $f(z) = \Im(z)$
   (c) $f(z) = |z|$
   (d) $f(z) = 2x + ixy^2$
   (e) $f(z) = e^x(\cos y - i \sin y)$
2. Determine all points of $\mathbb{C}$ at which $f : \mathbb{C} \to \mathbb{C}$ is differentiable, where for $z = x + iy \in \mathbb{C}$ one has:
   (a) $f(z) = x^3 + i(1-y)^3$
   (b) $f(z) = z\Im(z)$

(c) $f(z) = x^2 + iy^2$

(d) $f(z) = x^2 + y + i(2y - x)$

(e) $f(z) = z^2\overline{z}$

(f) $f(z) = x^3y^2 + ix^2y^3$

(g) $f(z) = |z|^4$

(h) $f(z) = x^2 - y^2 + 2i|xy|$

3. Show that for each of the following functions $f : \mathbb{C} \to \mathbb{C}$, the Cauchy–Riemann equations are satisfied at 0 but $f'(0)$ does not exist.
   (a) $f(z) = \sqrt{|xy|}$ for all $z = x + iy \in \mathbb{C}$.
   (b) $f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$
   (c) $f(z) = \begin{cases} \dfrac{(1+i)\Im(z^2)}{|z|^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

4. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \begin{cases} \dfrac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} & \text{if } z = x + iy \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$ Show that f is continuous. Then show that the Cauchy–Riemann equations are satisfied at $(0,0)$, although f is not differentiable at 0.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \begin{cases} \dfrac{x^3y(y - ix)}{x^6 + y^2} & \text{if } z = x + iy \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$ Show that f is continuous. Then show that the Cauchy–Riemann equations are satisfied at $(0,0)$, although f is not differentiable at 0.

6. For each $z \in \mathbb{C}$, let $f(z) = \begin{cases} |z|^2 & \text{if } \Re(z), \Im(z) \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$ Determine all the points of $\mathbb{R}^2$ at which
   (a) at least one of the Cauchy–Riemann equations is satisfied.
   (b) both the Cauchy–Riemann equations are satisfied.

7. Show that $f : \mathbb{C} \to \mathbb{C}$ is not analytic at any point of $\mathbb{C}$, where for $z = x + iy \in \mathbb{C}$ one has:
   (a) $f(z) = z|z|$
   (b) $f(z) = \dfrac{z}{1 + |z|}$
   (c) $f(z) = xy + iy$
   (d) $f(z) = e^y(\cos x + i \sin x)$
   (e) $f(z) = x^2 + iy^3$

8. Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there is no point c on the line segment $y = 1 - x$ joining z_1 to z_2 such that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c).$$

   This shows that the real-variable mean value theorem does not extend to complex derivatives.

9. Let f be a real-valued function on a domain $D \subseteq \mathbb{C}$. Show that for each $z \in D$, either $f'(z) = 0$ or $f'(z)$ does not exist.
10. Let $f : G \to \mathbb{C}$ be differentiable at a point $z_0 \in G$, where G is an open set in $\mathbb{C}$, and let $g(z) = \overline{f(z)}$ for all $z \in G$. Show that $g : G \to \mathbb{C}$ is differentiable at z_0 if and only if $f'(z_0) = 0$.
11. Let Ω_1 and Ω_2 be nonempty open sets in $\mathbb{C}$. Let $f : \Omega_1 \to \mathbb{C}$ be continuous such that $f(\Omega_1) \subset \Omega_2$ and let $g : \Omega_2 \to \mathbb{C}$ be holomorphic such that $g'(z) \neq 0$ for all $z \in \Omega_2$. If $g \circ f : \Omega_1 \to \mathbb{C}$ is holomorphic, then show that f is holomorphic.
12. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$, and let $g(z) = \overline{f(z)}$ for all $z \in \Omega$. If $g : \Omega \to \mathbb{C}$ is analytic, then show that f is a constant function.
13. Let G be an open set in $\mathbb{C}$ and let $G^* = \{\overline{z} : z \in G\}$. Show that G^* is open in $\mathbb{C}$. If $f : G \to \mathbb{C}$ is analytic and $g(z) = \overline{f(\overline{z})}$ for all $z \in G^*$, then show that $g : G^* \to \mathbb{C}$ is analytic and that $g'(z) = \overline{f'(\overline{z})}$ for all $z \in G^*$.
14. Let Ω be a nonempty subset of $\mathbb{C}$ (considered to be $\mathbb{R}^2$ as a set). Show that a function $f : \Omega \to \mathbb{C}$ is differentiable as a function of two real variables at $z_0 \in \Omega^0$ if and only if there exist φ, $\psi : \Omega \to \mathbb{C}$ such that both φ, ψ are continuous at z_0 and $f(z) - f(z_0) = (z - z_0)\varphi(z) + (\overline{z} - \overline{z}_0)\psi(z)$ for all $z \in \Omega$.
15. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If L is a line in $\mathbb{C}$ and $f(\Omega) \subset L$, then show that f is a constant function.
16. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If P is the parabola in $\mathbb{C}$ given by the equation $y = x^2$ and $f(\Omega) \subset P$, then show that f is a constant function.
17. Let $f : \Omega \to \mathbb{C}$ be an analytic function, where Ω is a domain in $\mathbb{C}$. If $f' : \Omega \to \mathbb{C}$ is a constant function, then show that there exist $a, b \in \mathbb{C}$ such that $f(z) = az + b$ for all $z \in \Omega$.
18. Let $f = u + iv$ be an entire function on $\mathbb{C}$. If $u(x, y) = \phi(x)$ and $v(x, y) = \psi(y)$, prove that $f(z) = az + b$ for all $z \in \mathbb{C}$, for some constants $a, b \in \mathbb{C}$.
19. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If for each $z \in \Omega$, either $f(z) = 0$ or $f'(z) = 0$, then show that f is a constant function.
20. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If for each $z \in \Omega$, $\mathrm{Re}(f(z)) = 0$ or $\mathrm{Im}(f(z)) = 0$, then show that $f : \Omega \to \mathbb{C}$ is a constant function.
21. For $z = x + iy \in \mathbb{C}$, classify all entire functions $f(z) = u(x, y) + iv(x, y)$ that satisfy $u_y(x, y) = v_x(x, y)$ for each $x, y \in \mathbb{R}$.

22. For $z = x + iy \in \mathbb{C}$, find all the entire functions $f(z) = u(x, y) + iv(x, y)$ satisfying $2u(x, y) + 3v(x, y) > 5$ for each $x, y \in \mathbb{R}$.
23. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If $z_0 \in \Omega$ such that $f'(z_0) \neq 0$, then show that there exists $\delta > 0$ such that f is one-to-one on $B_\delta(z_0)$.

Part B. Core theory and proof-based exercises

24. Let u and v be nonconstant harmonic functions on $\mathbb{C}$.
    (a) If $U(x, y) = u(x, -y)$, is U also harmonic?
    (b) If v is a harmonic conjugate of u, is u a harmonic conjugate of v?
    (c) Is uv always harmonic? If not, produce an example.
25. If v is a harmonic conjugate of u (u, v real-valued), prove that the functions uv and $u^2 - v^2$ are also harmonic.
26. What are all real-valued harmonic functions u on D such that u^2 is also harmonic?
27. Find a harmonic conjugate, if it exists, of the following functions:
    (a) $u(x, y) = 2xy$.
    (b) $u(r, \theta) = r^n \cos n\theta$, $n \in \mathbb{N}$.
    (c) $u(x, y) = x^2 - y^2 + x + y - \frac{y}{x^2+y^2}$.
28. Define the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

    Let $f = u + iv$ be defined on an open set in $\mathbb{C}$. Show that:
    (a) f satisfies the Cauchy–Riemann equations if and only if $\frac{\partial}{\partial \bar{z}} f(z) = 0$.
    (b) If $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then show that $\Delta f = 4\frac{\partial^2}{\partial z \partial \bar{z}} f$.
    (c) Prove that the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}^n$ is harmonic for all $n \in \mathbb{N}$.
29. Let $u(x, y) = \begin{cases} \Im\left(\dfrac{1}{(x+iy)^2}\right) & \text{if } (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}, \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$ Determine whether $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic.
30. Determine whether there exists an analytic function $f : G \to \mathbb{C}$ on some open set $G \subseteq \mathbb{C}$ such that, for all $z = x + iy \in G$,
    (a) $\Re f(z) = x^2 - 2y$,
    (b) $\Im f(z) = x^3 - y^3$.
31. Determine all functions $v : \mathbb{R}^2 \to \mathbb{R}$ such that $f = u + iv : \mathbb{C} \to \mathbb{C}$ is analytic, where for all $(x, y) \in \mathbb{R}^2$,
    (a) $u(x, y) = y^3 - 3x^2 y$,
    (b) $u(x, y) = e^{-x}(x \sin y - y \cos y)$.

Also express the resulting function f in terms of $z \in \mathbb{C}$.

32. Determine all analytic functions $f = u+iv : \mathbb{C} \to \mathbb{C}$ such that $u(x,y)-v(x,y) = e^x(\cos y - \sin y)$ for all $(x,y) \in \mathbb{R}^2$.
33. Let $u : \Omega \to \mathbb{C}$ be a harmonic function, where Ω is a domain in $\mathbb{C}$. Show that $u_x - iu_y : \Omega \to \mathbb{C}$ is analytic.
34. Let $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$ be harmonic, where Ω is a domain in $\mathbb{C}$. Show that $(u_y - v_x) + i(u_x + v_y) : \Omega \to \mathbb{C}$ is analytic.
35. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. Show that
    (a) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\Re(f(z))^2 = 2|f'(z)|^2$
    (b) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})|f(z)|^2 = 4|f'(z)|^2$
    for all $z = x+iy \in \mathbb{C}$.
36. Determine the radius of convergence of each of the following power series:
    (a) $\sum\limits_{n=1}^{\infty} n^{(-1)^n} z^n$,
    (b) $\sum\limits_{n=0}^{\infty} z^{2^n}$,
    (c) $\sum\limits_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^{(-1)^n n^2} z^n$,
    (d) $\sum\limits_{n=0}^{\infty} \left(\frac{2+(-1)^n}{5+(-1)^{n+1}}\right)^n z^n$,
    (e) $\sum\limits_{n=0}^{\infty} a^{n^2} z^n$, where $a \in \mathbb{C}$.
37. For each $n \in \mathbb{N}$, let a_n be equal to the total number of (positive integer) divisors of n^{60}. Determine (with justification) the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} a_n z^n$.
38. Determine all $z \in \mathbb{C}$ for which the following power series are convergent. (a) $\sum\limits_{n=1}^{\infty} \frac{2^n}{n^2}(z-2-i)^n$ (b) $\sum\limits_{n=0}^{\infty} 2^n(z-2)^n$ (c) $\sum\limits_{n=0}^{\infty} \frac{z^{4n}}{4n+1}$ (d) $\sum\limits_{n=0}^{\infty} \frac{(2z-i)^n}{3n+1}$
39. Show that the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$ is 1. Also, examine the convergence of the power series for $z = 1$, -1, and i.
40. State TRUE or FALSE with justification: There exists a power series $\sum\limits_{n=0}^{\infty} a_n(z-1+2i)^n$ in $\mathbb{C}$ which converges at $z = -3+i$ and diverges at $z = -2+2i$.
41. Show that the radius of convergence R of a power series $\sum\limits_{n=0}^{\infty} a_n z^n$ in $\mathbb{C}$ is given by
    $R = \sup\{|z| : z \in \mathbb{C},\ a_n z^n \to 0\} = \sup\{|z| : z \in \mathbb{C},\ \text{the sequence } (a_n z^n) \text{ is bounded}\}$.
42. Let R_1 and R_2 be the radii of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} b_n z^n$ respectively. If R is the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} (a_n+b_n)z^n$, then show that $R \geq$

$\min\{R_1, R_2\}$. Is it necessary that $R = \min\{R_1, R_2\}$? Justify. If $R_1 \neq R_2$, then show that $R = \min\{R_1, R_2\}$.

43. Let R_1 and R_2 be the radii of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} b_n z^n$ respectively. If R is the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n b_n z^n$, then show that $R \geq R_1 R_2$. (It is assumed that $R_1 R_2$ is defined.) Is it necessary that $R = R_1 R_2$? Justify.

44. Let $a_n \in \mathbb{C} \setminus \{0\}$ for all $n \in \mathbb{N} \cup \{0\}$ and let R be the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$. Is it necessary that the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} \dfrac{1}{a_n} z^n$ is $\dfrac{1}{R}$? Justify.

45. Let R be the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$. Determine the radius of convergence of each of the following power series. (a) $\sum\limits_{n=0}^{\infty} a_n^2 z^n$ (b) $\sum\limits_{n=0}^{\infty} 2^n a_n z_n$ (c) $\sum\limits_{n=1}^{\infty} n^n a_n z^n$

46. Determine whether $\left\{ \sum\limits_{n=1}^{\infty} \frac{z^n}{n} : z \in \mathbb{C}, \frac{1}{4} \leq |z| \leq \frac{3}{4} \right\}$ is a closed set in $\mathbb{C}$.

47. Determine whether the series $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$ is uniformly convergent on $\mathbb{C}$.

Part C. Extensions, synthesis, and challenge problems

48. State TRUE or FALSE with justification: If $R > 0$ is the radius of convergence of a power series $\sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$, then the series $\sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$ cannot converge uniformly on $\{z \in \mathbb{C} : |z - z_0| < R\}$.

49. Show that $|1 - (1 - z)e^z| \leq |z|^2$ for all $z \in \mathbb{D}$.

50. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \begin{cases} e^{-\frac{1}{z^4}} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$ Show that the Cauchy–Riemann equations for f are satisfied at every point of $\mathbb{R}^2$ but f is not continuous (and hence not differentiable) at 0.

51. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f'(z) = f(z)$ for all $z \in \mathbb{C}$ and $f(0) = 1$. Show that $f(z) = e^z$ for all $z \in \mathbb{C}$.

52. Let $f : \mathbb{C} \to \mathbb{C}$ be differentiable at 0 and $f'(0) = 1$. If $f(z + w) = f(z)f(w)$ for all $z, w \in \mathbb{C}$, then show that $f(z) = e^z$ for all $z \in \mathbb{C}$.

53. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If $g(z) = e^{f(z)}$ for all $z \in \Omega$ and if $g : \Omega \to \mathbb{C}$ is a constant function, then show that f is a constant function. If f is assumed to be only continuous, then does a similar result hold? Justify.

54. For all $z \in \mathbb{C}$, show that
    (a) $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$.
    (b) $\cos(\pi + z) = -\cos z$ and $\sin(\frac{\pi}{2} + z) = \cos z$.

  - (c) $\overline{\sin z} = \sin \overline{z}$ and $\overline{\cos z} = \cos \overline{z}$.
  - (d) $\sin 2z = 2 \sin z \cos z$ and $\cos 2z = \cos^2 z - \sin^2 z$.
  - (e) $\sin 3z = 3 \sin z - 4 \sin^3 z$ and $\cos 3z = 4 \cos^3 z - 3 \cos z$.
  - (f) $|\sin z|^2 = \sin^2 x + \sinh^2 y$.
  - (g) $|\cos z|^2 = \cos^2 x + \sinh^2 y$.
  - (h) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$.
  - (i) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$.

55. For all $z, w \in \mathbb{C}$, show that
    - (a) $\sin(z + w) = \sin z \cos w + \cos z \sin w$.
    - (b) $\cos(z + w) = \cos z \cos w - \sin z \sin w$.
56. Show that $\{\sin z : z \in \mathbb{C}\} = \mathbb{C}$ and $\{\cos z : z \in \mathbb{C}\} = \mathbb{C}$.
57. Let $f(z) = \sin z$ and $g(z) = \cos z$ for all $z \in \mathbb{C}$. Determine all the periods of $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$.
58. Show that $\{z \in \mathbb{C} : \cos z = 1\} = 2\pi\mathbb{Z}$ and $\{z \in \mathbb{C} : \sin z = 1\} = \dfrac{\pi}{2} + 2\pi\mathbb{Z}$.
59. Show that $\{z \in \mathbb{C} : \overline{\cos(iz)} = \cos(i\overline{z})\} = \mathbb{C}$ and $\{z \in \mathbb{C} : \overline{\sin(iz)} = \sin(i\overline{z})\} = \pi i\mathbb{Z}$.
60. State TRUE or FALSE with justification: If f is an entire function such that $|f(x)| \le 10$ and $|f(iy)| \le 10$ for all $x, y \in \mathbb{R}$, then there must exist $\lambda \in \mathbb{R}$ such that $|f(z)| \le \lambda$ for all $z \in \mathbb{C}$.
61. If $z \in \mathbb{D}$, then show that the series $\sum\limits_{n=1}^{\infty} \sin(z^n)$ is absolutely convergent.
62. Let $f(z) = \begin{cases} |z|^2 \sin \dfrac{1}{|z|} & \text{if } z \in \mathbb{C} \setminus \{0\}, \\ 0 & \text{if } z = 0. \end{cases}$ Determine whether $f : \mathbb{C} \to \mathbb{C}$ is differentiable at 0.
63. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \begin{cases} \dfrac{\sin z}{z} & \text{if } z \ne 0, \\ 1 & \text{if } z = 0. \end{cases}$ Show that f is an entire function.
64. Solve for $z \in \mathbb{C}$ the following equations. (a) $e^z = 2i$ (b) $\cos^2 z = 4$ (c) $\tan z = i$
65. Let $f(z) = \tan z$ for all $z \in \Omega = \mathbb{C} \setminus \{(2n + 1)\frac{\pi}{2} : n \in \mathbb{Z}\}$. Determine the range of the function $f : \Omega \to \mathbb{C}$.
66. Show that $\log\left(\dfrac{1}{z}\right) = -\log z$ for all $z \in \mathbb{C} \setminus \{0\}$.
67. Determine whether (a) $\text{Log}(1 + i)^2 = 2\text{Log}(1 + i)$, (b) $\text{Log}(-1 + i)^2 = 2\text{Log}(-1 + i)$, (c) $\log(i^2) = 2 \log i$.
68. Let Ω_1 and Ω_2 be domains in $\mathbb{C}$ such that $\Omega_1 \cap \Omega_2$ is connected. If there exists a branch of the logarithm on Ω_1 and there exists a branch of the logarithm on Ω_2, then is it necessary that there exists a branch of the logarithm on $\Omega_1 \cup \Omega_2$? Justify.
69. If $z_1 = 1 + i$, $z_2 = 1 - i$ and $z_3 = -1 - i$, then examine whether $(z_1 z_2)^i = z_1^i z_2^i$ and $(z_2 z_3)^i = z_2^i z_3^i$, where only principal values are considered.

70. Show that all the values of $(1-i)^{\sqrt{2}i}$ lie on a straight line in the complex plane.
71. If an entire function f is such that $f(z)$ is real for all $z \in \mathbb{R}$, and satisfying $f\left(\frac{1}{2n+1}\right) = f\left(\frac{1}{2n}\right)$ for all $n \in \mathbb{N}$, then f is a constant function.
72. Let u and v be respectively the real part and the imaginary part of $f : E \subset \mathbb{C} \to \mathbb{C}$, and let $z_0 = x_0 + iy_0 \in E^0$. If
$$\lim_{z \to z_0} \Re\left(\frac{f(z) - f(z_0)}{z - z_0}\right)$$
exists (in $\mathbb{R}$), then show that both $u_x(x_0, y_0)$ and $v_y(x_0, y_0)$ exist (in $\mathbb{R}$).

Problem Set 4: Complex Integration, Cauchy Theory, and the Maximum Principle

Guide to the set.

> The central aim here is to internalize contour integration and the logic of the Cauchy theory. The final part emphasizes synthesis across several theorems rather than isolated computations. The problems are arranged progressively: begin with Part A, use Part B for course-level mastery, and treat Part C as extended written work.

Part A. Foundations and preliminary checks

1. Let $\gamma(0) = 0$ and $\gamma(t) = e^{\frac{i-1}{t}}$ for all $t \in (0, 1]$. Show that $\gamma : [0, 1] \to \mathbb{C}$ is a rectifiable path in $\mathbb{C}$. Also, determine the length of γ.
2. Let $\gamma(0) = 0$ and $\gamma(t) = t + it \sin \frac{1}{t}$ for all $t \in (0, 1]$. Show that $\gamma : [0, 1] \to \mathbb{C}$ is a path in $\mathbb{C}$ but γ is not rectifiable.
3. Let $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [a, b] \to \mathbb{C}$ be rectifiable paths in $\mathbb{C}$ such that $\gamma_1(b) = \gamma_2(a)$. Show that the path $\gamma_1 + \gamma_2$ is rectifiable and that $L(\gamma_1 + \gamma_2) = L(\gamma_1) + L(\gamma_2)$.
4. If γ is a rectifiable path in $\mathbb{C}$, then evaluate $\int\limits_\gamma |dz|$, with justification.
5. Let γ be the polygon $[1-i, 1+i, -1+i, -1-i, 1-i]$. Express γ as a path and hence evaluate $\displaystyle\int_\gamma \frac{1}{z}\, dz$.
6. Evaluate the integral $\displaystyle\int_\gamma |z|\bar{z} dz$ where γ is the circle $|z| = 2$.
7. Without evaluating the integral, show that
   (a) $\left|\displaystyle\int_\gamma \frac{z+4}{z^3-1}\, dz\right| \le \dfrac{6\pi}{7}$, where $\gamma(t) = 2e^{it}$ for all $t \in [0, \frac{\pi}{2}]$.
   (b) $\left|\displaystyle\int\limits_\gamma \frac{dz}{z^4}\right| \le 4\sqrt{2}$, where γ denotes the line segment in $\mathbb{C}$ from i to 1.
   (c) $\left|\int\limits_\gamma (e^z - \bar{z})\, dz\right| \le 60$, where γ denotes the triangle $[0, 3i, -4, 0]$ in $\mathbb{C}$.
   (d) $\left|\displaystyle\int_\gamma \frac{dz}{z^2+1}\right| \le \dfrac{1}{2\sqrt{5}}$, where γ is the straight line segment from 2 to $2+i$.

8. State TRUE or FALSE with proper justification: If $f : \mathbb{D} \to \mathbb{C}$ is continuous such that $|f(z)| < 2$ for all $z \in \mathbb{D}$ and if $\gamma(t) = \frac{1}{2} + \frac{1}{4}e^{2it}$ for all $t \in [0, 2\pi]$, then it is necessary that $\left|\int\limits_{\gamma} f\right| < 2\pi$.
9. Let $f : \Omega \to \mathbb{C}$ be continuous, where Ω is a domain in $\mathbb{C}$. Let $z_0 \in \Omega$ and for each $r > 0$, let $\gamma_r(t) = z_0 + re^{it}$ for all $t \in [0, 2\pi]$. Show that $\lim\limits_{r\to 0} \int_{\gamma_r} f(z)\, dz = 0$ and $\lim\limits_{r\to 0} \int_{\gamma_r} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$.
10. Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded continuous function and for each $r > 0$, let $\gamma_r(t) = re^{it}$ for all $t \in [0, 2\pi]$. Show that $\lim\limits_{r\to \infty} \int_{\gamma_r} \frac{f(z)}{(z - z_0)^2}\, dz = 0$ for all $z_0 \in \mathbb{C}$.
11. For each $r > 0$, let $\gamma_r(t) = re^{it}$ for all $t \in [0, \pi]$. Show that $\lim\limits_{r\to 0} \int_{\gamma_r} \frac{e^{iz}}{z}\, dz = 0$.
12. For each $r > 0$, let $\gamma_r(t) = re^{it}$ for all $t \in [0, 2\pi]$. If $p(z)$ and $q(z)$ are polynomials with $\deg q(z) \geq \deg p(z) + 2$, then show that $\lim\limits_{r\to \infty} \int_{\gamma_r} \frac{p(z)}{q(z)}\, dz = 0$.
13. Let $f : G \to \mathbb{C}$ be continuous, where G is an open set in $\mathbb{C}$. If γ is a smooth path in G such that $0 \notin \text{range}(\gamma)$, then show that
$$\left|\int_{\gamma} \frac{f(z)}{z} dz\right|^2 \leq L(\gamma) \left(\max_{z\in\text{range}(\gamma)} \frac{1}{|z|^2}\right) \int_{\gamma} |f(z)|^2\, |dz|.$$
14. If $f(z) = \int_0^1 \frac{dt}{t - z}$ for all $z \in \mathbb{C} \setminus [0, 1]$, then show that $f : \mathbb{C} \setminus [0, 1] \to \mathbb{C}$ is continuous.
15. Let $f(z) = |z|^2$ for all $z \in \mathbb{C}$. Evaluate $\int\limits_{\gamma_1} f(z)\, dz$ and $\int\limits_{\gamma_2} f(z)\, dz$, where $\gamma_1 = [1, i]$ and $\gamma_2 = [1, 1 + i, i]$. Hence show that $f : \mathbb{C} \to \mathbb{C}$ does not have any primitive on $\mathbb{C}$.
16. Let $z_0 \in \mathbb{C}$ and let γ be a closed rectifiable path in $\mathbb{C}$ such that $z_0 \notin \text{range}(\gamma)$. If $n \in \mathbb{Z}$ and $n \neq 1$, then show that $\int_{\gamma} \frac{dz}{(z - z_0)^n} = 0$.
17. Let $f : G \to \mathbb{C}$ and $g : G \to \mathbb{C}$ be analytic, where G is an open set in $\mathbb{C}$. If γ is a rectifiable path in G joining $z_1 \in G$ to $z_2 \in G$, then show that $\int\limits_{\gamma} fg' = f(z_2)g(z_2) - f(z_1)g(z_1) - \int\limits_{\gamma} f'g$.
18. Evaluate $\int\limits_{\gamma} z^2 \sin z\, dz$, where $\gamma(t) = e^{it}$ for all $t \in [0, \frac{\pi}{2}]$.
19. If $z, w \in \mathbb{C}$ such that $\text{Re}(z) \leq 0$ and $\text{Re}(w) \leq 0$, then show that $|e^z - e^w| \leq |z - w|$.
20. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If $|f(z) - 1| < 1$ for all $z \in \Omega$ and if γ is a closed rectifiable path in Ω, then show that $\int_{\gamma} \frac{f'(z)}{f(z)}\, dz = 0$.
21. Determine whether the following functions have primitives.
    (a) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, defined by $f(z) = \frac{1}{z}$ for all $z \in \mathbb{C} \setminus \{0\}$.
    (b) $f : \mathbb{C} \to \mathbb{C}$, defined by $f(z) = e^{-z^2}$ for all $z \in \mathbb{C}$.

(c) $f : \mathbb{C} \setminus \{i, -i\} \to \mathbb{C}$, defined by $f(z) = \dfrac{1}{z^2+1}$ for all $z \in \mathbb{C} \setminus \{i, -i\}$.

(d) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, defined by $f(z) = \dfrac{\sin z}{z^2}$ for all $z \in \mathbb{C} \setminus \{0\}$.

22. If $\gamma(t) = 1 + 2e^{it}$ for all $t \in [0, 2\pi]$, then explain clearly why Cauchy's theorem for star-shaped domain cannot be applied to get $\int\limits_\gamma \frac{dz}{z-1} = 0$.

23. If $\gamma(t) = e^{it}$ for all $t \in [0, 2\pi]$, then evaluate (a) $\displaystyle\int_\gamma \frac{\text{Re}(z)}{2z-1}\,dz$ (b) $\displaystyle\int_\gamma \frac{|dz|}{|z-a|^2}$, where $a \in \mathbb{D}$ (c) $\int\limits_\gamma |z-1|\,|dz|$ (d) $\displaystyle\int_\gamma \Big(\frac{z-2}{2z-1}\Big)^3\,dz$

Part B. Core theory and proof-based exercises

24. Let $r \in \mathbb{R} \setminus \{1, 2\}$ and $r > 0$. If $\gamma(t) = re^{it}$ for all $t \in [0, 2\pi]$, then determine all possible values of $\displaystyle\int_\gamma \frac{e^{\sin z^2}}{(z^2+1)(z-2i)^3}\,dz$.

25. Let $r \in \mathbb{R} \setminus \{2\}$ and $r > 0$. If $\gamma(t) = re^{it}$ for all $t \in [0, 2\pi]$, then determine all possible values of $\displaystyle\int_\gamma \frac{z^2+1}{z(z^2+4)}\,dz$.

26. Evaluate $\int\limits_0^{2\pi} e^{e^{2it}-3it}\,dt$.

27. State TRUE or FALSE with justification: There exists a branch of the logarithm on $\mathbb{C} \setminus [-10, 10]$.

28. Let $f(z) = \dfrac{2z^3+1}{z^2+z}$ for all $z \in \mathbb{C} \setminus \{-1, 0\}$. Determine the Taylor series of $f : \mathbb{C} \setminus \{-1, 0\} \to \mathbb{C}$ about i.

29. Let $f : G \to \mathbb{C}$ be analytic, where G is an open set in $\mathbb{C}$. Let $z_0 \in G$ and $g(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} & \text{if } z \in G \setminus \{z_0\}, \\ f'(z_0) & \text{if } z = z_0. \end{cases}$ Determine whether $g : G \to \mathbb{C}$ is analytic.

30. State TRUE or FALSE with justification: If $f : \mathbb{C} \to \mathbb{C}$ is a non-constant analytic function, then there must exist a sequence (z_n) in $\mathbb{C}$ such that $|z_n| > n$ and $|f(z_n)| > n$ for all $n \in \mathbb{N}$.

31. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. If there exist $M, r > 0$ and $n \in \mathbb{N}$ such that $|f(z)| \le M|z|^n$ for all $z \in \mathbb{C}$ with $|z| > r$, then show that f is a polynomial of degree at most n.

32. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(0) = 0$ and $\lim\limits_{|z|\to\infty} f(z) = 0$. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.

33. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(0) = 0$ and $\lim\limits_{|z|\to\infty} \Re(f(z)) = 0$. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.

34. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\Re(f(z)) > 0$ for all $z \in \mathbb{C}$. Show that f is a constant function.

35. If both f and $\sqrt{f}$ are entire functions, show that f is constant.
36. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic such that $|\Re(f(z)) + \Im(f(z))| \leq 1$ for all $z \in \mathbb{C}$. Show that f is a constant function.
37. State TRUE or FALSE with justification: There exists a non-constant bounded analytic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.
38. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(z+1) = f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Show that f is a constant function.
39. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.
40. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that for each $z \in \mathbb{C}$, either $|f(z)| \leq 1$ or $|f'(z)| \leq 1$. Show that there exist $a, b \in \mathbb{C}$ such that $f(z) = az + b$ for all $z \in \mathbb{C}$.
41. If $f(z) = \sum\limits_{n=1}^{\infty} \dfrac{nz^n}{1 - z^n}$ for all $z \in \mathbb{D}$, then show that $f : \mathbb{D} \to \mathbb{C}$ is analytic.
42. Let G be an open set in $\mathbb{C}$. For each $n \in \mathbb{N}$, let $f_n : G \to \mathbb{C}$ be analytic and let $f : G \to \mathbb{C}$. If $f_n \to f$ uniformly on each compact subset of G, then show that for each $k \in \mathbb{N}$, $f_n^{(k)} \to f^{(k)}$ uniformly on each compact subset of G.
43. State TRUE or FALSE with justification:
    (a) There exists an entire function f such that $f(z)^2 = \sin z$ for all $z \in \mathbb{C}$.
    (b) If $f : \mathbb{D} \to \mathbb{C}$ is analytic such that $f(1 - \frac{1}{n}) = (1 - \frac{1}{n})^3$ for all $n \in \mathbb{N}$, then it is necessary that $f(z) = z^3$ for all $z \in \mathbb{D}$.
44. Let $f : \Omega \to \mathbb{C}$ be an analytic function, where Ω is a domain in $\mathbb{C}$. If $\{z \in \Omega : f(z) = 0\}$ is uncountable, then show that $f(z) = 0$ for all $z \in \Omega$.
45. Let $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If $f(z)g(z) = 0$ for all $z \in \Omega$, then show that $f(z) = 0$ for all $z \in \Omega$ or $g(z) = 0$ for all $z \in \Omega$.
46. Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ and $g : \overline{\mathbb{D}} \to \mathbb{C}$ be continuous such that f and g are analytic on $\mathbb{D}$. If $f(z) = g(z)$ for all $z \in \mathbb{C}$ with $|z| = 1$, then show that $f(z) = g(z)$ for all $z \in \mathbb{D}$.
47. Prove the fundamental theorem of algebra using the maximum modulus theorem.

Part C. Extensions, synthesis, and challenge problems

48. Let $f : \mathbb{D} \to \mathbb{D}$ be a bijective holomorphic function such that $f(0) = 0$. If $f^{-1} : \mathbb{D} \to \mathbb{D}$ is holomorphic, then show that there exists $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $f(z) = \alpha z$ for all $z \in \mathbb{D}$.
49. Let $f : \mathbb{D} \to \mathbb{C}$ be a non-constant analytic function such that $f(0) = 1$. Show that there exist infinitely many $z \in \mathbb{D}$ such that $|f(z)| = 1$.
50. Let $f : G \to \mathbb{C}$ be continuous, where G is an open set in $\mathbb{C}$. If the function $f^2 : G \to \mathbb{C}$, defined by $f^2(z) = f(z)^2$ for all $z \in G$, is analytic, then show that f is analytic.
51. Let G be an open set in $\mathbb{C}$ and let $f : G \to \mathbb{C}$ be such that both the functions $f^2 : G \to \mathbb{C}$ and $f^3 : G \to \mathbb{C}$ are analytic, where $f^2(z) = f(z)^2$ and $f^3(z) = f(z)^3$ for all $z \in G$. Show that f is analytic.

52. Let f and g be entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there exists $\alpha \in \mathbb{C}$ such that $f(z) = \alpha g(z)$ for all $z \in \mathbb{C}$.
53. Determine all entire functions f such that $f(z^2) = (f(z))^2$ for all $z \in \mathbb{C}$.
54. Let $f = u + iv$ be an entire function satisfying $u_x v_y - u_y v_x = 1$ throughout $\mathbb{C}$. Show that $f(z) = az + b$, where $a, b \in \mathbb{C}$ are constants and $|a| = 1$.
55. Let f be entire and assume that $|f(0)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Show that either $f(0) = 0$ or f is constant.
56. Let $f : \mathbb{D} \to \mathbb{C}$ be analytic and assume that $|f'(z)| \leq k$ for all $z \in \mathbb{D}$. Prove that $|f(z_1) - f(z_2)| \leq k|z_1 - z_2|$ for every pair of points $z_1, z_2 \in \mathbb{D}$.
57. Let $f : \mathbb{D} \to \mathbb{C}$ be analytic and assume that $\Re f'(z) \neq 0$ for all $z \in \mathbb{D}$. Show that f is injective on $\mathbb{D}$.
58. Let f be entire. Assume that $f(x) \in \mathbb{R}$ for every $x \in \mathbb{R}$ and that $f\left(\frac{1}{2n+1}\right) = f\left(\frac{1}{2n}\right)$ for all $n \in \mathbb{N}$. Show that f is constant.
59. Prove that any non-constant harmonic function on a non-empty open set $D \subseteq \mathbb{C}$ is infinitely differentiable (partial derivatives of all orders exist and are continuous on D).
60. Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\mathbb{C} \setminus \{0\}$ and bounded on $B(0, \frac{1}{2})$. Show that $\displaystyle\int_{|z|=R} f(z)\, dz = 0$ for every $R > 0$.
61. If g is entire and satisfies $|g(z) - 2z| \leq 1$ on $|z| = 1$, show that $|g'(0)| \leq 3$.
62. Suppose that f is analytic on the open unit disk D and satisfies $|f(z)| \leq 1$ for all $z \in D$. Show that $|f'(0)| \leq 1$.
63. Let $f : \mathbb{C} \to \mathbb{C}$ be continuous and analytic on $\mathbb{C} \setminus [-1, 1]$. Show that f is entire.
64. Define $F(z) = \int_0^1 \sin(t^2) e^{-itz}\, dt$. Show that F is entire and satisfies $|F(z)| \leq Ae^{B|y|}$ for $z = x + iy$ and for some positive constants A and B.
65. Find all entire functions f such that $f(x) = e^x$ for all $x \in \mathbb{R}$.
66. Show that an entire function whose image meets neither the real axis nor the imaginary axis must be constant.
67. Let f and g be analytic on a domain $D \subset \mathbb{C}$. If $\overline{f}\, g$ is analytic on D, show that either f is constant or $g \equiv 0$.
68. Let f be entire and assume that $\lim_{z \to \infty} \left|\frac{f(z)}{z}\right| = 0$. Show that f is constant.
69. If f and g are entire and $g\overline{f}$ is entire, show that either f is constant or $g \equiv 0$.
70. Use the maximum modulus principle to prove the fundamental theorem of algebra.
71. Let f be bounded and analytic on the right half-plane (RHP). Assume that f extends continuously to the imaginary axis and that $\sup_{y \in \mathbb{R}} |f(iy)| \leq M$. Show that $|f(z)| \leq M$ throughout the RHP. (Hint: apply the maximum modulus theorem to $g_\varepsilon(z) = (z+1)^{-\varepsilon} f(z)$ on a suitable semidisk.)

72. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. If either $\Re(f) : \mathbb{C} \to \mathbb{R}$ or $\Im(f) : \mathbb{C} \to \mathbb{R}$ is bounded, then show that f is a constant function.

Problem Set 5: Laurent Series, Singularities, and the Residue Theorem

Guide to the set.

> These problems are organized around annular expansions, isolated singularities, and the extraction of local information from Laurent coefficients and residues. The problems are arranged progressively: begin with Part A, use Part B for course-level mastery, and treat Part C as extended written work.

Part A. Foundations and preliminary checks

1. Determine all the singularities and their nature for
   (a) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, defined by $f(z) = \dfrac{\sin^2 z}{z^3}$ for all $z \in \mathbb{C} \setminus \{0\}$.
   (b) $f : \mathbb{C} \setminus S \to \mathbb{C}$, defined by $f(z) = \dfrac{e^z}{z(1-e^{-z})}$ for all $z \in \mathbb{C} \setminus S$, where $S = \{z \in \mathbb{C} : e^{-z} = 1\}$.
   (c) $f : \mathbb{C} \setminus S \to \mathbb{C}$, defined by $f(z) = \tan\frac{1}{z}$ for all $z \in \mathbb{C} \setminus S$, where $S = \{z \in \mathbb{C} \setminus \{0\} : \cos\frac{1}{z} = 0\}$.
2. Find the order of zero at $z = 0$ for $f : \mathbb{C} \to \mathbb{C}$, where for each $z \in \mathbb{C}$,
   (a) $f(z) = z^2(e^{z^2} - 1)$ (b) $f(z) = 6\sin z^3 + z^3(z^6 - 6)$.
3. Find all the zeros together with their orders for
   (a) $f : \mathbb{C} \to \mathbb{C}$, defined by $f(z) = (1 - e^z)(z^2 - 4)^3$ for all $z \in \mathbb{C}$.
   (b) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, defined by $f(z) = \dfrac{z^2 + 9}{z^4}$ for all $z \in \mathbb{C} \setminus \{0\}$.
4. Let $f : G \to \mathbb{C}$ and $g : G \to \mathbb{C}$ be analytic, where G is an open set in $\mathbb{C}$. If $z_0 \in G$ is a zero of both f and g of orders m and n respectively, then show that z_0 is a zero of $fg : G \to \mathbb{C}$ of order $m + n$.
5. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be analytic such that $|f(z)| \le 1$ for all $z \in \mathbb{C} \setminus \{0\}$ with $|z| < 1$. If $r > 0$ and $\gamma(t) = re^{it}$ for all $t \in [0, 2\pi]$, then determine (with justification) $\int_\gamma f$.
6. If $f(z) = \dfrac{1}{z(z-1)(z-2)}$ for all $z \in \mathbb{C} \setminus \{0, 1, 2\}$, then determine the Laurent series expansion of f in (a) $\operatorname{ann}(0; 0, 1)$, (b) $\operatorname{ann}(0; 1, 2)$, and (c) $\operatorname{ann}(0; 2, \infty)$.
7. If $f(z) = \dfrac{1}{z(z^2 - 1)}$ for all $z \in \mathbb{C} \setminus \{0, 1, -1\}$, then find the residue of $f : \mathbb{C} \setminus \{0, 1, -1\} \to \mathbb{C}$ at 1 using Laurent series expansion of f.

Part B. Core theory and proof-based exercises

8. Let f be an entire function such that $f(n) = 0$ for all $n \in \mathbb{Z}$. If $g(z) = \dfrac{f(z)}{\sin(\pi z)}$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$, then show that each $n \in \mathbb{Z}$ is a removable singularity of $g : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$.
9. Let G be an open set in $\mathbb{C}$ and let $z_0 \in G$ be a zero of order $m \in \mathbb{N}$ of an analytic function $f : G \to \mathbb{C}$. Determine $\operatorname{Res}\Big(\dfrac{f'}{f}; z_0\Big)$.
10. If $u : \mathbb{R}^2 \to \mathbb{R}$ is a bounded harmonic function, then show that u is a constant function.
11. Evaluate $\int\limits_{\gamma} \tan z \, dz$, where $\gamma(t) = 2e^{it}$ for all $t \in [0, 2\pi]$.
12. Determine all possible values of $\displaystyle\int_{\gamma} \frac{dz}{z(z^2+1)}$, where γ is any closed rectifiable path in $\mathbb{C} \setminus \{0, i, -i\}$.
13. Let $f : G \to \mathbb{C}$ be analytic and one-to-one, where G is an open set in $\mathbb{C}$. Show that $f'(z) \neq 0$ for all $z \in G$.
14. Show that all the roots of the equation $z^7 - 5z^3 + 12 = 0$ lie in $\{z \in \mathbb{C} : 1 < |z| < 2\}$.
15. If $\lambda \in \mathbb{R}$ such that $\lambda > 1$, then show that the equation $ze^{\lambda - z} = 1$ has exactly one root in $\mathbb{D}$ and that this root is real and positive.

Part C. Extensions, synthesis, and challenge problems

16. If $a \in \mathbb{R}$ such that $a > 1$, then show that $\displaystyle\int_0^{\pi} \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}}$.
17. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx = \frac{\pi}{\sqrt{2}}$.
18. Show that $\displaystyle\int_{\gamma} \frac{e^{az}}{z^2 + 1} \, dz = 2\pi i \sin a$, where $\gamma(t) = 2e^{it}$ for $t \in [0, 2\pi]$.
19. Let g be analytic on $B(0, 2)$. Compute $\displaystyle\int_{|z|=1} f(z) \, dz$ when

$$f(z) = \frac{a_k}{z^k} + \cdots + \frac{a_1}{z} + a_0 + g(z),$$

where the a_i are complex constants.

20. Does a primitive (antiderivative) of $\dfrac{1}{z}$ exist on $\mathbb{C} \setminus \{0\}$? If not, describe a maximal domain in $\mathbb{C}$ on which such a primitive does exist.
21. Show that for $m \neq -1$, the function z^m has a primitive on $\mathbb{C} \setminus \{0\}$.
22. Find the Laurent series of the function $f(z) = \exp\left(z + \frac{1}{z}\right)$ around 0. Further, show that for all $n \geq 0$,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{2\cos\theta} \cos(n\theta) \, d\theta = \sum_{k=0}^{\infty} \frac{1}{(n+k)! \, k!}.$$

23. Find the Laurent series expansions of the following functions about the indicated points $z = z_0$ or in the indicated regions. Specify the region of validity in each case.

(a) $z^2 \exp(1/z)$ in a neighborhood of $z = 0$.

(b) $\dfrac{1}{z^2+1}$ in a neighborhood of $z = -i$.

(c) $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ for $0 < |z| < 1$ and for $1 < |z| < 2$.

Problem Set 6: Meromorphic Theory, Argument Principle, and Conformal Mapping

Guide to the set.

> The last set combines global zero-counting arguments with conformal geometry. It is designed to encourage theorem-level reasoning rather than only local calculation. The problems are arranged progressively: begin with Part A, use Part B for course-level mastery, and treat Part C as extended written work.

Part A. Foundations and preliminary checks

1. ($\star$) **Winding number and index.** Let $\gamma : [0,1] \to \mathbb{C}$ be a piecewise C^1 closed curve (so $\gamma(0) = \gamma(1)$) and let $a \notin \gamma([0,1])$. Define

$$\operatorname{Ind}(\gamma, a) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz.$$

   (a) Show that $\operatorname{Ind}(\gamma, a) \in \mathbb{Z}$.

   (b) Compute $\operatorname{Ind}(\gamma, a)$ for $\gamma(t) = a + re^{2\pi i t}$, $0 \le t \le 1$.

   (c) Show that $\operatorname{Ind}(\gamma, a)$ is constant on each connected component of $\mathbb{C} \setminus \gamma([0,1])$.

2. ($\star$) **Argument principle.** Let f be meromorphic on a neighborhood of $\overline{D(a,r)}$ and assume that f has no zeros or poles on the circle $\partial D(a,r)$. Prove the argument principle in the form

$$\frac{1}{2\pi i}\int_{\partial D(a,r)} \frac{f'(z)}{f(z)}\,dz = N - P,$$

   where N (resp. P) is the number of zeros (resp. poles) of f in $D(a,r)$ counted with multiplicity.

3. Use the argument principle to determine the number of zeros (counted with multiplicity) of $f(z) = z^4 - 2z + 2$ in the disks:

   (a) $|z| < 1$,

   (b) $|z| < 2$.

4. ($\star$) **Rouché's theorem.** State and prove Rouché's theorem on $\partial D(0,r)$. Then use it to count the number of zeros (with multiplicity) of the following functions in $|z| < 1$:

   (a) $z^5 + 3z^2 + 2$,

   (b) $z^7 - 5z + 1$,

   (c) $e^z - z - 1$.

Part B. Core theory and proof-based exercises

5. Show that for each $n \in \mathbb{N}$ the polynomial $p_n(z) = z^n + z + 1$ has exactly one zero in the disk $|z| < 1$.
6. Let f be holomorphic on a neighborhood of $\overline{\mathbb{D}}$ and assume that $|f(z) - z^n| < |z^n|$ for all $|z| = 1$ and some fixed $n \in \mathbb{N}$. Show that f has exactly n zeros in $\mathbb{D}$ (counted with multiplicity).
7. ($\star$) Let f be entire and suppose that there exist constants $A, B > 0$ and $\alpha < 1$ such that $|f(z)| \le Ae^{B|z|^\alpha}$ for all $z \in \mathbb{C}$. Show that f must be a polynomial.
8. **Schwarz lemma.** Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Prove that $|f(z)| \le |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \le 1$, and characterize the equality case.
9. ($\star$) **Schwarz–Pick.** Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that for all $z, w \in \mathbb{D}$,

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)}\, f(z)} \right| \le \left| \frac{z - w}{1 - \overline{w}\, z} \right|.$$

Deduce that every holomorphic automorphism of $\mathbb{D}$ is of the form

$$\phi_{a,\theta}(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}\, z}, \qquad a \in \mathbb{D},\ \theta \in \mathbb{R}.$$

Part C. Extensions, synthesis, and challenge problems

10. **Möbius geometry.** Let $T(z) = \dfrac{az + b}{cz + d}$ with $ad - bc \neq 0$.
    (a) Show that T maps generalized circles (circles and straight lines) to generalized circles.
    (b) Find the image under T of the line $\Re z = 0$ when $T(z) = \dfrac{z-1}{z+1}$.
    (c) Determine the image of the circle $|z - 1| = 1$ under $T(z) = \dfrac{z-1}{z+1}$.
11. Find an explicit conformal bijection from the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ onto $\mathbb{D}$ and describe the images of the real axis and of horizontal/vertical lines.
12. Find a conformal bijection from the strip $S = \{z \in \mathbb{C} : 0 < \Im z < \pi\}$ onto $\mathbb{H}$ and hence onto $\mathbb{D}$.
13. ($\star$) Let $\Omega \subset \mathbb{C}$ be simply connected and not equal to $\mathbb{C}$. State the Riemann mapping theorem. Assuming it, explain how one can reduce the Dirichlet problem on Ω to the Dirichlet problem on $\mathbb{D}$.
14. **Harmonic conjugates.**
    (a) Verify that $u(x, y) = x^2 - y^2$ is harmonic on $\mathbb{R}^2$ and find a harmonic conjugate v on $\mathbb{C}$.
    (b) Find an analytic function f with $\Re f(z) = u(x, y)$.
15. ($\star$) **Mean value and maximum principle for harmonic functions.** Let u be harmonic on a neighborhood of $\overline{D(a, r)}$.

(a) Prove the mean value property

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt.$$

(b) Deduce that u cannot attain a strict maximum in $D(a, r)$ unless it is constant.

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