Department of Mathematics

Indian Institute of Technology Guwahati

Functional Analysis

Lecture Notes

MA543

Banach spaces, Hilbert spaces, and bounded linear operators

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Contents

Introduction

Functional analysis studies linear phenomena in infinite-dimensional settings by combining algebraic structure, metric or topological structure, and the action of linear operators. The central principle running through these notes is that a carefully chosen notion of size—typically a norm or inner product—turns qualitative questions about convergence, approximation, extension, compactness, and solvability into precise structural statements. From this point of view, completeness, duality, and spectral theory are not isolated topics; they are successive manifestations of the same general program.

The manuscript is organized as a progressive introduction to that program. It begins with normed spaces and the classical model examples ℓ^p, L^p, and spaces of continuous functions, and then passes to Banach spaces, where completeness becomes the decisive hypothesis behind the Baire category method and the three fundamental principles of operator theory. The Hahn–Banach theorem is treated as the main extension mechanism and as the starting point of duality. The later chapters develop weak and weak* topologies, reflexivity, spectral theory, compact operators, Hilbert space geometry, and finally the spectral theorem for compact self-adjoint operators. The ordering reflects both the internal logic of the subject and the demands of a first serious course in functional analysis.

Standing conventions. Throughout, F denotes either $\mathbb{R}$ or $\mathbb{C}$. For a normed space X, its continuous dual is denoted by X^*. If $T : X \to Y$ is a bounded linear operator, then $\|T\|$ denotes its operator norm. For a subset M of a normed space, $\overline{M}$ denotes norm closure unless another topology is explicitly specified. In Hilbert-space chapters, inner products are written in the standard bracket notation and all orthogonality statements are understood with respect to that inner product.

Expository policy. These notes are written as a rigorous course manuscript rather than as an encyclopedic reference. Proofs are presented in a form intended to be both fully reliable and pedagogically useful: standard reductions are made explicit when they clarify the mechanism of an argument, key examples are inserted near the relevant definitions, and closely related results are grouped so that their logical dependence remains visible. Where several proofs are possible, a single coherent route is chosen in order to preserve continuity of method across chapters.

Syllabus alignment and reading strategy. The notes may be read linearly from beginning to end, but they are also modular enough for use in a one-semester course. A first reading should concentrate on the definitions, the classical examples, and the four indispensable pillars of the subject: completeness, Baire category, Hahn–Banach extension, and bounded inverse phenomena. A second reading may then emphasize weak compactness, reflexivity, adjoints, and spectral questions. The problem sets are designed to support both readings by converting standard arguments into techniques the reader can reuse independently.

Course roadmap (syllabus alignment). The material is organized to align with a standard first graduate or advanced undergraduate syllabus in functional analysis:

- **Normed spaces and classical examples:** norms, metric/topology, completeness, ℓ^p, L^p, $C(K)$ .... **Chapter 1**
- **Banach spaces:** closed subspaces, quotients, separability, Baire category **Chapter 2**
- **Bounded linear operators:** operator norm, extensions, and the three fundamental principles (uniform boundedness, open mapping, closed graph) .... **Chapter 3**
- **Hahn–Banach:** extension, separation, annihilators; concrete dual identifications **Chapter 4**
- **Duality and weak topologies:** weak/weak* convergence, Banach–Alaoglu, reflexivity .... **Chapter 5**
- **Spectral theory in Banach spaces:** resolvent, spectrum, spectral radius **Chapter 6**
- **Compact operators:** compactness, basic spectral structure .... **Chapter 7**
- **Hilbert space theory:** orthogonality, projections, Riesz representation, orthonormal bases, adjoints .... **Chapter 8**
- **Spectral theorem:** compact self-adjoint operators .... **Chapter 9**

The *Problem Sets* at the end are designed to reinforce definitions and standard proof techniques; they may be used either as homework assignments or as guided practice while reading the chapters.

Chapter 1

Normed Spaces and Fundamental Examples

This chapter lays the foundational language of functional analysis by introducing seminorms, norms, and the metric and topological structures they induce on vector spaces. Its main purpose is to show how analytic geometry emerges from the interaction of algebraic structure and quantitative control. The standard sequence spaces and function spaces are treated systematically, with particular attention to the mechanisms that make them workable examples rather than merely formal constructions. The continuity of vector addition, scalar multiplication, and the norm is established in a unified manner, thereby placing normed spaces within the broader framework of topological vector spaces. The chapter culminates in Hölder's and Minkowski's inequalities, which provide the structural estimates underlying the basic ℓ^p and L^p theories developed throughout the remainder of the course.

1.1 Topological motivation and seminorms

Functional analysis begins when linear structure is studied together with topology. Seminorms and norms provide the basic bridge between these two viewpoints, and they explain how algebraic operations acquire continuity, convergence, and local geometric meaning.

We begin by placing normed spaces inside the broader framework of topological vector spaces. A normed linear space combines the linear structure of a vector space with a compatible notion of "size which induces a natural topology. Concretely, let X be a vector space over F. The algebraic operations

$$(x, y) \longmapsto x + y \quad (X \times X \to X), \qquad (a, x) \longmapsto ax \quad (F \times X \to X)$$

suggest studying topologies on X for which addition and scalar multiplication are continuous. A vector space equipped with such a topology is called a *topological vector space*.

In the normed setting, the topology is induced by a function $p : X \to [0, \infty)$ satisfying homogeneity and the triangle inequality. If one defines $\operatorname{dist}(x, y) = p(x-y)$, then translation invariance and homogeneity are built into the geometry.

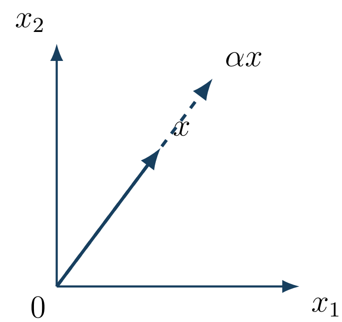

Figure 1.1: Scalar multiplication: the vector αx lies on the ray generated by x.

Definition 1.1 (Seminorm and norm)**.** A map $p : X \to [0, \infty)$ is called a *seminorm* if, for all $x, y \in X$ and $a \in F$,

$$p(ax) = |a|\, p(x), \qquad p(x + y) \le p(x) + p(y), \qquad p(0) = 0.$$

A seminorm p is a *norm* if, in addition, $p(x) = 0$ implies $x = 0$. In this case (X, p) is a *normed linear space* and $\|x\| := p(x)$ denotes the norm of x.

The metric induced by a norm is given by $d(x, y) = \|x - y\|$, and the induced topology is the *norm topology*. Unless explicitly stated otherwise, all closures and convergences below are taken with respect to the norm topology.

Proposition 1.2 (Continuity of the basic operations)**.** *Let $(X, \|\cdot\|)$ be a normed linear space. Then the maps*

$$(x, y) \longmapsto x+y \quad (X\times X \to X), \qquad (a, x) \longmapsto ax \quad (F\times X \to X), \qquad x \longmapsto \|x\| \quad (X \to \mathbb{R})$$

are continuous. In particular, every normed space is a topological vector space.

Proof. For addition, let $(x, y), (x_0, y_0) \in X \times X$. Then

$$\|(x + y) - (x_0 + y_0)\| \le \|x - x_0\| + \|y - y_0\|,$$

which establishes continuity. For scalar multiplication, let $(a, x), (a_0, x_0) \in F \times X$. Then

$$\begin{aligned} \|ax - a_0x_0\| &\leq \|ax - a_0x\| + \|a_0x - a_0x_0\| \\ &= |a - a_0|\,\|x\| + |a_0|\,\|x - x_0\|. \end{aligned}$$

If $(a, x) \to (a_0, x_0)$, then $x \to x_0$, so x remains bounded in a neighborhood of x_0; hence the right-hand side tends to 0. Finally, the reverse triangle inequality gives

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|,$$

so the norm map is Lipschitz, hence continuous. □

1.2 Minkowski's inequality

The triangle inequality is the decisive structural property behind the definition of a norm. In the present context, Minkowski's inequality is precisely the statement that guarantees stability of the spaces ℓ^p and L^p under addition and therefore allows these spaces to be treated as genuine normed linear spaces.

We state first the discrete form, whose proof contains the essential mechanism and whose integral counterpart follows by the same principle.

Theorem 1.3 (Minkowski's inequality in ℓ^p)**.** *Let $1 \leq p \leq \infty$ and let $x = (x_i)_{i\geq 1}$, $y = (y_i)_{i\geq 1} \in \ell^p$. Then $x + y \in \ell^p$ and*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p. \tag{1.1}$$

Proof. For $p = 1$ the inequality follows from $|x_i + y_i| \leq |x_i| + |y_i|$ by summation. For $p = \infty$ it follows from

$$\|x + y\|_\infty = \sup_i |x_i + y_i| \leq \sup_i |x_i| + \sup_i |y_i| = \|x\|_\infty + \|y\|_\infty.$$

Assume $1 < p < \infty$ and let q be the conjugate exponent, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1.2}$$

Using $|x_i + y_i| \leq |x_i| + |y_i|$ and the elementary identity

$$(|x_i| + |y_i|)^p = (|x_i| + |y_i|)^{p-1}|x_i| + (|x_i| + |y_i|)^{p-1}|y_i|,$$

we obtain

$$\begin{aligned}\sum_{i=1}^{\infty}|x_i+y_i|^p &\le \sum_{i=1}^{\infty}(|x_i|+|y_i|)^p \\ &= \sum_{i=1}^{\infty}(|x_i|+|y_i|)^{p-1}|x_i| + \sum_{i=1}^{\infty}(|x_i|+|y_i|)^{p-1}|y_i|. \qquad (1.3)\end{aligned}$$

By Hölder's inequality (with exponents p and q as in (1.2)),

$$\begin{aligned}\sum_{i=1}^{\infty}(|x_i|+|y_i|)^{p-1}|x_i| &\le \left(\sum_{i=1}^{\infty}(|x_i|+|y_i|)^p\right)^{\frac{p-1}{p}}\left(\sum_{i=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^{\infty}(|x_i|+|y_i|)^p\right)^{\frac{p-1}{p}}\|x\|_p, \qquad (1.4)\end{aligned}$$

and similarly

$$\sum_{i=1}^{\infty}(|x_i|+|y_i|)^{p-1}|y_i| \le \left(\sum_{i=1}^{\infty}(|x_i|+|y_i|)^p\right)^{\frac{p-1}{p}}\|y\|_p. \qquad (1.5)$$

Let

$$A := \sum_{i=1}^{\infty}(|x_i|+|y_i|)^p.$$

Combining (1.3), (1.4), and (1.5) gives

$$A \le A^{\frac{p-1}{p}}\Big(\|x\|_p + \|y\|_p\Big).$$

If $A=0$, then the conclusion is immediate. Otherwise, dividing by $A^{(p-1)/p}$ yields

$$A^{1/p} \le \|x\|_p + \|y\|_p.$$

Since $|x_i+y_i| \le |x_i|+|y_i|$, the last inequality implies (1.1). □

Remark 1.4 (Equality case). Assume $1<p<\infty$ and $x,y\ne 0$. Equality holds in (1.1) if and only if there exists $\lambda \ge 0$ such that

$$|x_i| = \lambda|y_i| \quad \text{for all } i,$$

and $x_i\overline{y_i} \ge 0$ for all i (in the real case this reduces to $x_iy_i \ge 0$).

Remark 1.5 (Normed-space consequence). In particular, (1.1) shows that $(\ell^p, \|\cdot\|_p)$ is a normed linear space for each $1 \le p \le \infty$.

1.3 Sequence spaces under the supremum norm

The supremum norm measures the largest coordinate of a sequence and therefore captures uniform control rather than summability. The spaces ℓ^∞, c, and c_0 provide the first important examples in which boundedness, convergence, and decay at infinity must be carefully distinguished.

The space

$$\ell^\infty := \{x = (x_n)_{n\geq 1} : \sup_{n\geq 1} |x_n| < \infty\}$$

becomes a normed linear space when equipped with the supremum norm

$$\|x\|_\infty := \sup_{n\geq 1} |x_n|.$$

Two important subspaces are

$$c := \{x = (x_n) : x_n \text{ converges}\}, \qquad c_0 := \{x = (x_n) : x_n \to 0\}.$$

Both are linear subspaces of ℓ^∞, and $c_0 \subset c \subset \ell^\infty$.

Proposition 1.6. *The following inclusions are strict:*

$$\ell^1 \subsetneq \ell^2 \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty.$$

Proof. If $x \in \ell^1$, then $\sum_{n=1}^\infty |x_n| < \infty$, hence $x_n \to 0$ and $\sup_n |x_n| \leq \|x\|_1$. Therefore $x \in \ell^\infty$ and

$$\sum_{n=1}^\infty |x_n|^2 \leq \left(\sup_{n\geq 1} |x_n|\right) \sum_{n=1}^\infty |x_n| \leq \|x\|_1^2,$$

so $x \in \ell^2$. Thus $\ell^1 \subset \ell^2$.

Next, if $x \in \ell^2$, then $\sum |x_n|^2 < \infty$, so necessarily $x_n \to 0$; hence $\ell^2 \subset c_0$. The sequence $(1/n)_{n\geq 1}$ belongs to $c_0 \setminus \ell^2$, so the inclusion is strict.

The constant sequence $\mathbf{1} = (1, 1, 1, \dots)$ belongs to $c \setminus c_0$, while the bounded oscillatory sequence $((-1)^n)_{n\geq 1}$ belongs to $\ell^\infty \setminus c$. This establishes the remaining strict inclusions. □

Proposition 1.7 (Comparison of the standard norms on F^n)**.** *For $x = (x_1, \dots, x_n) \in F^n$ one has*

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty, \qquad \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty.$$

In particular, all standard norms on a finite-dimensional space are equivalent.

Proof. The inequalities $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ are immediate from

$$\max_i |x_i|^2 \leq \sum_{i=1}^n |x_i|^2 \leq \left(\sum_{i=1}^n |x_i|\right)^2.$$

The estimate $\|x\|_1 \leq n\|x\|_\infty$ follows from $\sum_{i=1}^n |x_i| \leq \sum_{i=1}^n \|x\|_\infty = n\|x\|_\infty$, and then $\|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$ follows by squaring. □

1.4 Geometry of spheres in ℓ^p

Although all norms on a finite-dimensional space induce the same topology, they need not encode the same geometry. Examining the unit balls for different values of p helps build geometric intuition for convexity, extremal points, and the very different analytic behaviour that the same underlying vector space may exhibit under different norms.

For $0 \leq p \leq \infty$, consider $(\mathbb{R}^n, \|\cdot\|_1), (\mathbb{R}^n, \|\cdot\|_2), (\mathbb{R}^n, \|\cdot\|_p), (\mathbb{R}^n, \|\cdot\|_\infty)$.

The geometry of the unit ball varies significantly with p. In dimension two, the basic shapes are illustrated in Figure 1.2.

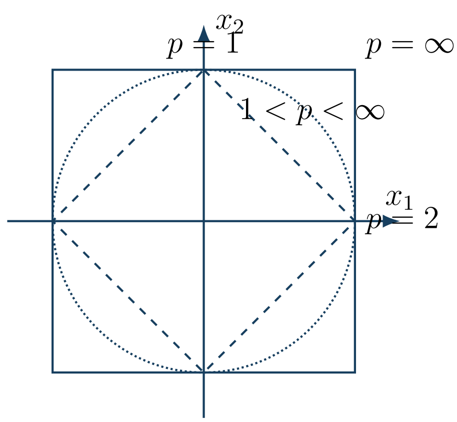

Figure 1.2: Comparison of the unit balls in $\mathbb{R}^2$ for the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$, together with the intermediate geometry for $1 < p < \infty$.

1.5 The space of finitely supported sequences c_{00}

The space c_{00} consists of elementary sequences with only finitely many nonzero coordinates. It plays the role of a tractable algebraic core inside the classical sequence spaces, and its density explains why many arguments can first be carried out for finitely supported sequences and then extended by approximation.

Let

$$c_{00} := \{x = (x_n)_{n\geq 1} : x_n = 0 \text{ for all but finitely many } n\}.$$

This is a linear subspace of every sequence space introduced above. In particular, it is a subspace of ℓ^p for each $1 \le p < \infty$ and of ℓ^∞.

Definition 1.8 (Dense subset)**.** Let $(X, \|\cdot\|)$ be a normed space and let $A \subset X$. We say that A is *dense in* X if for every $x \in X$ and every $\varepsilon > 0$ there exists $a \in A$ such that $\|x - a\| < \varepsilon$. Equivalently, $\overline{A} = X$.

Proposition 1.9. *For $1 \le p < \infty$, the space c_{00} is dense in $(\ell^p, \|\cdot\|_p)$.*

Proof. Let $x = (x_n) \in \ell^p$ and define the truncations

$$x^{(N)} := (x_1, \dots, x_N, 0, 0, \dots) \in c_{00}.$$

Then

$$\|x - x^{(N)}\|_p^p = \sum_{n>N} |x_n|^p \xrightarrow[N\to\infty]{} 0,$$

so $x^{(N)} \to x$ in ℓ^p. □

Proposition 1.10. *In $(\ell^\infty, \|\cdot\|_\infty)$ one has $\overline{c_{00}} = c_0$. In particular, c_{00} is not dense in ℓ^∞.*

Proof. If $x = (x_n) \in c_0$, then $x_n \to 0$. Given $\varepsilon > 0$, choose N such that $|x_n| < \varepsilon$ for all $n > N$ and let

$$x^{(N)} := (x_1, \dots, x_N, 0, 0, \dots) \in c_{00}.$$

Then

$$\|x - x^{(N)}\|_\infty = \sup_{n>N} |x_n| < \varepsilon,$$

so $c_0 \subset \overline{c_{00}}$.

Conversely, c_0 is closed in ℓ^∞ (see Example 2.4), and since $c_{00} \subset c_0$ it follows that $\overline{c_{00}} \subset c_0$. Hence $\overline{c_{00}} = c_0$. □

Remark 1.11 (Density versus openness)*.* Because every proper linear subspace of a normed space has empty interior, c_{00} is not open in either ℓ^p $(1 \le p < \infty)$ or ℓ^∞. Thus, in infinite-dimensional spaces, density and openness are fundamentally different notions.

1.6 Basic function spaces

Spaces of functions furnish the most important examples in functional analysis. Even at this preliminary stage, it is useful to compare different ways of measuring the size of a function, since the resulting norm or seminorm determines which analytic features—uniform behaviour, integrability, or mere boundedness—are being emphasized.

Let $[a, b] \subset \mathbb{R}$.

- $C[a, b]$ denotes the space of continuous functions on $[a, b]$, equipped with the supremum norm
$$\|f\|_\infty := \sup_{t \in [a,b]} |f(t)|.$$
- $R[a, b]$ denotes the space of Riemann integrable functions on $[a, b]$. The quantity
$$\|f\|_1 := \int_a^b |f(t)|\, dt$$
satisfies the triangle inequality, but it is only a *seminorm* on $R[a, b]$ (it may vanish for nonzero functions).
- $B[a, b]$ denotes the space of bounded functions on $[a, b]$, equipped with $\|\cdot\|_\infty$.

Thus $C[a, b] \subset R[a, b] \subset B[a, b]$.

1.7 Lebesgue spaces and the fundamental inequalities

Lebesgue spaces form one of the principal arenas of modern analysis. The inequalities established here, especially Hölder's and Minkowski's inequalities, are the basic estimates that make L^p spaces workable and prepare the ground for later results on completeness, duality, and operator theory.

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. For $1 \le p < \infty$ we define
$$L^p(\Omega) := \left\{ f \text{ measurable} : \int_\Omega |f|^p \, d\mu < \infty \right\} / \sim,$$
where $f \sim g$ if and only if $f = g$ μ-a.e., and we set
$$\|f\|_p := \left(\int_\Omega |f|^p \, d\mu \right)^{1/p}.$$
For $p = \infty$, we set
$$L^\infty(\Omega) := \{ f \text{ measurable} : \operatorname{ess\,sup}_\Omega |f| < \infty \} / \sim, \qquad \|f\|_\infty := \operatorname{ess\,sup}_\Omega |f|.$$

Lemma 1.12. *If $f \in L^1(\Omega)$ and $\|f\|_1 = 0$, then $f = 0$ μ-a.e.*

Proof. Let $E = \{x \in \Omega : |f(x)| > 0\} = \bigcup_{n \ge 1} E_n$ with $E_n = \{x : |f(x)| \ge 1/n\}$. Then
$$\mu(E_n) = \int_{E_n} 1 \, d\mu \le n \int_{E_n} |f| \, d\mu \le n \int_\Omega |f| \, d\mu = 0,$$

so $\mu(E) = 0$. $\square$

Theorem 1.13 (Hölder's inequality). *Let $1 < p < \infty$ and let q be the conjugate exponent, $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \le \|f\|_p \, \|g\|_q.$$

Proof. Young's inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ for $a, b \ge 0$ gives, after setting

$$a = \frac{|f|}{\|f\|_p}, \qquad b = \frac{|g|}{\|g\|_q},$$

that

$$\frac{|fg|}{\|f\|_p \|g\|_q} \le \frac{1}{p} \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g|^q}{\|g\|_q^q}.$$

Integrating over Ω yields

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \le \frac{1}{p} + \frac{1}{q} = 1.$$

$\square$

Theorem 1.14 (Minkowski's inequality). *Let $1 \le p < \infty$ and let $f, g \in L^p(\Omega)$. Then $f + g \in L^p(\Omega)$ and*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

Proof. The case $p = 1$ follows from $\int |f + g| \le \int |f| + \int |g|$. Assume $1 < p < \infty$ and let $q = \frac{p}{p-1}$. Then

$$\|f + g\|_p^p = \int |f + g| \, |f + g|^{p-1} \le \int |f| \, |f + g|^{p-1} + \int |g| \, |f + g|^{p-1}.$$

Applying Hölder's inequality to each term gives

$$\|f + g\|_p^p \le (\|f\|_p + \|g\|_p) \left\| |f + g|^{p-1} \right\|_q.$$

Since $(p-1)q = p$, we have $\left\| |f + g|^{p-1} \right\|_q = \|f + g\|_p^{p-1}$, and dividing yields the claim. $\square$

A remark on inclusions between L^p spaces

On a finite measure space (Ω, μ) (for instance $\Omega = [0, 1]$ with Lebesgue measure), one has the continuous inclusion

$$L^q(\Omega) \hookrightarrow L^p(\Omega) \qquad (1 \le p \le q \le \infty),$$

with the estimate $\|f\|_p \le \mu(\Omega)^{\frac{1}{p}-\frac{1}{q}}\|f\|_q$.

On $\Omega = \mathbb{R}$ equipped with Lebesgue measure, *no inclusion holds* between $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$:

- $f(x) = x^{-1/2}\,\chi_{(0,1)}(x)$ belongs to $L^1(\mathbb{R})$ but not to $L^2(\mathbb{R})$.
- $g(x) = (1+|x|)^{-1}$ belongs to $L^2(\mathbb{R})$ but not to $L^1(\mathbb{R})$ since

$$\int_{\mathbb{R}} |g(x)|\,dx = 2\int_0^\infty \frac{1}{1+x}\,dx = \infty.$$

Example 1.15 (A disjoint-translate construction)**.** Let $f(x) = x^{-1/2}\,\chi_{(0,1)}(x)$ and define $f_n(x) = f(x-n+1)$, so that $\operatorname{supp}(f_n) \subset (n-1,n)$ are pairwise disjoint. Set

$$h(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x).$$

Then $h \in L^1(\mathbb{R})$ but $h \notin L^2(\mathbb{R})$. Indeed,

$$\|h\|_1 = \sum_{n=1}^{\infty} 2^{-n}\|f_n\|_1 = \|f\|_1 \sum_{n=1}^{\infty} 2^{-n} < \infty,$$

while

$$\|h\|_2^2 = \sum_{n=1}^{\infty} 2^{-2n}\|f_n\|_2^2 = \sum_{n=1}^{\infty} 2^{-2n}\cdot\infty = \infty,$$

since $\|f\|_2 = \infty$.

Chapter 2

Banach Spaces and the Baire Category Method

The organizing principle of this chapter is completeness and its far-reaching consequences. After defining Banach spaces and establishing completeness for the standard examples, the discussion turns to the structural permanence properties of completeness, including closed subspaces, quotient spaces, and related constructions. The chapter then develops the Baire category theorem in a form suited to functional analysis and shows how completeness yields global information that is invisible at the level of pointwise arguments. This transition from local control to global structure is one of the decisive themes of the subject, and it prepares the ground for the major operator-theoretic principles proved in subsequent chapters.

2.1 Banach spaces

Completeness is the point at which a normed space becomes robust enough for limiting procedures to behave properly. In practice, many of the deepest global theorems of functional analysis depend not merely on linearity and continuity, but on the assurance that Cauchy sequences genuinely converge inside the space under consideration.

The basic concept is formalized in the following definition.

Definition 2.1 (Completeness and Banach spaces)**.** A normed linear space $(X, \|\cdot\|)$ is *complete* if every Cauchy sequence in X converges to a point of X. A complete normed linear space is called a *Banach space*.

Proposition 2.2 (Closed subspaces of Banach spaces)**.** *If X is a Banach space and $M \subset X$ is a closed linear subspace, then M is a Banach space with the induced norm.*

Proof. Let (x_n) be a Cauchy sequence in M. Then (x_n) is also a Cauchy sequence in X, and since X is complete, there exists $x \in X$ such that $x_n \to x$ in X. Because M is closed in X and each x_n belongs to M, it follows that $x \in M$. Hence every Cauchy sequence in M converges in M, and therefore M is complete. □

Example 2.3 (The scalar field)**.** The space $(\mathbb{R}, |\cdot|)$ is a Banach space because every Cauchy sequence in $\mathbb{R}$ converges.

Example 2.4 (The space c_0 is Banach)**.** Let

$$c_0 := \Big\{x = (x_n)_{n\geq 1} : \ x_n \to 0\Big\} \subset \ell^\infty, \qquad \|x\|_\infty := \sup_{n\geq 1} |x_n|.$$

Then $(c_0, \|\cdot\|_\infty)$ is a Banach space.

Proof. It suffices to show that c_0 is a closed subspace of the Banach space ℓ^∞, and then invoke Proposition 2.2.

Let $x^{(k)} \in c_0$ and assume $x^{(k)} \to x$ in ℓ^∞, i.e.,

$$\|x^{(k)} - x\|_\infty = \sup_{n\geq 1} |x_n^{(k)} - x_n| \xrightarrow[k\to\infty]{} 0. \tag{2.1}$$

Fix $\varepsilon > 0$ and choose k_0 such that $\sup_n |x_n^{(k_0)} - x_n| < \varepsilon/2$. Since $x^{(k_0)} \in c_0$, there exists N such that $|x_n^{(k_0)}| < \varepsilon/2$ for all $n \geq N$. Hence, for every $n \geq N$,

$$|x_n| \leq |x_n - x_n^{(k_0)}| + |x_n^{(k_0)}| < \varepsilon.$$

Thus $x_n \to 0$ as $n \to \infty$, so $x \in c_0$. Therefore c_0 is closed in ℓ^∞. $\square$

Example 2.5 (Continuous functions on a compact interval)**.** Let $C([a, b])$ denote the space of continuous functions on $[a, b]$ equipped with the supremum norm

$$\|f\|_\infty := \sup_{t\in[a,b]} |f(t)|.$$

Then $(C([a, b]), \|\cdot\|_\infty)$ is a Banach space.

Proof. Let (f_n) be Cauchy in $(C([a, b]), \|\cdot\|_\infty)$. Then, for every $\varepsilon > 0$ there exists N such that

$$\|f_m - f_n\|_\infty < \varepsilon \qquad \text{for all } m, n \geq N. \tag{2.2}$$

Fix $t \in [a, b]$. From (2.2) we obtain

$$|f_m(t) - f_n(t)| \leq \|f_m - f_n\|_\infty < \varepsilon \qquad \text{for all } m, n \geq N.$$

Thus $(f_n(t))$ is a Cauchy sequence in $\mathbb{R}$, and hence convergent. Define $f(t) := \lim_{n\to\infty} f_n(t)$.

Letting $n \to \infty$ in (2.2) (for fixed $m \geq N$) gives

$$|f_m(t) - f(t)| \leq \varepsilon \qquad \text{for all } t \in [a,b] \text{ and all } m \geq N,$$

which implies $\|f_m - f\|_\infty \leq \varepsilon$ for all $m \geq N$. Hence $f_n \to f$ uniformly on $[a,b]$. Since a uniform limit of continuous functions is continuous, $f \in C([a,b])$, so $C([a,b])$ is complete. □

Example 2.6 (An incomplete norm on continuous functions)**.** Equip $C([0,1])$ with the norm

$$\|f\|_1 := \int_0^1 |f(t)|\, dt.$$

Then $(C([0,1]), \|\cdot\|_1)$ is not complete.

Proof. For $n \in \mathbb{N}$ define $f_n \in C([0,1])$ by

$$f_n(t) := \begin{cases} nt, & 0 \leq t < \frac{1}{n}, \\ 1, & \frac{1}{n} \leq t \leq 1. \end{cases} \tag{2.3}$$

If $m > n$, then a direct computation gives

$$\|f_n - f_m\|_1 = \frac{1}{2}\Big(\frac{1}{n} - \frac{1}{m}\Big). \tag{2.4}$$

Therefore (f_n) is Cauchy in $(C([0,1]), \|\cdot\|_1)$.

Let $f := \mathbf{1}_{(0,1]}$ (with the convention $f(0) = 0$). Then $f \in L^1([0,1])$ but $f \notin C([0,1])$, and

$$\|f_n - f\|_1 = \int_0^{1/n} (1 - nt)\, dt = \frac{1}{2n} \xrightarrow[n\to\infty]{} 0. \tag{2.5}$$

Thus (f_n) converges in $L^1([0,1])$ to a function f which is not continuous, showing that $(C([0,1]), \|\cdot\|_1)$ is not complete. □

Proposition 2.7 (Inclusion on finite measure spaces)**.** *Let (Ω, μ) be a measure space with $\mu(\Omega) < \infty$, and let $1 \leq p \leq q \leq \infty$. Then $L^q(\Omega) \subset L^p(\Omega)$ and*

$$\|f\|_p \leq \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q, \qquad f \in L^q(\Omega), \tag{2.6}$$

where we interpret $\frac{1}{q} = 0$ when $q = \infty$.

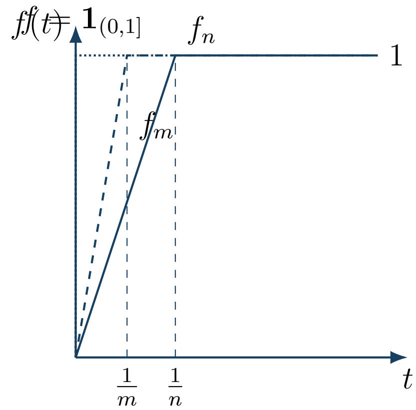

Figure 2.1: The functions f_n and f_m from (2.3) (here $m > n$), together with their pointwise limit $f = \mathbf{1}_{(0,1]}$.

Proof. First assume that $q < \infty$ and let $r := \frac{q}{p} \geq 1$, so that the conjugate exponent satisfies $\frac{1}{r} + \frac{1}{r'} = 1$. Writing $|f|^p = |f|^p \cdot 1$ and applying Hölder's inequality gives

$$\int_\Omega |f|^p \, d\mu \leq \Big(\int_\Omega |f|^{pr} \, d\mu \Big)^{1/r} \Big(\int_\Omega 1^{r'} \, d\mu \Big)^{1/r'} = \|f\|_q^p \, \mu(\Omega)^{1-\frac{p}{q}}.$$

Taking pth roots yields (2.6). The case $q = \infty$ follows similarly from $\int_\Omega |f|^p \leq \|f\|_\infty^p \, \mu(\Omega)$. □

Theorem 2.8 (Completeness of $L^p(\mathbb{R})$)**.** *Let $1 \leq p < \infty$. Then $L^p(\mathbb{R})$ is a Banach space. Moreover, if $f_n \to f$ in $L^p(\mathbb{R})$, then there exists a subsequence (f_{n_k}) such that $f_{n_k}(x) \to f(x)$ for almost every $x \in \mathbb{R}$.*

Proof. Let (f_n) be a Cauchy sequence in $L^p(\mathbb{R})$. Choose a subsequence (f_{n_k}) such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k} \qquad \text{for all } k \geq 1. \tag{2.7}$$

Define

$$g(x) := |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|, \qquad x \in \mathbb{R}. \tag{2.8}$$

Let $g_K(x) := |f_{n_1}(x)| + \sum_{k=1}^{K} |f_{n_{k+1}}(x) - f_{n_k}(x)|$. Then $g_K \uparrow g$ pointwise. By Minkowski's inequality and (2.7),

$$\|g_K\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{K} \|f_{n_{k+1}} - f_{n_k}\|_p \leq \|f_{n_1}\|_p + 1. \tag{2.9}$$

Applying Fatou's lemma to $g_K^p \uparrow g^p$ and using (2.9) gives $g \in L^p(\mathbb{R})$.

Now consider the partial sums

$$s_K := f_{n_1} + \sum_{k=1}^{K}(f_{n_{k+1}} - f_{n_k}) = f_{n_{K+1}}. \tag{2.10}$$

By (2.8), the series $\sum_{k\geq 1}(f_{n_{k+1}} - f_{n_k})(x)$ is absolutely convergent for almost every x (because $g(x) < \infty$ a.e.). Therefore $s_K(x) = f_{n_{K+1}}(x)$ converges pointwise a.e. to some measurable function f.

Finally, we show $f_{n_k} \to f$ in $L^p(\mathbb{R})$. From (2.8) and (2.10) we have, for each K,

$$|f_{n_{K+1}}(x) - f(x)| \leq \sum_{k=K+1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| \leq g(x) \quad \text{a.e.}$$

Moreover, $|f_{n_{K+1}} - f|^p \to 0$ pointwise a.e. as $K \to \infty$. Since $g^p \in L^1(\mathbb{R})$, the dominated convergence theorem yields

$$\|f_{n_{K+1}} - f\|_p^p = \int_{\mathbb{R}} |f_{n_{K+1}} - f|^p \xrightarrow[K\to\infty]{} 0.$$

Thus $f_{n_k} \to f$ in $L^p(\mathbb{R})$. As (f_n) was Cauchy, it follows that the whole sequence $f_n \to f$ in $L^p(\mathbb{R})$.

The subsequence assertion is proved by applying the construction above to the convergent sequence (f_n): we can choose (f_{n_k}) satisfying (2.7), which then converges pointwise a.e. to f by the argument following (2.10). $\square$

2.2 Functions vanishing at infinity

The space $C_0(\mathbb{R})$ is a natural substitute for continuous functions on a compact interval when the domain is noncompact. It retains the analytic convenience of the supremum norm while encoding the decay condition that many problems on unbounded domains require.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to *vanish at infinity* if

$$\lim_{|x|\to\infty} f(x) = 0.$$

If f is continuous and vanishes at infinity, then f is bounded: choose $R > 0$ such that $|f(x)| < 1$ for all $|x| > R$, and note that f is bounded on the compact interval $[-R, R]$ by continuity. Hence f is bounded on $\mathbb{R}$.

Let

$$C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous and } \lim_{|x|\to\infty} f(x) = 0\}.$$

Then $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ is a complete normed space, where

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$

If (f_n) is a Cauchy sequence in $\Big(C_0(\mathbb{R}), \|\cdot\|_\infty\Big)$, then for each fixed $x \in \mathbb{R}$ the sequence $(f_n(x))$ is Cauchy in $\mathbb{R}$, hence convergent. Define

$$f(x) := \lim_{n\to\infty} f_n(x).$$

Moreover, the Cauchy property in the sup norm implies uniform convergence: given $\varepsilon > 0$, choose n_0 such that $\|f_n - f_m\|_\infty < \varepsilon$ for all $n, m \geq n_0$, and let $m \to \infty$ to obtain $\|f_n - f\|_\infty \leq \varepsilon$ for all $n \geq n_0$. Thus $f_n \to f$ uniformly on $\mathbb{R}$, and in particular f is continuous.

Finally, we show that f vanishes at infinity. Fix $\varepsilon > 0$ and choose n_0 as above so that $\|f - f_{n_0}\|_\infty \leq \varepsilon$. Since $f_{n_0} \in C_0(\mathbb{R})$, there exists $R > 0$ such that $|f_{n_0}(x)| \leq \varepsilon$ whenever $|x| \geq R$. For such x,

$$|f(x)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x)| \leq 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\lim_{|x|\to\infty} f(x) = 0$, i.e. $f \in C_0(\mathbb{R})$. Hence $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ is complete.

2.3 The space $L^\infty(E)$

In measure-theoretic analysis one often needs a notion of boundedness that ignores exceptional null sets. The space $L^\infty(E)$ captures exactly this idea through the essential supremum and therefore complements the scale of L^p spaces in a manner that is indispensable for duality and interpolation arguments.

Let $E \subset \mathbb{R}$ be Lebesgue measurable. A measurable function $f : E \to F$ is said to be *essentially bounded* if there exists $M \geq 0$ such that

$$m\Big(\{x \in E : |f(x)| > M\}\Big) = 0.$$

The smallest such bound is the *essential supremum* of $|f|$, denoted by

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x\in E} |f(x)| = \inf\Big\{ M \geq 0 : m\Big(\{x \in E : |f(x)| > M\}\Big) = 0 \Big\}.$$

As in the case of L^p, functions are identified when they agree almost everywhere.

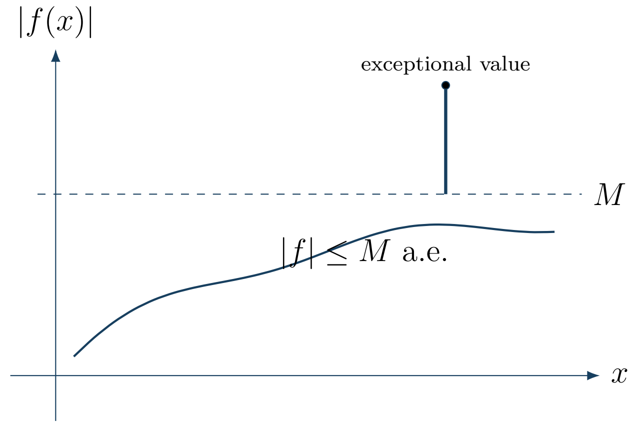

Figure 2.2: Essential boundedness ignores values attained only on a null set.

Remark 2.9 (Superlevel sets below the essential supremum). If $0 < \alpha < \|f\|_\infty$, then the set $\{x \in E : |f(x)| > \alpha\}$ has positive measure. Otherwise, α would already be an admissible essential upper bound.

Proposition 2.10 (Comparison with L^p on finite measure sets)**.** *If $m(E) < \infty$ and $1 \leq p < \infty$, then $L^\infty(E) \subset L^p(E)$ and*

$$\|f\|_p \leq m(E)^{1/p}\|f\|_\infty \qquad (f \in L^\infty(E)).$$

Proof. Since $|f(x)| \leq \|f\|_\infty$ almost everywhere,

$$\int_E |f|^p\, dx \leq \|f\|_\infty^p\, m(E),$$

and the claimed estimate follows on taking pth roots. □

Theorem 2.11. *For every measurable set $E \subset \mathbb{R}$, the space $L^\infty(E)$ is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $L^\infty(E)$. Choose a subsequence (f_{n_k}) such that

$$\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k} \qquad (k \geq 1).$$

For each k let

$$A_k := \{x \in E : |f_{n_{k+1}}(x) - f_{n_k}(x)| > 2^{-k}\}.$$

Then $m(A_k) = 0$. Hence the exceptional set $A := \bigcup_{k\geq 1} A_k$ also has measure zero, and for every $x \in E \setminus A$ we have

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| \leq 2^{-k} \qquad (k \geq 1).$$

Therefore the series $\sum_{k\geq 1}(f_{n_{k+1}}(x) - f_{n_k}(x))$ converges absolutely for $x \in E \setminus A$, so $(f_{n_k}(x))$ converges pointwise there to a measurable function f.

Moreover, for $m > k$ and $x \in E \setminus A$,

$$|f_{n_m}(x) - f_{n_k}(x)| \leq \sum_{j=k}^{m-1} 2^{-j} \leq 2^{-k+1}.$$

Letting $m \to \infty$ gives

$$|f(x) - f_{n_k}(x)| \leq 2^{-k+1} \qquad \text{for a.e. } x \in E.$$

Hence

$$\|f - f_{n_k}\|_\infty \leq 2^{-k+1} \xrightarrow[k\to\infty]{} 0.$$

Since (f_n) is Cauchy in norm and a subsequence converges to f, the entire sequence converges to f in $L^\infty(E)$. Thus $L^\infty(E)$ is complete. $\square$

Proposition 2.12 (Limit of L^p-norms as $p \to \infty$)**.** *Let $E \subset \mathbb{R}$ be measurable with $0 < m(E) < \infty$, and let $f \in L^\infty(E)$. Then*

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Proof. The upper bound follows from the previous proposition:

$$\|f\|_p \leq m(E)^{1/p}\|f\|_\infty,$$

so $\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty$.

Fix $\varepsilon > 0$ and consider the set

$$G_\varepsilon := \{x \in E : |f(x)| > \|f\|_\infty - \varepsilon\}.$$

By the defining property of the essential supremum, $m(G_\varepsilon) > 0$. Therefore

$$\|f\|_p^p = \int_E |f|^p \, dx \geq \int_{G_\varepsilon} |f|^p \, dx \geq (\|f\|_\infty - \varepsilon)^p m(G_\varepsilon).$$

Taking pth roots and letting $p \to \infty$ yields

$$\liminf_{p \to \infty} \|f\|_p \geq \|f\|_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows. □

2.4 Characterization of Banach spaces

A useful way to test completeness is to ask whether every absolutely convergent series converges in the space. This criterion is conceptually important because it translates the metric notion of completeness into a statement about infinite linear combinations, thereby linking topology and algebra in a very effective form.

In $\mathbb{R}$ or $\mathbb{C}$, absolute convergence implies convergence. The same principle characterizes completeness in a normed space.

Example 2.13 (A non-complete normed space)**.** Let $x^{(k)} \in c_{00}$ be the sequence whose kth coordinate is $1/k^2$ and all other coordinates are 0; thus

$$x^{(k)} = (0, \dots, 0, \tfrac{1}{k^2}, 0, \dots).$$

Then

$$\sum_{k=1}^{\infty} \|x^{(k)}\|_\infty = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

so the series is absolutely convergent with respect to $\|\cdot\|_\infty$. Its partial sums are

$$S_n = \sum_{k=1}^{n} x^{(k)} = (1, \tfrac{1}{2^2}, \dots, \tfrac{1}{n^2}, 0, 0, \dots),$$

and hence

$$S_n \to (1, \tfrac{1}{2^2}, \tfrac{1}{3^2}, \dots)$$

in ℓ^∞. The limit belongs to c_0 but not to c_{00}. Therefore c_{00} is not complete in the supremum norm, and

$$\overline{c_{00}}^{\|\cdot\|_\infty} = c_0.$$

Theorem 2.14 (Series characterization of completeness)**.** *A normed linear space $(X, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series in X converges in X.*

Proof. First assume that X is complete, and let $(x_k) \subset X$ satisfy

$$\sum_{k=1}^{\infty} \|x_k\| < \infty. \tag{2.11}$$

Define

$$y_n := \sum_{k=1}^{n} x_k.$$

If $m > n$, then by the triangle inequality,

$$\|y_m - y_n\| = \left\| \sum_{k=n+1}^{m} x_k \right\| \le \sum_{k=n+1}^{m} \|x_k\| \xrightarrow[m,n\to\infty]{} 0,$$

so (y_n) is Cauchy. Since X is complete, $y_n \to y \in X$, and therefore $\sum_{k=1}^{\infty} x_k = y$.

Conversely, assume that every absolutely convergent series in X converges. Let (y_n) be a Cauchy sequence in X. Choose inductively integers

$$n_1 < n_2 < n_3 < \cdots$$

such that

$$\|y_{n_{k+1}} - y_{n_k}\| < 2^{-k} \qquad (k \ge 1).$$

Set

$$x_k := y_{n_{k+1}} - y_{n_k}.$$

Then

$$\sum_{k=1}^{\infty} \|x_k\| \le \sum_{k=1}^{\infty} 2^{-k} < \infty,$$

so the series $\sum_{k=1}^{\infty} x_k$ converges by hypothesis. Since its partial sums telescope,

$$\sum_{k=1}^{N} x_k = y_{n_{N+1}} - y_{n_1},$$

we conclude that (y_{n_k}) converges in X. A Cauchy sequence with a convergent subsequence must converge to the same limit; hence (y_n) converges. Thus X is complete. $\square$

Definition 2.15 (Equivalent norms)**.** Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a linear space X. We say that they are *equivalent* if there exist constants $\alpha, \beta > 0$ such that

$$\alpha\|x\|_2 \le \|x\|_1 \le \beta\|x\|_2 \qquad (x \in X).$$

Proposition 2.16 (All norms are equivalent in finite dimensions)**.** *Every two norms on a finite-dimensional vector space are equivalent.*

Proof. Let X be finite dimensional, and fix a basis $e_1, \dots, e_n$. Identify X with F^n via

$$(x_1, \dots, x_n) \longmapsto x_1 e_1 + \cdots + x_n e_n.$$

Let $\|\cdot\|_2$ denote the Euclidean norm on F^n, and let $\|\cdot\|_1$ be any other norm on X. Consider the map

$$\Phi : F^n \to \mathbb{R}, \qquad \Phi(x_1, \dots, x_n) := \|x_1 e_1 + \cdots + x_n e_n\|_1.$$

By the triangle inequality, Φ is continuous. On the Euclidean unit sphere

$$S := \{x \in F^n : \|x\|_2 = 1\},$$

which is compact, Φ attains its minimum and maximum:

$$m := \min_{x \in S} \Phi(x), \qquad M := \max_{x \in S} \Phi(x).$$

Because $\Phi(x) = 0$ only when $x = 0$, we have $m > 0$. If $x \neq 0$, then $x/\|x\|_2 \in S$, and therefore

$$m \leq \Phi\left(\frac{x}{\|x\|_2}\right) \leq M.$$

Multiplying by $\|x\|_2$ yields

$$m\|x\|_2 \leq \|x\|_1 \leq M\|x\|_2.$$

Thus the two norms are equivalent. □

Remark 2.17 (Topological equivalence of norms). Equivalent norms generate the same topology. Indeed, if

$$\alpha\|x\|_2 \leq \|x\|_1 \leq \beta\|x\|_2,$$

then for every $r > 0$,

$$B^{(2)}_{r/\beta}(0) \subset B^{(1)}_{r}(0) \qquad \text{and} \qquad B^{(1)}_{r/\alpha}(0) \subset B^{(2)}_{r}(0).$$

Hence a set is open for one norm if and only if it is open for the other.

Remark 2.18 (Non-equivalent norms in infinite dimensions). In infinite-dimensional spaces, norms need not be equivalent. For example, on $C([0,1])$ the norms

$$\|f\|_1 = \int_0^1 |f(t)|\,dt \qquad \text{and} \qquad \|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$$

are not equivalent. We always have $\|f\|_1 \le \|f\|_\infty$, but there is no constant $B > 0$ such that

$$\|f\|_\infty \le B\|f\|_1 \qquad (f \in C([0,1])).$$

Indeed, for $f_n(t) = t^n$ we have $\|f_n\|_\infty = 1$ while

$$\|f_n\|_1 = \int_0^1 t^n\,dt = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0.$$

Thus no such B can exist.

Exercises. Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent on $L^2([0,1])$, and likewise that $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent on $L^1([0,1])$.

2.5 Quotient spaces

Quotient spaces allow us to collapse a subspace to zero and thereby isolate the genuinely new directions that remain. They appear naturally whenever one studies extensions, annihilators, or solvability modulo constraints, and they provide one of the standard mechanisms for passing from local information to global structure.

Let X be a normed space and let $M \subset X$ be a linear subspace. Define an equivalence relation on X by

$$x \sim y \quad \Longleftrightarrow \quad x - y \in M.$$

The equivalence class of x is the coset $x + M = \{x + m : m \in M\}$, and the set of all cosets is the *quotient space*

$$X/M := \{x + M : x \in X\}.$$

It is a linear space with $(x + M) + (y + M) = (x + y) + M$ and $\lambda(x + M) = (\lambda x) + M$.

The quotient norm. For $x \in X$ define

$$\|x + M\|_{X/M} := \operatorname{dist}(x, M) := \inf_{m\in M} \|x - m\|.$$

This quantity depends only on the coset $x + M$ and is always a seminorm on X/M. If, in addition, M is closed, then $\|\cdot\|_{X/M}$ is a norm.

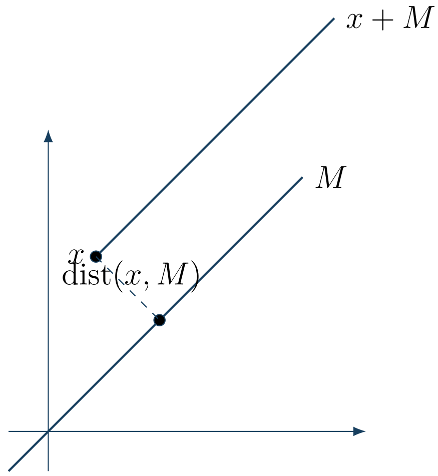

Figure 2.3: Geometric interpretation of the quotient norm: the norm of the coset $x + M$ is the distance from x to the subspace M.

Proposition 2.19. *Let X be a normed space and let $M \subset X$ be a closed subspace. Then*

$$\|x + M\|_{X/M} := \inf_{m \in M} \|x - m\|$$

defines a norm on X/M.

Proof. We verify the norm axioms.

Well-definedness. Suppose $x + M = y + M$. Then $y = x + m_0$ for some $m_0 \in M$. Hence

$$\inf_{m \in M} \|y - m\| = \inf_{m \in M} \|x - (m - m_0)\| = \inf_{m \in M} \|x - m\|,$$

because $m \mapsto m - m_0$ is a bijection of M onto itself.

Homogeneity. For $\lambda \in F$ and $\lambda \neq 0$,

$$\|\lambda(x+M)\|_{X/M} = \inf_{m \in M} \|\lambda x - m\| = \inf_{m \in M} \|\lambda x - \lambda m\| = |\lambda| \inf_{m \in M} \|x - m\| = |\lambda| \, \|x + M\|_{X/M}.$$

The case $\lambda = 0$ is immediate.

Triangle inequality. Given $\varepsilon > 0$, choose $m_1, m_2 \in M$ such that

$$\|x - m_1\| < \|x + M\|_{X/M} + \varepsilon, \qquad \|y - m_2\| < \|y + M\|_{X/M} + \varepsilon.$$

Then $m_1 + m_2 \in M$ and therefore

$$\begin{aligned}\|(x+y)+M\|_{X/M} &\le \|(x+y)-(m_1+m_2)\| \\ &\le \|x-m_1\| + \|y-m_2\| \\ &< \|x+M\|_{X/M} + \|y+M\|_{X/M} + 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the triangle inequality follows.

Definiteness. If $\|x+M\|_{X/M} = 0$, then there exists a sequence $(m_n) \subset M$ with $\|x-m_n\| \to 0$. Hence $m_n \to x$. Because M is closed, $x \in M$, so $x + M = M$, the zero vector of X/M. □

Remark 2.20 (The quotient map)*.* The quotient map

$$q : X \to X/M, \qquad q(x) = x + M,$$

is linear, continuous, and surjective. Moreover,

$$\|q(x)\|_{X/M} = \operatorname{dist}(x, M) \le \|x\|,$$

so $\|q\| \le 1$.

Example 2.21 (A one-dimensional quotient)**.** Let $X = C([0,1])$ with the supremum norm, and let

$$M := \{f \in C([0,1]) : f(0) = 0\}.$$

Then M is a closed subspace of X, and X/M is naturally isometrically isomorphic to F.

Proof. Define

$$\Phi : X/M \to F, \qquad \Phi(f+M) = f(0).$$

This is well defined: if $f + M = g + M$, then $f - g \in M$, so $(f-g)(0) = 0$, whence $f(0) = g(0)$. The map is linear, injective, and surjective. Finally,

$$\|f+M\|_{X/M} = \inf\{\|h\|_\infty : h \in C([0,1]),\ h(0) = f(0)\}.$$

Every such h satisfies $\|h\|_\infty \ge |h(0)| = |f(0)|$, while the constant function $h \equiv f(0)$ attains this bound. Therefore

$$\|f+M\|_{X/M} = |f(0)| = |\Phi(f+M)|,$$

so Φ is an isometric isomorphism. □

Example 2.22 (The quotient ℓ^∞/c_0)**.** For $x = (x_n) \in \ell^\infty$, let $\widetilde{x} = x + c_0 \in \ell^\infty/c_0$. Then

$$\|\widetilde{x}\| = \limsup_{n\to\infty} |x_n|.$$

Proof. Set $L := \limsup_{n\to\infty} |x_n|$. If $y = (y_n) \in c_0$, then $y_n \to 0$, so

$$\limsup_{n\to\infty} |x_n - y_n| = \limsup_{n\to\infty} |x_n| = L.$$

Since the supremum norm dominates the limsup,

$$\|x - y\|_\infty \geq L \qquad (y \in c_0),$$

and hence $\|\widetilde{x}\| \geq L$.

Conversely, fix $\varepsilon > 0$. By the definition of limsup, there exists N such that $|x_n| \leq L + \varepsilon$ for all $n \geq N$. Define

$$y_n := \begin{cases} x_n, & n < N, \\ 0, & n \geq N. \end{cases}$$

Then $y \in c_0$, and $|(x - y)_n| \leq L + \varepsilon$ for all n. Therefore

$$\|x - y\|_\infty \leq L + \varepsilon.$$

Taking the infimum over $y \in c_0$ and then letting $\varepsilon \downarrow 0$ gives $\|\widetilde{x}\| \leq L$. $\square$

Example 2.23 (The quotient c/c_0)**.** If $x = (x_n) \in c$ and $x_n \to \ell$, then for the coset $\widetilde{x} = x + c_0 \in c/c_0$ we have

$$\|\widetilde{x}\| = |\ell|.$$

In particular, the map

$$\Psi : c/c_0 \to F, \qquad \Psi(x + c_0) = \lim_{n\to\infty} x_n,$$

is an isometric isomorphism.

Proof. Since $x_n \to \ell$, the sequence $x - \ell\mathbf{1}$ belongs to c_0, where $\mathbf{1} = (1, 1, \dots)$. Hence

$$\widetilde{x} = \ell\mathbf{1} + c_0.$$

Applying the previous example to the bounded sequence $\ell\mathbf{1}$ yields

$$\|\widetilde{x}\| = \|\ell\mathbf{1} + c_0\| = \limsup_{n\to\infty} |\ell| = |\ell|.$$

It follows that Ψ is linear and norm-preserving, and therefore an isometric isomorphism from c/c_0 onto F. □

Theorem 2.24. *If M is a closed subspace of a Banach space X, then the quotient space X/M is a Banach space.*

Proof. By Theorem 2.14, it is enough to show that every absolutely convergent series in X/M converges. Let $(\widetilde{x}_n) \subset X/M$ satisfy

$$\sum_{n=1}^{\infty} \|\widetilde{x}_n\| < \infty.$$

Choose representatives $x_n \in X$ of the cosets $\widetilde{x}_n$, and for each n pick $m_n \in M$ such that

$$\|x_n - m_n\| \le \|\widetilde{x}_n\| + 2^{-n}.$$

Then

$$\sum_{n=1}^{\infty} \|x_n - m_n\| < \infty.$$

Since X is Banach, the series $\sum_{n=1}^{\infty}(x_n - m_n)$ converges to some $y \in X$. Therefore, in X/M,

$$\sum_{n=1}^{N} \widetilde{x}_n = \sum_{n=1}^{N}(x_n + M) = \sum_{n=1}^{N}(x_n - m_n + M) \longrightarrow y + M.$$

Thus $\sum_{n=1}^{\infty} \widetilde{x}_n$ converges in X/M. □

Theorem 2.25. *Let M be a complete subspace of a normed space X. If X/M is complete, then X is complete.*

Proof. Again using Theorem 2.14, let $(x_n) \subset X$ satisfy

$$\sum_{n=1}^{\infty} \|x_n\| < \infty.$$

Write $s_N := \sum_{n=1}^{N} x_n$ and let $q : X \to X/M$ be the quotient map. Since

$$\sum_{n=1}^{\infty} \|q(x_n)\| \le \sum_{n=1}^{\infty} \|x_n\| < \infty,$$

the series $\sum q(x_n)$ converges in X/M to some $q(y)$. Hence

$$q(s_N - y) \to 0 \qquad (N \to \infty).$$

For each N choose $m_N \in M$ such that

$$\|s_N - y + m_N\| \le \|q(s_N - y)\| + 2^{-N}.$$

Then $\|s_N - y + m_N\| \to 0$. For $N > K$ we therefore have

$$\begin{aligned}\|m_N - m_K\| &\le \|m_N + y - s_N\| + \|s_N - s_K\| + \|s_K - y - m_K\| \\ &\le \|s_N - y + m_N\| + \sum_{n=K+1}^{N} \|x_n\| + \|s_K - y + m_K\|.\end{aligned}$$

The right-hand side tends to 0 as $K, N \to \infty$, so (m_N) is Cauchy in M. Since M is complete, $m_N \to m \in M$. Finally,

$$\|s_N - (y - m)\| \le \|s_N - y + m_N\| + \|m_N - m\| \xrightarrow[N\to\infty]{} 0.$$

Thus $\sum x_n$ converges in X, and X is complete. $\square$

Example 2.26 (The space c is Banach)**.** Since $c/c_0 \cong F$ and c_0 is Banach, Theorem 2.25 implies that c is Banach.

Proposition 2.27 (The quotient map is open)**.** *Let M be a closed subspace of a normed space X, and let $q : X \to X/M$ be the quotient map. Then q is linear, continuous, surjective, and open.*

Proof. Continuity and surjectivity are clear. To prove openness, it suffices to show that the image of every open ball is open. Let $r > 0$. Then

$$q(B_X(0, r)) = \{x + M \in X/M : \|x + M\|_{X/M} < r\} = B_{X/M}(0, r).$$

Indeed, if $x \in B_X(0, r)$, then $\|q(x)\|_{X/M} \le \|x\| < r$. Conversely, if $\|x + M\|_{X/M} < r$, then by definition of the quotient norm there exists $m \in M$ such that $\|x - m\| < r$, and hence

$$x + M = (x - m) + M = q(x - m) \in q(B_X(0, r)).$$

By translation, $q(B_X(x_0, r)) = B_{X/M}(q(x_0), r)$ for every $x_0 \in X$, so q is open. $\square$

Proposition 2.28 (First isomorphism theorem for normed spaces)**.** *Let X and Y be normed spaces, and let $T : X \to Y$ be a bounded surjective linear map. Then the induced map*

$$\widetilde{T} : X/\ker T \to Y, \qquad \widetilde{T}(x + \ker T) = Tx,$$

is a well-defined bounded linear bijection. If, in addition, X and Y are Banach spaces, then $\widetilde{T}$ is an isomorphism of Banach spaces.

Proof. If $x + \ker T = y + \ker T$, then $x - y \in \ker T$, so $Tx = Ty$; thus $\widetilde{T}$ is well defined. Linearity is immediate, surjectivity follows from surjectivity of T, and injectivity is clear from the definition of the kernel.

If $q : X \to X/\ker T$ denotes the quotient map, then $T = \widetilde{T} \circ q$. Hence, for $x \in X$,

$$\|\widetilde{T}(q(x))\| = \|Tx\| \leq \|T\| \, \|x\|.$$

Taking the infimum over all representatives of the coset $q(x)$ yields

$$\|\widetilde{T}(x + \ker T)\| \leq \|T\| \, \|x + \ker T\|,$$

so $\widetilde{T}$ is bounded. If X and Y are Banach, then $X/\ker T$ is Banach by Theorem 2.24, and the open mapping theorem implies that $\widetilde{T}^{-1}$ is also bounded. □

Remark 2.29 (Riesz's lemma as a substitute for nearest-point projection). In finite-dimensional Euclidean geometry, a point outside a closed subspace admits a nearest point in that subspace. In general normed spaces this need not be true. Riesz's lemma provides a useful substitute: although an exact nearest-point projection may fail to exist, one can still find unit vectors that stay uniformly away from a given proper closed subspace.

Theorem 2.30 (Riesz lemma)**.** *Let M be a proper closed subspace of a normed linear space X. Then for every $0 < t < 1$, there exists a unit vector $x_t \in X$ such that*

$$\operatorname{dist}(x_t, M) > t.$$

Proof. Choose $u \in X \setminus M$ and set

$$\delta := \inf_{m \in M} \|u - m\|.$$

Because M is closed and $u \notin M$, we have $\delta > 0$. Fix $0 < t < 1$. Since $\delta < \delta/t$, the definition of the infimum gives $m_0 \in M$ such that

$$\delta \le \|u - m_0\| < \frac{\delta}{t}.$$

Define

$$x_t := \frac{u - m_0}{\|u - m_0\|}.$$

Then $\|x_t\| = 1$. For any $m \in M$, the vector $m_0 + \|u - m_0\| m$ also belongs to M, and hence

$$\|x_t - m\| = \frac{1}{\|u - m_0\|} \Big\| u - (m_0 + \|u - m_0\| m) \Big\| \ge \frac{\delta}{\|u - m_0\|} > t.$$

Taking the infimum over $m \in M$ proves the claim. □

Remark 2.31 (Noncompactness of the unit ball in infinite dimensions). A standard consequence of Riesz's lemma is that the closed unit ball of an infinite-dimensional normed space cannot be compact. Indeed, one can recursively choose unit vectors (x_n) such that

$$\operatorname{dist}\Big(x_n, \operatorname{span}\{x_1, \dots, x_{n-1}\}\Big) > \frac{1}{2} \qquad (n \ge 2),$$

which implies $\|x_n - x_m\| > \frac{1}{2}$ for $n \neq m$. Thus (x_n) has no convergent subsequence.

Theorem 2.32. *The closed unit ball of a normed linear space X is compact if and only if X is finite dimensional.*

Proof. First assume that the closed unit ball

$$\overline{B(0)} = \{x \in X : \|x\| \le 1\}$$

is compact. Suppose, toward a contradiction, that X is infinite-dimensional. Choose $x_1 \in X$ with $\|x_1\| = 1$, and recursively define

$$M_n := \operatorname{span}\{x_1, \dots, x_n\}.$$

Since each M_n is a proper finite-dimensional subspace, Riesz's lemma yields a unit vector $x_{n+1} \in X$ such that

$$\operatorname{dist}(x_{n+1}, M_n) > \tfrac{1}{2}.$$

In particular, for $m < n$ we have $x_m \in M_{n-1}$, so

$$\|x_n - x_m\| > \tfrac{1}{2}.$$

Thus $(x_n) \subset \overline{B(0)}$ has no convergent subsequence, contradicting compactness. Hence X must be finite dimensional.

Conversely, assume that $\dim X = n < \infty$. Choose a basis $e_1, \ldots, e_n$ of X, and define the coordinate map

$$T : X \to F^n, \qquad T\Big(\sum_{i=1}^{n} x_i e_i\Big) = (x_1, \ldots, x_n).$$

Then T is a linear bijection. By Proposition 2.16, both T and T^{-1} are continuous, so T is a homeomorphism. The set $T(\overline{B(0)})$ is closed and bounded in F^n, hence compact by Heine–Borel. Therefore $\overline{B(0)} = T^{-1}(T(\overline{B(0)}))$ is compact. □

2.6 Separable normed and Banach spaces

Separability is a fundamental smallness condition in analysis. A countable dense set permits approximation by discrete data, makes diagonal arguments possible, and often allows general theorems to be reduced to constructions on countable families.

Separability is a basic "size" invariant for metric spaces.

Definition 2.33 (Separable space)**.** A normed space $(X, \|\cdot\|)$ is *separable* if it contains a countable dense subset.

Example 2.34 (Finite-dimensional spaces are separable)**.** The set $\mathbb{Q}$ is countable and dense in $\mathbb{R}$. More generally, $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}^n$, so every finite-dimensional normed space is separable.

Proposition 2.35. *For $1 \le p < \infty$, the space $(\ell^p, \|\cdot\|_p)$ is separable.*

Proof. By Proposition 1.9, the subspace c_{00} is dense in ℓ^p. Let $D \subset c_{00}$ denote the set of finitely supported sequences whose coordinates belong to $\mathbb{Q}$ in the real case, and to $\mathbb{Q} + i\mathbb{Q}$ in the complex case. Then D is countable. Since every element of c_{00} has finite support, its nonzero coordinates may be approximated arbitrarily well by rational (respectively, Gaussian rational) numbers, so D is dense in c_{00}, and therefore dense in ℓ^p. □

Proposition 2.36. *$(c_0, \|\cdot\|_\infty)$ is separable.*

Proof. By Proposition 1.10, we have $\overline{c_{00}} = c_0$ in ℓ^∞. Let $D \subset c_{00}$ be the set of finitely supported sequences whose coordinates are rational in the real case, and belong to $\mathbb{Q} + i\mathbb{Q}$ in the complex case. Then D is countable and dense in c_{00}, hence dense in c_0. □

Proposition 2.37. *$(C([0, 1]), \|\cdot\|_\infty)$ is separable.*

Proof. By the Weierstrass approximation theorem, the polynomials are dense in $C([0,1])$ with respect to the supremum norm. Let D be the set of polynomials whose coefficients lie in $\mathbb{Q}$ in the real case, and in $\mathbb{Q} + i\mathbb{Q}$ in the complex case. Then D is countable. Since the coefficients of any fixed polynomial may be approximated arbitrarily well by rational (respectively, Gaussian rational) numbers, D is dense in the set of all polynomials, and therefore dense in $C([0,1])$. Hence $C([0,1])$ is separable. □

Proposition 2.38. *$(\ell^\infty, \|\cdot\|_\infty)$ is not separable.*

Proof. Let
$$S = \{x = (x_n) \in \ell^\infty : x_n \in \{0,1\}\}.$$

Then S is uncountable. Moreover, if $x \neq y$ in S, then $\|x - y\|_\infty = 1$, so the open balls $\{B(x, \frac{1}{2}) : x \in S\}$ are pairwise disjoint.

If $A \subset \ell^\infty$ were countable and dense, then every ball $B(x, \frac{1}{2})$ would meet A, which is impossible because the balls are disjoint while A is countable. □

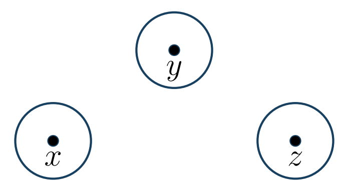

Figure 2.4: A schematic picture of pairwise disjoint balls used in the non-separability of ℓ^∞.

Example 2.39 (Exercises on separability)**.** (i) Show that $\overline{c_{00}}^{\|\cdot\|_\infty} = c_0$ and deduce that $(c_0, \|\cdot\|_\infty)$ is separable.

(ii) Show that $(C([0,1]), \|\cdot\|_\infty)$ is separable (use Weierstrass approximation and polynomials with rational coefficients).

(iii) Show that $L^p([0,1])$ is separable for $1 \leq p < \infty$, but $L^\infty([0,1])$ is not separable (hint: consider $f_t = \chi_{[0,t]}$ for $t \in (0,1)$).

Proposition 2.40. *Let M be a closed subspace of a normed space X. Then X is separable if and only if both M and the quotient space X/M are separable.*

Proof. First assume that X is separable. Any subspace of a separable metric space is separable, hence M is separable. Moreover, the quotient map $\pi : X \to X/M$, $\pi(x) = x+M$, is continuous and onto, so $\pi(X) = X/M$ is separable as the continuous image of a separable space.

Conversely, assume that M and X/M are separable. Let $\{m_j\}_{j\in\mathbb{N}} \subset M$ be dense in M, and let $\{x_n + M\}_{n\in\mathbb{N}} \subset X/M$ be dense in X/M, where we choose representatives $x_n \in X$. Consider the countable set

$$E := \{x_n + m_j : \ n, j \in \mathbb{N}\} \subset X.$$

We now show that $\overline{E} = X$. Fix $x \in X$ and $\varepsilon > 0$. Choose n such that

$$\| (x + M) - (x_n + M) \|_{X/M} < \varepsilon/2,$$

i.e. $\operatorname{dist}(x - x_n, M) < \varepsilon/2$. Then there exists $m \in M$ with $\|x - x_n - m\| < \varepsilon/2$. Choose j with $\|m - m_j\| < \varepsilon/2$. Hence,

$$\|x - (x_n + m_j)\| \leq \|x - x_n - m\| + \|m - m_j\| < \varepsilon.$$

Thus E is dense, and X is separable. □

Dense subspaces of $L^p(\mathbb{R})$

Density questions are central in analysis because they allow one to pass from simple model functions to arbitrary elements of a Banach space by approximation. For $L^p(\mathbb{R})$, the most useful dense classes are the simple functions, the step functions, and the compactly supported continuous functions.

Proposition 2.41 (Density of simple functions)**.** *For $1 \leq p < \infty$, the set of simple functions with finite-measure support is dense in $L^p(\mathbb{R})$.*

Proof. Let $f \in L^p(\mathbb{R})$. By the standard approximation theorem for measurable functions, there exists a sequence of simple measurable functions (φ_n) such that $\varphi_n \to f$ pointwise almost everywhere and $|\varphi_n| \leq |f|$ almost everywhere for every n. Then

$$|f - \varphi_n|^p \leq 2^p |f|^p \in L^1(\mathbb{R}),$$

so the dominated convergence theorem yields

$$\|f - \varphi_n\|_p^p = \int_{\mathbb{R}} |f - \varphi_n|^p \, dx \xrightarrow[n\to\infty]{} 0.$$

Replacing each φ_n by $\varphi_n \chi_{[-n,n]}$ if necessary, we may moreover assume that the support has finite measure. □

Theorem 2.42 (Density of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$)**.** *For $1 \le p < \infty$, the space $C_c(\mathbb{R})$ of compactly supported continuous functions is dense in $L^p(\mathbb{R})$.*

Proof. Fix $f \in L^p(\mathbb{R})$ and $\varepsilon > 0$. By the preceding proposition, choose a simple function

$$\psi = \sum_{j=1}^{m} a_j \chi_{E_j}$$

with measurable sets E_j of finite measure such that $\|f - \psi\|_p < \varepsilon/2$.

By regularity of Lebesgue measure, for each j there exist a compact set K_j and an open set O_j such that

$$K_j \subset E_j \subset O_j \qquad \text{and} \qquad m(O_j \setminus K_j) < \left(\frac{\varepsilon}{2m \max\{1, |a_j|\}}\right)^p.$$

By Urysohn's lemma on $\mathbb{R}$, there exists $g_j \in C_c(\mathbb{R})$ such that $0 \le g_j \le 1$, $g_j = 1$ on K_j, and $g_j = 0$ on O_j^c. Then $|\chi_{E_j} - g_j| \le 1$ and the difference vanishes outside $O_j \setminus K_j$, so

$$\|\chi_{E_j} - g_j\|_p^p \le m(O_j \setminus K_j).$$

Set

$$g := \sum_{j=1}^{m} a_j g_j \in C_c(\mathbb{R}).$$

Then

$$\|\psi - g\|_p \le \sum_{j=1}^{m} |a_j| \, \|\chi_{E_j} - g_j\|_p < \frac{\varepsilon}{2}.$$

Hence

$$\|f - g\|_p \le \|f - \psi\|_p + \|\psi - g\|_p < \varepsilon,$$

which proves the density of $C_c(\mathbb{R})$. $\square$

Corollary 2.43 (Separability of $L^p(\mathbb{R})$)**.** *For $1 \le p < \infty$, the space $L^p(\mathbb{R})$ is separable.*

Proof. By the preceding theorem, it is enough to approximate elements of $C_c(\mathbb{R})$. Every function in $C_c(\mathbb{R})$ can be approximated uniformly on its compact support by piecewise linear functions whose breakpoints are rational and whose values at those breakpoints are rational (or belong to $\mathbb{Q} + i\mathbb{Q}$ in the complex case). The collection of all such functions is countable. Since uniform approximation on a fixed compact support implies convergence in $L^p(\mathbb{R})$, this countable family is dense in $L^p(\mathbb{R})$. Hence $L^p(\mathbb{R})$ is separable. $\square$

Remark 2.44 (A bounded-interval analogue). On a bounded interval $[a, b]$, the same argument shows that the Riemann integrable functions are dense in $L^p([a, b])$ for $1 \le p < \infty$. By contrast, $C_c(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$ with respect to the essential-supremum norm.

Proposition 2.45 (Closure of $C_c(\mathbb{R})$ in $C_0(\mathbb{R})$)**.** *The closure of $C_c(\mathbb{R})$ in $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ is all of $C_0(\mathbb{R})$. In particular, $C_0(\mathbb{R})$ is separable.*

Proof. Let $f \in C_0(\mathbb{R})$ and fix $\varepsilon > 0$. Since f vanishes at infinity, there exists $R > 0$ such that $|f(x)| < \varepsilon$ whenever $|x| \ge R$. Choose $\eta \in C_c(\mathbb{R})$ with $0 \le \eta \le 1$ and $\eta(x) = 1$ for $|x| \le R$. Then $h := \eta f \in C_c(\mathbb{R})$ and

$$|f(x) - h(x)| = |1 - \eta(x)|\,|f(x)| \le \varepsilon \qquad (x \in \mathbb{R}),$$

so $\|f - h\|_\infty \le \varepsilon$. Hence $\overline{C_c(\mathbb{R})}^{\|\cdot\|_\infty} = C_0(\mathbb{R})$. Since $C_c(\mathbb{R})$ itself contains a countable sup-norm dense subset given by rational piecewise linear compactly supported functions, $C_0(\mathbb{R})$ is separable. □

We conclude this section with two standard remarks that will be useful later.

Banach–Mazur theorem. Every separable Banach space X is linearly isometric to a subspace of $C([0, 1])$. A proof may be found, for example, in Fabian, p. 240, Theorem 5.8. We shall return to this result when discussing the weak* topology.

Schauder basis. Let $(X, \|\cdot\|)$ be a normed space. A sequence $(e_n)_{n\ge1}$ in X is called a *Schauder basis* if every $x \in X$ admits a unique expansion

$$x = \sum_{i=1}^{\infty} a_i e_i,$$

where the series converges in norm, that is,

$$\left\| \sum_{i=1}^{k} a_i e_i - x \right\| \to 0 \qquad \text{as } k \to \infty.$$

In finite-dimensional spaces, Schauder bases coincide with Hamel bases. In contrast, for $(c_0, \|\cdot\|_\infty)$, the canonical unit vectors form a Schauder basis but not a Hamel basis; indeed, every Hamel basis of c_0 is uncountable.

Theorem 2.46. *If a normed space $(X, \|\cdot\|)$ has a Schauder basis, then $(X, \|\cdot\|)$ is separable.*

Proof. Let $(e_j)_{j\geq 1}$ be a Schauder basis for X. Consider the set

$$D := \left\{ \sum_{j=1}^{m} q_j e_j : m \in \mathbb{N},\ q_j \in \mathbb{Q} \right\}$$

in the real case, and the analogous set with $q_j \in \mathbb{Q} + i\mathbb{Q}$ in the complex case. Since it is a countable union of countable sets, D is countable.

We now show that D is dense in X. Fix $x \in X$ and $\varepsilon > 0$. Since (e_j) is a Schauder basis, there exist scalars (a_j) such that

$$x = \sum_{j=1}^{\infty} a_j e_j,$$

and the partial sums converge to x. Hence there exists m such that

$$\left\| x - \sum_{j=1}^{m} a_j e_j \right\| < \frac{\varepsilon}{2}.$$

Choose rational numbers $q_1, \dots, q_m$ (or elements of $\mathbb{Q} + i\mathbb{Q}$ in the complex case) so close to $a_1, \dots, a_m$ that

$$\left\| \sum_{j=1}^{m} (a_j - q_j) e_j \right\| < \frac{\varepsilon}{2}.$$

Then

$$\left\| x - \sum_{j=1}^{m} q_j e_j \right\| \leq \left\| x - \sum_{j=1}^{m} a_j e_j \right\| + \left\| \sum_{j=1}^{m} (a_j - q_j) e_j \right\| < \varepsilon.$$

Thus D is countable and dense in X, so X is separable. □

2.7 Baire category theorem

The Baire category theorem is the first major structural consequence of completeness and lies behind several of the cornerstone results of operator theory. Its force comes from the principle that a complete metric space cannot be exhausted by countably many sets that are topologically negligible.

The Baire category theorem is one of the first genuinely global consequences of completeness. Informally, it says that a complete metric space cannot be assembled from countably many "small" pieces. In particular, an infinite-dimensional Banach space cannot behave like a countable-dimensional algebraic object.

Definition 2.47 (Nowhere dense, meager, and second category)**.** Let X be a topological space.

(i) A set $A \subset X$ is called *nowhere dense* if $\operatorname{int}(\overline{A}) = \varnothing$.

(ii) A set $M \subset X$ is called *meager* (or *of first category*) if it can be written as a countable union of nowhere dense sets.

(iii) A set is said to be *of second category* if it is not meager.

Example 2.48 (Standard meager and nowhere dense sets)**.** The set $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$. The rational numbers $\mathbb{Q}$ are meager in $\mathbb{R}$ because

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \frac{1}{n}\mathbb{Z},$$

and each set $n^{-1}\mathbb{Z}$ is nowhere dense. The Cantor set is another standard example of a nowhere dense set: it is closed, uncountable, and has empty interior.

Theorem 2.49 (Baire category theorem, Version I)**.** *Let X be a complete metric space. Then every countable union of nowhere dense subsets of X has empty interior. Equivalently, X cannot be written as a countable union of nowhere dense sets.*

Proof. Let $X = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense. To prove that the union has empty interior, fix a nonempty open set $W \subset X$ and show that $W \not\subset \bigcup_{n=1}^{\infty} A_n$.

Replacing A_n by $\overline{A_n}$ if necessary, we may assume that each A_n is closed and has empty interior. Since A_1 has empty interior and W is nonempty open, the set $W \setminus A_1$ is nonempty and open. Hence we may choose a closed ball

$$\overline{B_1} = \overline{B_{\gamma_1}(x_1)} \subset W \setminus A_1, \qquad \gamma_1 < 1.$$

Assume inductively that closed balls $\overline{B_j} = \overline{B_{\gamma_j}(x_j)}$ have been chosen for $1 \le j \le n$ so that

$$\overline{B_j} \subset B_{j-1} \setminus A_j, \qquad \gamma_j < 2^{-j} \qquad (j \ge 2),$$

with $B_0 := W$. Because A_{n+1} has empty interior, the open set $B_n \setminus A_{n+1}$ is nonempty. Therefore we may choose a closed ball

$$\overline{B_{n+1}} = \overline{B_{\gamma_{n+1}}(x_{n+1})} \subset B_n \setminus A_{n+1}, \qquad \gamma_{n+1} < 2^{-(n+1)}.$$

Thus the balls are nested: $\overline{B_{n+1}} \subset B_n$ for every n.

If $m > n$, then $x_m, x_n \in B_n$, so

$$\|x_m - x_n\| \le 2\gamma_n < 2^{1-n}.$$

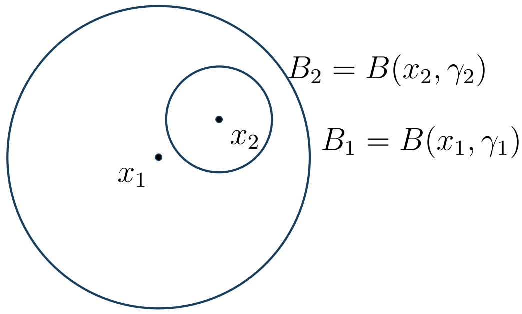

Figure 2.5: The nested-ball construction in the proof of the Baire category theorem.

Therefore (x_n) is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x_n \to x$. Because all but finitely many terms of the sequence lie in each $\overline{B_n}$ and $\overline{B_n}$ is closed, we have

$$x \in \bigcap_{n=1}^{\infty} \overline{B_n}.$$

In particular, $x \in W$, and since $\overline{B_n} \subset X \setminus A_n$ for every n, we also have $x \notin A_n$ for all n. Thus

$$x \in W \setminus \bigcup_{n=1}^{\infty} A_n.$$

Since every nonempty open set W contains a point outside $\bigcup_{n=1}^{\infty} A_n$, the union has empty interior. □

Corollary 2.50. *Let X be a Banach space. If $X = \bigcup_{n=1}^{\infty} A_n$, then there exists $n_0 \in \mathbb{N}$ such that* $\operatorname{int}(\overline{A_{n_0}}) \neq \varnothing$.

Proof. If every $\overline{A_n}$ had empty interior, then each A_n would be nowhere dense. This would imply that $X = \bigcup_{n=1}^{\infty} A_n$ is a countable union of nowhere dense sets, contradicting Theorem 2.49. Therefore $\overline{A_n}$ has nonempty interior for at least one index n. □

Theorem 2.51 (Baire category theorem, Version II)**.** *Let X be a complete metric space. Then every countable intersection of dense open subsets of X is dense.*

Proof. Let $(V_n)_{n\geq 1}$ be dense open subsets of X. To prove density of $\bigcap_{n=1}^{\infty} V_n$, fix a nonempty open set $W \subset X$ and show that

$$W \cap \bigcap_{n=1}^{\infty} V_n \neq \varnothing.$$

Since V_1 is dense and open, $W \cap V_1$ is nonempty and open. Choose a closed ball

$$\overline{B_1} = \overline{B_{\gamma_1}(x_1)} \subset W \cap V_1, \qquad \gamma_1 < 1.$$

Assume inductively that closed balls $\overline{B_j} = \overline{B_{\gamma_j}(x_j)}$ have been chosen for $1 \leq j \leq n$ so that

$$\overline{B_j} \subset B_{j-1} \cap V_j, \qquad \gamma_j < 2^{-j} \qquad (j \geq 2),$$

with $B_0 := W$. Because V_{n+1} is dense and open, the set $B_n \cap V_{n+1}$ is again nonempty and open. Hence we may choose a closed ball

$$\overline{B_{n+1}} = \overline{B_{\gamma_{n+1}}(x_{n+1})} \subset B_n \cap V_{n+1}, \qquad \gamma_{n+1} < 2^{-(n+1)}.$$

As in the proof of Theorem 2.49, the centers (x_n) form a Cauchy sequence. Let $x = \lim x_n$. Then $x \in \overline{B_n} \subset V_n \cap W$ for every n, and therefore

$$x \in W \cap \bigcap_{n=1}^{\infty} V_n.$$

Since every nonempty open set W meets the intersection, that intersection is dense in X. $\square$

Proposition 2.52 (Infinite-dimensional Banach spaces have no countable Hamel basis)**.** *Let X be an infinite-dimensional Banach space. Then X does not admit a countable Hamel basis.*

Proof. Suppose that X had a countable Hamel basis $(e_n)_{n \geq 1}$. For each $n \in \mathbb{N}$ let

$$F_n := \operatorname{span}\{e_1, \ldots, e_n\}.$$

Then each F_n is finite-dimensional, hence closed. Because X is infinite-dimensional, every F_n is a proper subspace of X; therefore F_n has empty interior. Indeed, if a proper linear subspace contained a nonempty open ball, translation and scaling would show that it equals all of X.

Since (e_n) is a Hamel basis, every vector of X belongs to some F_n, so

$$X = \bigcup_{n=1}^{\infty} F_n.$$

This expresses the Banach space X as a countable union of nowhere dense sets, contradicting Theorem 2.49. $\square$

Remark 2.53 (On the Hamel dimension of ℓ^p). For $1 \le p < \infty$, the space ℓ^p is infinite-dimensional and separable, so Proposition 2.52 shows that every Hamel basis of ℓ^p is uncountable. A sharper result, proved later using Hahn–Banach arguments, is that every Hamel basis of an infinite-dimensional Banach space has cardinality at least $\mathfrak{c}$. Since ℓ^p itself has cardinality $\mathfrak{c}$, it follows that the Hamel dimension of ℓ^p is exactly $\mathfrak{c}$.

Exercise 2.54 (Interior and closure identities)**.** Let $A \subseteq X$ be a subset of a topological space X. Prove that

(i) $(X \setminus A)^\circ = X \setminus \overline{A}$;

(ii) $\overline{X \setminus A} = X \setminus A^\circ$.

Example 2.55 (Deriving Version II from Version I)**.** Let $(V_n)_{n\ge1}$ be dense open subsets of a complete metric space X, and set

$$A_n := X \setminus V_n.$$

By part (ii) of the preceding exercise, $\overline{A_n} = X \setminus V_n^\circ = X \setminus V_n$, because V_n is open. Since V_n is also dense, we have $\operatorname{int}(\overline{A_n}) = \varnothing$, so each A_n is nowhere dense. Applying Theorem 2.49 to the complement of $\bigcap_{n=1}^\infty V_n$ yields Theorem 2.51.

Example 2.56 (A smooth function that is locally a polynomial)**.** Let $f \in C^\infty(\mathbb{R})$ and suppose that for each $t \in \mathbb{R}$ there exists $n_t \in \mathbb{N}$ such that $f^{(n_t)}(t) = 0$. Then f agrees with a polynomial on some nonempty open interval.

Proof. For each $m \in \mathbb{N}$ let

$$E_m := \{x \in \mathbb{R} : f^{(m)}(x) = 0\}.$$

Each E_m is closed, and the hypothesis gives $\mathbb{R} = \bigcup_{m=1}^\infty E_m$. By Theorem 2.49, some E_{m_0} has nonempty interior. Hence there exists a nonempty open interval (a, b) such that

$$f^{(m_0)}(t) = 0 \qquad (t \in (a, b)).$$

Therefore f is a polynomial of degree at most $m_0 - 1$ on (a, b). □

Remark 2.57 (Failure of Version II without completeness). Enumerate $\mathbb{Q} = \{q_1, q_2, \dots\}$ and define

$$V_n := \mathbb{Q} \setminus \{q_n\}.$$

Each V_n is open and dense in $\mathbb{Q}$ (with the subspace topology inherited from $\mathbb{R}$), yet

$$\bigcap_{n=1}^{\infty} V_n = \varnothing.$$

This does not contradict Theorem 2.51, because $\mathbb{Q}$ is not complete.

Exercise 2.58 (A meager dense subspace)**.** Show that $C_c(\mathbb{R})$ is a dense meager subspace of $(C_0(\mathbb{R}), \|\cdot\|_\infty)$.

Chapter 3

Bounded Linear Operators and the Fundamental Principles

This chapter develops the basic analytic theory of bounded linear operators between normed and Banach spaces. After introducing the operator norm and the natural modes of convergence for operators, it establishes the extension and continuity results that allow operator-theoretic arguments to be carried out efficiently. Its central achievement is the proof of the three fundamental principles of functional analysis—the uniform boundedness principle, the open mapping theorem, and the closed graph theorem—each of which reveals a deep interaction between linear structure, continuity, and completeness. Together these theorems form the principal technical engine of the subject and will be invoked repeatedly in the chapters that follow.

3.1 Continuous linear transformations

In functional analysis the relevant morphisms are linear maps that respect the topology induced by the norm. The present section shows that, for linear operators, continuity, boundedness, and local control at the origin are equivalent viewpoints, a fact that makes operator theory both flexible and computationally effective.

Let X and Y be normed linear spaces over F. For linear maps, continuity is completely determined by the behaviour at the origin, and this in turn is equivalent to the existence of a global linear bound.

Definition 3.1 (Bounded linear map). A linear map $T : X \to Y$ is said to be *bounded* if there exists $M \geq 0$ such that

$$\|Tx\|_Y \leq M\|x\|_X \qquad (x \in X).$$

Proposition 3.2. *For a linear map $T : X \to Y$, the following are equivalent:*

(i) T is continuous on X;

(ii) T is continuous at 0;

(iii) T is bounded.

Proof. The implication (i)⇒(ii) is immediate. Assume (ii). Then there exists $\delta > 0$ such that

$$\|x\|_X < \delta \quad \Longrightarrow \quad \|Tx\|_Y < 1.$$

If $y \neq 0$, set $x := \frac{\delta}{2\|y\|_X} y$. Then $\|x\|_X = \delta/2 < \delta$, so

$$\left\| T\left(\frac{\delta}{2\|y\|_X} y \right) \right\|_Y < 1.$$

By linearity,

$$\|Ty\|_Y < \frac{2}{\delta}\|y\|_X,$$

so T is bounded. Finally, if T is bounded, say $\|Tx\|_Y \le M\|x\|_X$, then $x_n \to x$ implies

$$\|Tx_n - Tx\|_Y = \|T(x_n - x)\|_Y \le M\|x_n - x\|_X \to 0.$$

Therefore (iii)⇒(i). □

Definition 3.3 (Operator norm)**.** If $T \in \mathcal{B}(X, Y)$, its *operator norm* is

$$\|T\| := \inf\{M \ge 0 : \|Tx\|_Y \le M\|x\|_X \text{ for all } x \in X\}.$$

Lemma 3.4. *For $T \in \mathcal{B}(X, Y)$,*

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\| \le 1} \|Tx\|.$$

Proof. Set

$$\alpha := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Then $\|Tx\| \le \alpha\|x\|$ for every $x \neq 0$, and the same estimate holds for $x = 0$. Hence $\|T\| \le \alpha$. Conversely, by definition of the operator norm,

$$\frac{\|Tx\|}{\|x\|} \le \|T\| \qquad (x \neq 0),$$

so $\alpha \le \|T\|$. Thus $\alpha = \|T\|$.

If $\|x\| = 1$, then $\|Tx\| \leq \|T\|$, hence $\sup_{\|x\|=1} \|Tx\| \leq \|T\|$. On the other hand, for any $x \neq 0$, writing $u = x/\|x\|$ gives

$$\frac{\|Tx\|}{\|x\|} = \|Tu\| \leq \sup_{\|u\|=1} \|Tu\|.$$

Taking the supremum over $x \neq 0$ yields $\|T\| \leq \sup_{\|x\|=1} \|Tx\|$. The equality with $\sup_{\|x\|\leq 1} \|Tx\|$ is immediate. □

Proposition 3.5. *If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space with respect to the operator norm.*

Proof. Let (T_n) be a Cauchy sequence in $\mathcal{B}(X, Y)$. For each fixed $x \in X$,

$$\|T_n x - T_m x\| \leq \|T_n - T_m\| \, \|x\|,$$

so $(T_n x)$ is Cauchy in Y. Since Y is Banach, there exists $Tx \in Y$ such that $T_n x \to Tx$. This defines a map $T : X \to Y$. Since pointwise limits of linear maps are linear, T is linear.

Since (T_n) is Cauchy in norm, it is bounded: choose $M > 0$ such that $\|T_n\| \leq M$ for all n. Then

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq M\|x\|,$$

so $T \in \mathcal{B}(X, Y)$. Finally, if $\varepsilon > 0$, choose N such that $\|T_n - T_m\| < \varepsilon$ whenever $m, n \geq N$. For $n \geq N$ and $\|x\| \leq 1$, letting $m \to \infty$ gives

$$\|(T_n - T)x\| = \lim_{m\to\infty} \|(T_n - T_m)x\| \leq \varepsilon.$$

Taking the supremum over the unit ball yields $\|T_n - T\| \leq \varepsilon$ for all $n \geq N$. Hence $T_n \to T$ in $\mathcal{B}(X, Y)$. □

Example 3.6 (A diagonal operator on ℓ^p)**.** For $1 \leq p \leq \infty$, define

$$T(x_1, x_2, x_3, \dots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, \dots\right).$$

Then $T \in \mathcal{B}(\ell^p, \ell^p)$ and $\|T\| = 1$.

Proof. For $1 \leq p < \infty$,

$$\|Tx\|_p^p = \sum_{n=1}^{\infty} \left|\frac{x_n}{n}\right|^p \leq \sum_{n=1}^{\infty} |x_n|^p = \|x\|_p^p,$$

and similarly $\|Tx\|_\infty \le \|x\|_\infty$ when $p = \infty$. Thus $\|T\| \le 1$. Equality follows by evaluating T at $e_1 = (1, 0, 0, \dots)$. □

Example 3.7 (The Volterra operator on continuous functions)**.** Define $T : C([0, 1]) \to C([0, 1])$ by

$$(Tf)(x) = \int_0^x f(t)\, dt.$$

Then $T \in \mathcal{B}(C([0, 1]), C([0, 1]))$ and $\|T\| = 1$.

Proof. For $x \in [0, 1]$,

$$|(Tf)(x)| \le \int_0^x |f(t)|\, dt \le \|f\|_\infty,$$

so $\|Tf\|_\infty \le \|f\|_\infty$ and hence $\|T\| \le 1$. Taking $f \equiv 1$ gives $(Tf)(x) = x$, so $\|Tf\|_\infty = 1 = \|f\|_\infty$. □

Example 3.8 (Integral operators on function spaces)**.** Let $\varphi : [0, 1] \times [0, 1] \to \mathbb{C}$ be continuous, and define

$$(Tf)(x) = \int_0^1 \varphi(x, t) f(t)\, dt.$$

Then T defines a bounded operator on $C([0, 1])$, with

$$\|T\| \le \sup_{x \in [0,1]} \int_0^1 |\varphi(x, t)|\, dt.$$

Moreover, the same formula defines a bounded operator $T : L^2([0, 1]) \to L^2([0, 1])$, and

$$\|T\| \le \left(\int_0^1 \int_0^1 |\varphi(x, t)|^2\, dt\, dx \right)^{1/2}.$$

Proof. For $f \in C([0, 1])$,

$$|(Tf)(x)| \le \|f\|_\infty \int_0^1 |\varphi(x, t)|\, dt,$$

which gives the stated bound after taking the supremum in x.

For $f \in L^2([0, 1])$, the Cauchy–Schwarz inequality gives

$$|(Tf)(x)| \le \left(\int_0^1 |\varphi(x, t)|^2\, dt \right)^{1/2} \|f\|_2.$$

Squaring and integrating in x yields the L^2 estimate. □

Example 3.9 (Cesàro averaging operator)**.** Let $BC([0,\infty))$ denote the space of bounded continuous functions on $[0,\infty)$, equipped with the supremum norm. Define

$$(Tf)(t) = \begin{cases} \frac{1}{t} \int_0^t f(s)\,ds, & t > 0, \\ f(0), & t = 0. \end{cases}$$

Then $T \in \mathcal{B}(BC([0,\infty)), BC([0,\infty)))$ and $\|T\| = 1$.

Proof. For $t > 0$,

$$|(Tf)(t)| \le \frac{1}{t} \int_0^t |f(s)|\,ds \le \|f\|_\infty,$$

and the same bound holds at $t = 0$. Hence $\|Tf\|_\infty \le \|f\|_\infty$. Continuity at $t = 0$ follows from

$$\lim_{t \downarrow 0} \frac{1}{t} \int_0^t f(s)\,ds = f(0),$$

by continuity of f. Equality $\|T\| = 1$ follows from $T\mathbf{1} = \mathbf{1}$. □

Example 3.10 (Differentiation)**.** The differentiation operator need not be bounded unless the domain is equipped with a stronger norm.

(i) The map $D : C^1([0,1]) \to C([0,1])$, $Df = f'$, is bounded when $C^1([0,1])$ is endowed with the norm

$$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty.$$

In fact, $\|Df\|_\infty \le \|f\|_{C^1}$, so $\|D\| \le 1$, and equality holds.

(ii) If one instead regards differentiation as a map from $(C^1([0,1]), \|\cdot\|_\infty)$ to $(C([0,1]), \|\cdot\|_\infty)$, then it is unbounded. Indeed, for $f_n(t) = t^n$ we have $\|f_n\|_\infty = 1$ but $\|f_n'\|_\infty = n \to \infty$.

3.2 Extension of uniformly continuous functions

Density is useful only when data prescribed on a dense subset determine a unique object on the ambient space. Uniform continuity is exactly the hypothesis that allows such data to pass to limits unambiguously, and the resulting extension principle will serve as the model for extending bounded linear operators from dense subspaces.

A uniformly continuous function defined on a dense subset extends uniquely to the closure of that subset. This serves as the prototype for several extension principles used later.

Proposition 3.11. *Let A be a dense subset of a metric space (M, d), and let $f : A \to \mathbb{R}$ be uniformly continuous. Then there exists a unique uniformly continuous extension $\widehat{f} : \overline{A} \to \mathbb{R}$ such that $\widehat{f}|_A = f$.*

Proof. Fix $x \in \overline{A}$. Choose a sequence $(x_n) \subset A$ with $x_n \to x$. Since (x_n) is Cauchy and f is uniformly continuous, $(f(x_n))$ is Cauchy in $\mathbb{R}$, hence convergent. Define

$$\widehat{f}(x) := \lim_{n\to\infty} f(x_n).$$

A standard argument shows that this value does not depend on the approximating sequence, so $\widehat{f}$ is well defined.

To prove uniform continuity, let $\varepsilon > 0$, and choose $\delta > 0$ such that

$$d(u, v) < \delta \quad \Longrightarrow \quad |f(u) - f(v)| < \varepsilon \qquad (u, v \in A).$$

Take $x, y \in \overline{A}$ with $d(x, y) < \delta/3$, and choose sequences $x_n, y_n \in A$ converging to x and y, respectively. For large n we have $d(x_n, x) < \delta/3$ and $d(y_n, y) < \delta/3$, hence $d(x_n, y_n) < \delta$. Therefore $|f(x_n) - f(y_n)| < \varepsilon$, and letting $n \to \infty$ yields

$$|\widehat{f}(x) - \widehat{f}(y)| \le \varepsilon.$$

Thus $\widehat{f}$ is uniformly continuous.

Uniqueness follows because any two continuous extensions agreeing on the dense set A must agree on $\overline{A}$. □

3.3 Extension theorem

Many operators are first defined on a simple dense subspace, where formulas can be checked directly. The extension theorem explains when such preliminary definitions extend uniquely and continuously to the whole space, and it is therefore one of the basic tools for constructing operators in analysis.

The precise extension statement is the following.

Theorem 3.12 (Extension from a dense subspace)**.** *Let M be a dense subspace of a normed space X, and let Y be a Banach space. If $T : M \to Y$ is bounded and linear, then there exists a unique bounded linear extension $\widehat{T} : X \to Y$ such that*

$$\widehat{T}|_M = T \qquad \textit{and} \qquad \|\widehat{T}\| = \|T\|.$$

Proof. For $x \in X$, choose a sequence $(x_n) \subset M$ with $x_n \to x$. Since T is bounded,

$$\|Tx_n - Tx_m\| \leq \|T\| \, \|x_n - x_m\|,$$

so (Tx_n) is Cauchy in Y. As Y is Banach, (Tx_n) converges. Define

$$\widehat{T}(x) := \lim_{n\to\infty} Tx_n.$$

The limit is independent of the approximating sequence, so $\widehat{T}$ is well defined. Linearity follows from linearity of limits.

Moreover,

$$\|\widehat{T}(x)\| = \lim_{n\to\infty} \|Tx_n\| \leq \|T\| \lim_{n\to\infty} \|x_n\| = \|T\| \, \|x\|,$$

so $\widehat{T}$ is bounded and $\|\widehat{T}\| \leq \|T\|$. Since $\widehat{T}$ extends T, we also have $\|T\| \leq \|\widehat{T}\|$. Hence $\|\widehat{T}\| = \|T\|$. Uniqueness follows from continuity and density of M. □

Example 3.13 (Finite-dimensional domains)**.** Every linear map from a finite-dimensional normed space X into a normed space Y is bounded.

Proof. Choose a basis $e_1, \ldots, e_n$ of X. For $x = \sum_{j=1}^n a_j e_j$,

$$\|Tx\| \leq \sum_{j=1}^{n} |a_j| \, \|Te_j\|.$$

Since all norms on X are equivalent, there exists $C > 0$ such that $\sum_{j=1}^n |a_j| \leq C\|x\|$. Therefore

$$\|Tx\| \leq C\Big(\max_{1\leq j\leq n} \|Te_j\|\Big)\|x\|,$$

which proves boundedness. □

3.4 Open mapping theorem

A surjective bounded operator between Banach spaces carries a large amount of hidden quantitative information. The open mapping theorem reveals that surjectivity is not merely algebraic: once completeness is available, images of open sets remain open, and this principle quickly leads to strong inverse and closed-graph results.

The next result is one of the fundamental consequences of completeness in operator theory. It is *false* without linearity: there exist continuous surjections between Banach spaces that do not send open sets to open sets.

Theorem 3.14 (Open mapping theorem)**.** *Let X and Y be Banach spaces and let $T \in \mathcal{B}(X,Y)$ be surjective. Then T is an open map: for every open set $U \subset X$, the image $T(U)$ is open in Y.*

Proof. It suffices to prove that $T(B_X(0,1))$ contains a ball centered at 0; once this is established, scaling and translation yield openness for arbitrary open sets.

Since T is surjective,

$$Y = \bigcup_{n=1}^{\infty} T\Big(B_X(0,n)\Big) = \bigcup_{n=1}^{\infty} n\,T\Big(B_X(0,1)\Big).$$

Set $E := \overline{T(B_X(0,1))}$. Then $Y = \bigcup_{n\geq 1} nE$. Each nE is closed, so by the Baire category theorem there exist $n_0 \in \mathbb{N}$ and $y_0 \in Y$ and $r > 0$ such that

$$B_Y(y_0, r) \subset n_0 E.$$

By translation invariance of balls,

$$B_Y(0,r) \subset n_0E - n_0E \subset n_0\overline{T(B_X)} - n_0\overline{T(B_X)} \subset \overline{T(B_X(0,2n_0))}.$$

After rescaling X (replace $B_X(0,1)$ by $B_X(0,2n_0)$), we may assume that

$$B_Y(0,1) \subset \overline{T(B_X(0,1))}. \tag{3.1}$$

Claim. Under (3.1), one has $B_Y(0,\frac{1}{2}) \subset T(B_X(0,1))$.

Fix $y \in Y$ with $\|y\| < \frac{1}{2}$. To ensure that the eventual preimage lies in the *open* unit ball, we use the following scaled consequence of (3.1): for each $k \geq 1$,

$$B_Y(0,2^{-k}) \subset \overline{T\Big(B_X(0,2^{-k})\Big)}. \tag{3.2}$$

Indeed, multiplying (3.1) by 2^{-k} gives $2^{-k}B_Y(0,1) = B_Y(0,2^{-k}) \subset 2^{-k}\overline{T(B_X(0,1))} = \overline{T\Big(2^{-k}B_X(0,1)\Big)}$.

Define the residuals $r_0 := y$ and, having chosen $x_1,\ldots,x_k$, set $r_k := y - T(x_1 + \cdots + x_k)$. Since $\|y\| < 2^{-1}$, by (3.2) with $k=1$ we can choose $x_1 \in B_X(0,2^{-1})$ so that $\|r_0 - Tx_1\| < 2^{-2}$, hence $\|r_1\| < 2^{-2}$. Inductively, if $\|r_{k-1}\| < 2^{-k}$, then $r_{k-1} \in B_Y(0,2^{-k})$, and (3.2) yields an element $x_k \in B_X(0,2^{-k})$ with $\|r_{k-1} - Tx_k\| < 2^{-(k+1)}$, so that $\|r_k\| < 2^{-(k+1)}$.

Now the series $\sum_{k\geq 1} x_k$ converges in X because $\sum_{k\geq 1}\|x_k\| < \sum_{k\geq 1} 2^{-k} = 1$. Let $x := \sum_{k\geq 1} x_k$; then $\|x\| < 1$, and

$$\|y - T(x_1 + \cdots + x_n)\| = \|r_n\| \longrightarrow 0 \qquad (n \to \infty),$$

so $Tx = y$ by continuity of T. Hence $y \in T(B_X(0,1))$, proving the claim.

Returning to the original scaling, we conclude that $T(B_X(0,1))$ contains an open ball centered at 0, and hence T is open. □

3.5 Inverse mapping theorem

A bijective bounded operator need not, a priori, have a bounded inverse, since the inverse is initially defined only at the algebraic level. The inverse mapping theorem shows that in the Banach setting algebraic invertibility automatically upgrades to topological invertibility.

It is most natural to formulate the result as an immediate consequence of the open mapping theorem.

Corollary 3.15 (Inverse mapping theorem)**.** *Let X and Y be Banach spaces and let $T \in \mathcal{B}(X,Y)$ be bijective. Then $T^{-1} : Y \to X$ is bounded.*

Proof. By Theorem 3.14, T is an open map. Hence $T(B_X(0,1))$ contains a ball $B_Y(0,r)$ for some $r > 0$. If $y \in Y$ and $y \neq 0$, write $y = \|y\| \frac{y}{\|y\|}$ and note that $\frac{r}{\|y\|}y \in B_Y(0,r)$, so $\frac{r}{\|y\|}y = Tx$ for some x with $\|x\| \leq 1$. Therefore

$$\|T^{-1}y\| = \|\tfrac{\|y\|}{r}x\| \leq \tfrac{1}{r}\|y\|.$$

Thus $\|T^{-1}\| \leq 1/r$. □

3.6 Closed graph theorem

Continuity of a linear operator can be difficult to verify directly from its formula, especially when the operator is defined implicitly. The closed graph theorem provides a powerful criterion: it is enough to know that convergence of inputs and outputs is compatible with the graph of the operator.

The theorem below replaces direct norm estimates by a geometric condition on the graph.

Theorem 3.16 (Closed graph theorem)**.** *Let X and Y be Banach spaces and let $T : X \to Y$ be linear. Then T is bounded if and only if its graph*

$$\mathcal{G}(T) = \{(x, Tx) : x \in X\} \subset X \times Y$$

is closed (with respect to the product norm $\|(x, y)\| := \|x\| + \|y\|$).

Proof. If T is bounded and $(x_n, Tx_n) \to (x, y)$ in $X \times Y$, then $x_n \to x$ and $Tx_n \to y$. By continuity of T, we also have $Tx_n \to Tx$, hence $y = Tx$. Therefore $(x, y) \in \mathcal{G}(T)$, so the graph is closed.

Conversely, assume that $\mathcal{G}(T)$ is closed. Since $X \times Y$ is Banach, $\mathcal{G}(T)$ is Banach as a closed subspace. Consider the projection

$$P : \mathcal{G}(T) \to X, \qquad P(x, Tx) = x.$$

This map is linear, bijective, and bounded because

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|.$$

By the inverse mapping theorem, $P^{-1} : X \to \mathcal{G}(T)$ is bounded. Therefore there exists $C > 0$ such that

$$\|x\| + \|Tx\| = \|P^{-1}x\| \leq C\|x\| \qquad (x \in X).$$

Therefore $\|Tx\| \leq (C - 1)\|x\|$, so T is bounded. $\square$

Remark 3.17 (Why completeness is needed)*.* Completeness is essential in Theorems 3.14–3.16. For instance, the identity map

$$I : (\ell^1, \|\cdot\|_1) \to (\ell^1, \|\cdot\|_\infty)$$

is bounded, but its inverse is not bounded, because the ℓ^1 norm cannot be controlled by the ℓ^∞ norm on ℓ^1.

Example 3.18 (Coordinatewise multiplication on ℓ^p)**.** Let (a_n) be a scalar sequence with the property that for every $x = (x_n) \in \ell^p$, the sequence $(a_n x_n)$ again belongs to ℓ^p. Define

$$T : \ell^p \to \ell^p, \qquad T(x_n) = (a_n x_n).$$

Then T is bounded.

Proof. We verify that the graph of T is closed. Suppose $x^{(k)} \to x$ in ℓ^p and $Tx^{(k)} \to y$ in ℓ^p. Then for each coordinate n,

$$x_n^{(k)} \to x_n \qquad \text{and} \qquad a_n x_n^{(k)} \to y_n.$$

Therefore $y_n = a_n x_n$ for every n, so $y = Tx$. Therefore the graph of T is closed, and the closed graph theorem implies that T is bounded. □

Example 3.19 (Multiplication operators on $L^1(\mathbb{R})$)**.** If $\varphi \in L^\infty(\mathbb{R})$ and $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is defined by

$$T(f) = \varphi f,$$

then T is bounded and

$$\|T\| = \|\varphi\|_\infty.$$

Conversely, if φ is measurable and $\varphi f \in L^1(\mathbb{R})$ for every $f \in L^1(\mathbb{R})$, then the same formula defines a bounded operator on $L^1(\mathbb{R})$; in particular, necessarily $\varphi \in L^\infty(\mathbb{R})$.

Proof. If $\varphi \in L^\infty$, then

$$\|T(f)\|_1 = \int_{\mathbb{R}} |\varphi(x) f(x)|\, dx \le \|\varphi\|_\infty \|f\|_1,$$

so $\|T\| \le \|\varphi\|_\infty$. To obtain the reverse inequality, fix $\varepsilon > 0$ and consider

$$E_\varepsilon := \{x \in \mathbb{R} : |\varphi(x)| > \|\varphi\|_\infty - \varepsilon\}.$$

This set has positive measure. Choose a measurable subset $F \subset E_\varepsilon$ with $0 < m(F) < \infty$, and set

$$f_\varepsilon := \frac{\chi_F}{m(F)}.$$

Then $\|f_\varepsilon\|_1 = 1$, and

$$\|Tf_\varepsilon\|_1 = \frac{1}{m(F)} \int_F |\varphi(x)|\, dx \ge \|\varphi\|_\infty - \varepsilon.$$

Therefore $\|T\| \ge \|\varphi\|_\infty - \varepsilon$, and letting $\varepsilon \downarrow 0$ gives $\|T\| = \|\varphi\|_\infty$.

For the converse, assume merely that $\varphi f \in L^1(\mathbb{R})$ for every $f \in L^1(\mathbb{R})$. Define $T(f) = \varphi f$. If $f_n \to f$ in L^1 and $Tf_n \to g$ in L^1, then by passing to a subsequence we may assume that $f_n(x) \to f(x)$ and $Tf_n(x) \to g(x)$ almost everywhere. Thus $g(x) = \varphi(x) f(x)$ almost everywhere, so the graph of T is closed. By the closed graph theorem, T is bounded. Applying the first part now shows that necessarily $\varphi \in L^\infty(\mathbb{R})$ and $\|T\| = \|\varphi\|_\infty$. □

Remark 3.20 (The general L^p case). The same argument works on $L^p(\mathbb{R})$ for every $1 \le p < \infty$: whenever multiplication by a measurable function φ maps $L^p(\mathbb{R})$ into itself, the resulting operator is automatically bounded, and necessarily $\varphi \in L^\infty(\mathbb{R})$.

3.6.1 Banach–Steinhaus theorem (uniform boundedness principle)

Pointwise estimates for a family of operators are often straightforward to obtain, whereas a uniform operator-norm bound is typically much more delicate. The Banach–Steinhaus theorem shows that completeness bridges this gap and converts pointwise boundedness into a global statement on bounded subsets.

A family of continuous linear maps can be pointwise bounded without being uniformly bounded on the whole unit ball if the underlying space is incomplete. Completeness is precisely the ingredient that rules out this pathology.

Theorem 3.21 (Banach–Steinhaus theorem)**.** *Let X be a Banach space, let Y be a normed space, and let $\{T_i : i \in I\} \subset \mathcal{B}(X,Y)$. Define*

$$E := \Big\{x \in X : \sup_{i\in I} \|T_i x\| = \infty\Big\}.$$

Then exactly one of the following alternatives holds:

(i) $\sup_{i\in I} \|T_i\| < \infty$*;*

(ii) E is a dense G_δ subset of X.

Proof. For each $n \in \mathbb{N}$, let

$$E_n := \Big\{x \in X : \sup_{i\in I} \|T_i x\| > n\Big\}.$$

Then

$$X \setminus E_n = \bigcap_{i\in I} \{x \in X : \|T_i x\| \le n\},$$

which is closed because each T_i is continuous. Hence every E_n is open, and

$$E = \bigcap_{n=1}^{\infty} E_n.$$

First assume that each E_n is dense in X. By the Baire category theorem, $E = \bigcap_{n\ge1} E_n$ is a dense G_δ set, so alternative (ii) holds.

Now assume that some E_n is not dense. Then there exists a ball $B(x_0, r)$ such that

$$B(x_0, r) \cap E_n = \varnothing.$$

Equivalently,

$$\sup_{i \in I} \|T_i x\| \le n \qquad (x \in B(x_0, r)).$$

If $\|x\| < r$, then $x + x_0 \in B(x_0, r)$, and therefore

$$\|T_i x\| \le \|T_i(x + x_0)\| + \|T_i x_0\| \le 2n \qquad (i \in I).$$

Now let $y \in X$ be arbitrary, $y \neq 0$, and set $x = \frac{r}{2\|y\|} y$. Then $\|x\| = r/2 < r$, so

$$\left\| T_i \left(\frac{r}{2\|y\|} y \right) \right\| \le 2n.$$

Hence

$$\|T_i y\| \le \frac{4n}{r} \|y\| \qquad (i \in I),$$

which implies $\|T_i\| \le 4n/r$ for all $i \in I$. Thus alternative (i) holds. □

Corollary 3.22 (Uniform boundedness principle)**.** *Let $\{T_i\}_{i \in I} \subset \mathcal{B}(X, Y)$, where X is Banach, and suppose that for each $x \in X$,*

$$\sup_{i \in I} \|T_i x\| < \infty.$$

Then

$$\sup_{i \in I} \|T_i\| < \infty.$$

Proof. Under the pointwise boundedness hypothesis, the exceptional set E appearing in Theorem 3.21 is empty. Consequently, alternative (ii) in that theorem is impossible, and therefore alternative (i) holds. This is exactly the asserted conclusion. □

Remark 3.23 (Failure without completeness)*.* Completeness of X is essential. Consider the vector space $P(\mathbb{R})$ of all real polynomials, equipped with the norm

$$\|p\|_1 := \sup_{j \ge 0} |a_j|, \qquad p(x) = a_0 + a_1 x + \cdots + a_d x^d.$$

For $n \in \mathbb{N}$ define

$$T_n(p) := \sum_{j=0}^{n} a_j.$$

For each fixed polynomial p, the sequence $\{T_n(p)\}$ is eventually constant, so the family $\{T_n\}$ is pointwise bounded. On the other hand, if $p_n(x) = 1 + x + \cdots + x^n$, then $\|p_n\|_1 = 1$ and $T_n(p_n) = n + 1$, so $\|T_n\| = n + 1 \to \infty$. Thus uniform boundedness fails on this incomplete normed space.

Corollary 3.24 (Pointwise limits of bounded operators)**.** *Let X be a Banach space, let Y be a normed space, and let $(T_n) \subset \mathcal{B}(X, Y)$. Suppose that for every $x \in X$ the limit*

$$T(x) := \lim_{n\to\infty} T_n(x)$$

exists in Y. Then $T \in \mathcal{B}(X, Y)$ and

$$\|T\| \leq \limsup_{n\to\infty} \|T_n\|.$$

In particular, if $\|T_n\| \to L$, then $\|T\| \leq L$.

Proof. For each $x \in X$, the sequence $\{T_n x\}$ converges and is therefore bounded. By the uniform boundedness principle, there exists $M > 0$ such that $\|T_n\| \leq M$ for all n. Passing to the limit in

$$\|T_n x\| \leq M\|x\|$$

gives $\|Tx\| \leq M\|x\|$, so T is bounded.

Moreover, for every $x \in X$,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \Big(\limsup_{n\to\infty} \|T_n\|\Big)\|x\|.$$

Taking the supremum over $\|x\| \leq 1$ yields the stated bound. □

3.6.2 Application: divergence of Fourier series for some continuous functions

One of the strengths of the uniform boundedness principle is that it detects pathological behaviour from seemingly harmless pointwise formulas. The present application illustrates this vividly by showing that Fourier partial sum operators cannot remain uniformly controlled on $C([-\pi, \pi])$.

For $f \in C([-\pi, \pi])$, the Fourier coefficients of f are defined by

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ikt}\, dt, \qquad k \in \mathbb{Z},$$

and the nth symmetric partial sum of the Fourier series is

$$S_n(f)(x) := \sum_{k=-n}^{n} \widehat{f}(k)e^{ikx}.$$

Using the Dirichlet kernel

$$D_n(t) := \sum_{k=-n}^{n} e^{ikt} = \frac{\sin((n+\frac{1}{2})t)}{\sin(t/2)} \qquad (t \notin 2\pi\mathbb{Z}),$$

we may write

$$S_n(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)D_n(x-s)\,ds.$$

We now use the uniform boundedness principle to show that the Fourier series of a continuous function need not converge even at a single point.

Proposition 3.25. *For $f \in C([-\pi,\pi])$, define*

$$T_n(f) := S_n(f)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(t)\,dt.$$

Then $T_n \in \mathcal{B}(C([-\pi,\pi]),\mathbb{C})$ and

$$\|T_n\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt.$$

Proof. The upper bound follows immediately:

$$|T_n(f)| \le \|f\|_\infty \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt,$$

so

$$\|T_n\| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt.$$

For the reverse inequality, approximate the sign of D_n by continuous functions. For $\delta > 0$ define

$$f_{n,\delta}(t) := \frac{D_n(t)}{|D_n(t)|+\delta}.$$

Then $f_{n,\delta} \in C([-\pi,\pi])$ and $\|f_{n,\delta}\|_\infty \le 1$. Moreover,

$$f_{n,\delta}(t)D_n(t) = \frac{D_n(t)^2}{|D_n(t)|+\delta} \longrightarrow |D_n(t)| \qquad (\delta \downarrow 0),$$

and the integrands are dominated by $|D_n| \in L^1([-\pi,\pi])$. By dominated convergence,

$$\lim_{\delta \downarrow 0} T_n(f_{n,\delta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt.$$

Since $\|f_{n,\delta}\|_\infty \le 1$, this shows that $\|T_n\|$ is at least the right-hand side. Therefore equality holds. □

Lemma 3.26 (Growth of the L^1 norm of the Dirichlet kernel)**.** *There exists an absolute constant $c > 0$ such that, for all $n \ge 1$,*

$$\int_{-\pi}^{\pi} |D_n(t)|\,dt \ge c\log(n+1).$$

In particular,

$$\|D_n\|_{L^1([-\pi,\pi])} \to \infty.$$

Proof. Recall that

$$D_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin(t/2)}.$$

For $m = 0, 1, \dots, n-1$ set

$$t_m := \frac{(2m+1)\pi}{2n+1}, \qquad \delta := \frac{\pi}{4n+2}, \qquad I_m := [t_m - \delta,\, t_m + \delta] \subset (0,\pi).$$

The intervals I_m are pairwise disjoint. If $t \in I_m$, then

$$\Big|(n+\tfrac{1}{2})t - (n+\tfrac{1}{2})t_m\Big| \le (n+\tfrac{1}{2})\delta = \frac{\pi}{4},$$

so

$$\Big|\sin\Big((n+\tfrac{1}{2})t\Big)\Big| \ge \sin(\pi/4) = \frac{\sqrt{2}}{2}.$$

Also, for $t \in (0,\pi)$ we have $\sin(t/2) \le t/2$, hence $1/|\sin(t/2)| \ge 2/t$. Therefore

$$|D_n(t)| \ge \frac{\sqrt{2}}{t} \qquad (t \in I_m).$$

Integrating over these disjoint intervals gives

$$\int_0^{\pi} |D_n(t)|\,dt \ge \sqrt{2}\sum_{m=0}^{n-1}\int_{t_m-\delta}^{t_m+\delta}\frac{dt}{t} \ge \sqrt{2}\sum_{m=0}^{n-1}\frac{2\delta}{t_m+\delta}.$$

A direct computation shows that $\frac{2\delta}{t_m+\delta} = \frac{2}{4m+3}$. Hence

$$\int_0^\pi |D_n(t)|\,dt \geq \sqrt{2}\sum_{m=0}^{n-1} \frac{2}{4m+3} \geq \frac{\sqrt{2}}{2}\sum_{m=1}^{n} \frac{1}{m} \geq \frac{\sqrt{2}}{2}\log(n+1).$$

Doubling the estimate proves the claim. □

Theorem 3.27 (A dense set of continuous functions with divergent Fourier series)**.** *There exists a dense G_δ set $G \subset C([-\pi,\pi])$ such that for every $f \in G$,*

$$\sup_{n\geq 1} |S_n(f)(0)| = \infty.$$

In particular, the Fourier series of f diverges at 0*.*

Proof. By Proposition 3.25 and Lemma 3.26,

$$\|T_n\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt \xrightarrow[n\to\infty]{} \infty.$$

Thus the family $\{T_n\}_{n\geq 1} \subset \mathcal{B}(C([-\pi,\pi]),\mathbb{C})$ is not uniformly bounded. Since $C([-\pi,\pi])$ is Banach, Theorem 3.21 implies that the set

$$G := \left\{ f \in C([-\pi,\pi]) : \sup_{n\geq 1} |T_n(f)| = \infty \right\}$$

is a dense G_δ subset of $C([-\pi,\pi])$. But $T_n(f) = S_n(f)(0)$, so the asserted divergence follows. □

Summary: the three fundamental principles

In this chapter we have seen how completeness (via the Baire category theorem) forces strong operator-theoretic consequences. The *uniform boundedness principle* controls operator norms from pointwise bounds; the *open mapping theorem* upgrades surjectivity to quantitative openness; and the *closed graph theorem* turns graph-closedness into boundedness. These results will be used repeatedly in duality and spectral theory.

Chapter 4

The Hahn–Banach Theorem and Its Applications

The Hahn–Banach theorem is one of the decisive structural results of functional analysis. It shows that bounded linear functionals admit norm-preserving extensions and thereby makes duality into an effective tool rather than a merely formal notion. From this extension principle one derives the fundamental geometric consequences of the theory: separation of points and convex sets, identification of duals of quotient spaces and subspaces, and the canonical embedding of a normed space into its bidual. The chapter emphasizes both the abstract mechanism of the theorem and the concrete ways in which it converts local information into global linear functionals.

4.1 Sublinear functionals

The Hahn–Banach theorem is fundamentally a theorem about extending linear functionals while preserving domination by a larger, typically nonlinear gauge. Sublinear functionals supply exactly the right framework for expressing this domination in a flexible and geometrically meaningful way.

Throughout this chapter X is a real or complex vector space.

Definition 4.1 (Sublinear functional)**.** A function $p : X \to \mathbb{R}$ is *sublinear* if

$$p(x+y) \le p(x) + p(y), \qquad p(\alpha x) = \alpha\, p(x) \text{ for all } \alpha \ge 0.$$

A seminorm is a sublinear functional satisfying $p(-x) = p(x)$.

Example 4.2 (Norms and Minkowski functionals)**.** If X is a normed space, then $p(x) = \|x\|$ is sublinear. More generally, if $C \subset X$ is convex and absorbs X, the Minkowski functional $p_C(x) = \inf\{t > 0 : x \in tC\}$ is sublinear.

4.2 The extension step

The proof of Hahn–Banach is built from a deceptively simple local mechanism: extend the functional by one new direction while maintaining the required bound. Once this one-dimensional step is understood, the full theorem follows by a maximality argument.

The key local lemma is the following.

Lemma 4.3 (One-dimensional extension)**.** *Let $M \subset X$ be a linear subspace, let $p : X \to \mathbb{R}$ be sublinear, and let $f_0 : M \to \mathbb{R}$ be linear with $f_0(x) \le p(x)$ for all $x \in M$. Fix $x_0 \in X \setminus M$. Then there exists a linear extension $f_1 : M + \mathbb{R}x_0 \to \mathbb{R}$ of f_0 such that $f_1(x) \le p(x)$ for all $x \in M + \mathbb{R}x_0$.*

Proof. For $m \in M$ set

$$A := \sup_{m\in M}\Big(f_0(m) - p(m - x_0)\Big), \qquad B := \inf_{m\in M}\Big(p(m + x_0) - f_0(m)\Big).$$

Sublinearity gives $A \le B$. Choose any $\alpha \in [A, B]$ and define $f_1(m + tx_0) := f_0(m) + t\alpha$. Then f_1 is linear, extends f_0, and the choice $\alpha \in [A, B]$ is exactly the condition that $f_1 \le p$ on $M + \mathbb{R}x_0$. □

4.3 Hahn–Banach theorem

The Hahn–Banach theorem is one of the foundational results of functional analysis. It ensures that continuous linear functionals are abundant enough to detect geometry, separate points from subspaces, and provide a workable duality theory even in infinite-dimensional settings.

We record first the real version, which contains the essential extension argument.

Theorem 4.4 (Hahn–Banach, real version)**.** *Let X be a real vector space, let $p : X \to \mathbb{R}$ be sublinear, and let $M \subset X$ be a subspace. If $f_0 : M \to \mathbb{R}$ is linear and $f_0(x) \le p(x)$ for all $x \in M$, then there exists a linear $f : X \to \mathbb{R}$ extending f_0 such that $f(x) \le p(x)$ for all $x \in X$.*

Proof. Let $\mathcal{F}$ be the collection of pairs (N, f_N) where N is a subspace with $M \subset N \subset X$ and $f_N : N \to \mathbb{R}$ is linear, extends f_0, and satisfies $f_N \le p$ on N. Partially order $\mathcal{F}$ by extension. Every chain has an upper bound given by the union of domains and the compatible linear functionals, so by Zorn's lemma there is a maximal element $(N_{\max}, f_{\max})$.

If $N_{\max} \ne X$, pick $x_0 \in X \setminus N_{\max}$. By Lemma 4.3, $f_{\max}$ extends to $N_{\max} + \mathbb{R}x_0$ while preserving the inequality $f \le p$, contradicting maximality. Hence $N_{\max} = X$. □

Theorem 4.5 (Hahn–Banach, norm-preserving form)**.** *Let X be a normed space over $F \in \{\mathbb{R}, \mathbb{C}\}$, let $M \subset X$ be a subspace, and let $f_0 \in M^*$. Then there exists $f \in X^*$ extending f_0 with $\|f\| = \|f_0\|$.*

Proof. First assume $F = \mathbb{R}$. Apply Theorem 4.4 to $p(x) = \|f_0\|\,\|x\|$. Then $f_0(x) \le \|f_0\|\|x\| = p(x)$ on M, so we obtain an extension f with $f(x) \le p(x)$ on X. Replacing x by $-x$ yields $|f(x)| \le \|f_0\|\|x\|$, hence $\|f\| \le \|f_0\|$. Since f extends f_0, also $\|f\| \ge \|f_0\|$.

For $F = \mathbb{C}$, apply the real version to the real-linear functional $\Re f_0$ and then recover a complex-linear extension by the standard formula $f(x) := \Re f(x) - i\,\Re f(ix)$. The norm equality follows from $\|f\| = \|\Re f\|$. $\square$

4.4 Separation and supporting functionals

The power of Hahn–Banach becomes especially visible in its geometric consequences. Separation theorems show that linear functionals can distinguish points and convex sets, while supporting functionals encode tangent hyperplanes and furnish sharp norm-attaining information.

We begin with the separation of a point from a closed subspace by means of a continuous linear functional.

Theorem 4.6 (Separation from a closed subspace)**.** *Let X be a normed space, let $M \subset X$ be a closed subspace, and let $x_0 \notin M$. Then there exists $f \in X^*$ such that $f|_M = 0$, $\|f\| = 1$, and*

$$f(x_0) = \operatorname{dist}(x_0, M).$$

Proof. Let $\delta = \operatorname{dist}(x_0, M) > 0$. Define g on the subspace $M + \operatorname{span}\{x_0\}$ by

$$g(m + tx_0) := t\,\delta \qquad (m \in M,\ t \in F).$$

Then g is linear and $g|_M = 0$. Moreover, for $t \ne 0$,

$$|g(m + tx_0)| = |t|\,\delta = |t|\ \operatorname{dist}(x_0, M) = |t|\ \operatorname{dist}\Big(x_0, -\frac{m}{t} + M\Big) \le \|m + tx_0\|,$$

while the case $t = 0$ is trivial. Thus $|g(z)| \le \|z\|$ for every $z \in M + \operatorname{span}\{x_0\}$, so $\|g\| \le 1$.

On the other hand, for every $m \in M$ we have

$$\delta = |g(x_0 - m)| \le \|g\|\,\|x_0 - m\|.$$

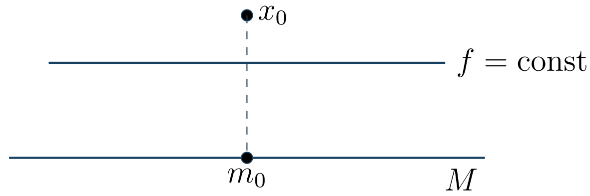

Figure 4.1: Geometric separation: a functional f vanishes on M and separates x_0 from M at distance $\operatorname{dist}(x_0, M)$.

Taking the infimum over $m \in M$ yields $\delta \le \|g\| \, \delta$, and since $\delta > 0$ we obtain $\|g\| \ge 1$. Hence $\|g\| = 1$.

By Theorem 4.5, g extends to a functional $f \in X^*$ with $\|f\| = \|g\| = 1$. Since f extends g, we still have $f|_M = 0$ and

$$f(x_0) = g(x_0) = \delta = \operatorname{dist}(x_0, M).$$

□

Corollary 4.7 (Norming functionals)**.** *For every $x \in X$ with $x \neq 0$ there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$.*

Proof. Apply Theorem 4.6 with $M = \{0\}$ and $x_0 = x$. Since $\operatorname{dist}(x, \{0\}) = \|x\|$, the resulting functional $f \in X^*$ satisfies $\|f\| = 1$ and $f(x) = \|x\|$. □

4.5 Annihilators, subspaces, and quotients

Once the dual space is available, subspaces of X naturally correspond to orthogonality-type conditions in X^*. The language of annihilators clarifies this correspondence and explains how quotient constructions on the primal side reappear as subspaces on the dual side.

For a subspace $M \subset X$ define its *annihilator*

$$M^\perp := \{f \in X^* : f(m) = 0 \text{ for all } m \in M\}.$$

Proposition 4.8 (Dual of a subspace)**.** *Let $M \subset X$ be a subspace. The restriction map* $\operatorname{Res} : X^* \to M^*$*,* $\operatorname{Res}(f) = f|_M$*, is surjective and* $\ker(\operatorname{Res}) = M^\perp$*. Consequently, M^* is isometrically isomorphic to the quotient $X^*/M^\perp$.*

Proof. Surjectivity of Res is precisely Theorem 4.5. Its kernel consists of those functionals in X^* that vanish on M, namely $M^\perp$. Therefore Res induces a well-defined bijection

$$\widehat{\mathrm{Res}} : X^*/M^\perp \to M^*.$$

The induced map is isometric because Theorem 4.5 provides norm-preserving extensions from M^* to X^*. □

Proposition 4.9 (Dual of a quotient)**.** *If $M \subset X$ is a closed subspace and $\pi : X \to X/M$ is the quotient map, then the pullback*

$$\Phi : (X/M)^* \to M^\perp, \qquad \Phi(\varphi) = \varphi \circ \pi$$

is an isometric isomorphism.

Proof. If $\varphi \in (X/M)^*$, then $\varphi \circ \pi$ vanishes on M, so Φ indeed maps into $M^\perp$. To prove injectivity, suppose $\Phi(\varphi) = 0$. Then $\varphi(\pi(x)) = 0$ for every $x \in X$, and since π is surjective, it follows that $\varphi = 0$.

For surjectivity, let $f \in M^\perp$. Define

$$\varphi(x + M) := f(x).$$

This is well defined because if $x + M = y + M$, then $x - y \in M$, and therefore $f(x - y) = 0$. It is also bounded, since

$$|\varphi(x + M)| = |f(x)| \le \|f\| \, \|x\|$$

for every representative x, and taking the infimum over all representatives gives

$$|\varphi(x + M)| \le \|f\| \, \|x + M\|.$$

Finally, $\Phi(\varphi) = f$, and the norm computation shows that $\|\varphi\| = \|f\|$. Hence Φ is an isometric isomorphism. □

4.6 The dual of ℓ^p

Abstract duality becomes much more transparent once it is computed in concrete examples. The identification of $(\ell^p)^*$ with ℓ^q is a model result: it exhibits how analytic inequalities translate into exact descriptions of bounded linear functionals.

Let $1 \le p \le \infty$ and write q for the conjugate exponent ($\frac{1}{p} + \frac{1}{q} = 1$, with the convention $q = \infty$ if $p = 1$).

Theorem 4.10. *For $1 \le p < \infty$, the dual $(\ell^p)^*$ is isometrically isomorphic to ℓ^q via the pairing*

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

For $p = \infty$, the map $\ell^1 \hookrightarrow (\ell^\infty)^*$ given by the same pairing is an isometric embedding, but not onto.

Proof. Assume $1 \le p < \infty$. For $y \in \ell^q$ define $f_y(x) = \langle x, y \rangle$. Hölder's inequality yields $|f_y(x)| \le \|x\|_p \|y\|_q$, hence $f_y \in (\ell^p)^*$ and $\|f_y\| \le \|y\|_q$. Taking x supported where y is large (standard extremal choice) shows $\|f_y\| = \|y\|_q$.

Conversely, let $f \in (\ell^p)^*$ and set $y_n = f(e_n)$, where (e_n) is the standard basis. For $x \in c_{00}$ (finitely supported sequences), linearity gives $f(x) = \sum x_n y_n$. Using boundedness of f on ℓ^p and the standard finite-dimensional duality argument shows $(y_n) \in \ell^q$ and $\|y\|_q \le \|f\|$. Since c_{00} is dense in ℓ^p, $f = f_y$.

For $p = \infty$, the same construction gives an isometric embedding $\ell^1 \to (\ell^\infty)^*$. It is not surjective because ℓ^∞ is not separable whereas $(\ell^1)^* \cong \ell^\infty$ is. $\square$

Chapter 5

Duality, Weak Topologies, and Reflexivity

This chapter develops the weak forms of convergence and compactness that dominate the modern structure theory of Banach spaces. Beginning with adjoint operators and annihilators, it introduces the weak and weak topologies as the natural topologies generated by the dual pairing. The Banach–Alaoglu theorem is then proved carefully through Tychonoff compactness, showing that bounded sets in dual spaces possess compactness properties unavailable in the norm topology. The discussion concludes with metrization results in the separable setting and with the standard criteria and consequences for reflexivity, thereby clarifying how duality, compactness, and geometric structure fit together.*

5.1 Dual spaces and adjoint operators

Duality provides a systematic way to test vectors by functionals and operators by their action on functionals. The adjoint operator is the natural bridge between these two viewpoints, and it will reappear repeatedly in the study of weak topologies, quotient spaces, and spectral theory.

We begin by fixing the basic dual-space language.

Definition 5.1 (Dual space)**.** Let X be a normed linear space over F. The *continuous dual* of X is

$$X^* := \{\varphi : X \to F : \varphi \text{ is linear and bounded}\}.$$

It becomes a normed linear space with the operator norm

$$\|\varphi\| = \sup\{|\varphi(x)| : \|x\| \le 1\}.$$

Proposition 5.2. *For every normed linear space X, the dual space X^* is a Banach space.*

Proof. Let (φ_n) be a Cauchy sequence in X^*. Then for each fixed $x \in X$,

$$|\varphi_n(x) - \varphi_m(x)| \le \|\varphi_n - \varphi_m\| \, \|x\|,$$

so $(\varphi_n(x))$ is a Cauchy sequence in F. Since F is complete, the limit

$$\varphi(x) := \lim_{n\to\infty} \varphi_n(x)$$

exists for every $x \in X$.

Since pointwise limits of linear maps are linear, φ is linear. We now show that φ is bounded and that $\varphi_n \to \varphi$ in norm. Since (φ_n) is Cauchy in X^*, it is bounded: choose $M > 0$ such that $\|\varphi_n\| \le M$ for all n. Then

$$|\varphi(x)| = \lim_{n\to\infty} |\varphi_n(x)| \le M\|x\| \qquad (x \in X),$$

so $\varphi \in X^*$ and $\|\varphi\| \le M$.

Finally, fix $\varepsilon > 0$. Choose N such that $\|\varphi_n - \varphi_m\| < \varepsilon$ whenever $m, n \ge N$. For $n \ge N$ and $\|x\| \le 1$ we may let $m \to \infty$ to obtain

$$|\varphi_n(x) - \varphi(x)| = \lim_{m\to\infty} |\varphi_n(x) - \varphi_m(x)| \le \varepsilon.$$

Taking the supremum over $\|x\| \le 1$ gives $\|\varphi_n - \varphi\| \le \varepsilon$ for $n \ge N$. Hence $\varphi_n \to \varphi$ in X^*. $\square$

Definition 5.3 (Adjoint (transpose) operator)**.** Let $T : X \to Y$ be a bounded linear map between normed spaces. Its *adjoint* (or *transpose*) is the map

$$T^* : Y^* \to X^*, \qquad (T^*g)(x) := g(Tx).$$

Theorem 5.4. *If $T \in \mathcal{B}(X, Y)$, then $T^* \in \mathcal{B}(Y^*, X^*)$ and*

$$\|T^*\| = \|T\|.$$

Moreover, if $S \in \mathcal{B}(Y, Z)$, then

$$(ST)^* = T^*S^*.$$

If T is invertible, then T^* is invertible and

$$(T^*)^{-1} = (T^{-1})^*.$$

Proof. For $g \in Y^*$ and $x \in X$,

$$|(T^*g)(x)| = |g(Tx)| \le \|g\|\,\|Tx\| \le \|g\|\,\|T\|\,\|x\|,$$

hence $T^*g \in X^*$ and $\|T^*g\| \le \|T\| \, \|g\|$. Therefore $\|T^*\| \le \|T\|$.

If $T = 0$, there is nothing further to prove. Assume $T \neq 0$. By definition of the operator norm, for each $\varepsilon > 0$ there exists $x_\varepsilon \in X$ with $\|x_\varepsilon\| = 1$ and

$$\|Tx_\varepsilon\| > \|T\| - \varepsilon.$$

By the Hahn–Banach theorem there exists $g_\varepsilon \in Y^*$ with $\|g_\varepsilon\| = 1$ and

$$g_\varepsilon(Tx_\varepsilon) = \|Tx_\varepsilon\|.$$

Hence

$$\|T^*\| \ge \|T^*g_\varepsilon\| \ge |(T^*g_\varepsilon)(x_\varepsilon)| = |g_\varepsilon(Tx_\varepsilon)| = \|Tx_\varepsilon\| > \|T\| - \varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives $\|T^*\| \ge \|T\|$, and so $\|T^*\| = \|T\|$.

The identity $(ST)^* = T^*S^*$ is immediate from the definition:

$$((ST)^*h)(x) = h(STx) = (S^*h)(Tx) = (T^*S^*h)(x).$$

If T is invertible, then

$$T^*(T^{-1})^* = (T^{-1}T)^* = I_{Y^*}, \qquad (T^{-1})^*T^* = (TT^{-1})^* = I_{X^*},$$

so $(T^*)^{-1} = (T^{-1})^*$. $\square$

5.2 Annihilators and quotient duals

The formal similarities between subspaces of X and subspaces of X^* become sharper when one studies annihilators on both sides simultaneously. This viewpoint leads to clean dual descriptions of quotients and clarifies how geometric constraints are encoded by linear functionals.

The fundamental construction is the annihilator of a subspace.

Definition 5.5 (Annihilator)**.** If $M \subset X$ is a subspace, its *annihilator* is

$$M^\perp := \{\varphi \in X^* : \varphi(m) = 0 \text{ for all } m \in M\}.$$

If $N \subset X^*$ is a subspace, we define

$${}^\perp N := \{x \in X : \varphi(x) = 0 \text{ for all } \varphi \in N\}.$$

Proposition 5.6. *Let M be a subspace of a normed space X. Then $M^\perp$ is a closed subspace of X^*. Moreover, if M is dense in X, then $M^\perp = \{0\}$. If M is a proper closed subspace, then $M^\perp \neq \{0\}$.*

Proof. The fact that $M^\perp$ is a linear subspace is immediate. If $\varphi_n \in M^\perp$ and $\varphi_n \to \varphi$ in X^*, then for every $m \in M$,

$$\varphi(m) = \lim_{n\to\infty} \varphi_n(m) = 0,$$

so $\varphi \in M^\perp$. Thus $M^\perp$ is closed.

If M is dense and $\varphi \in M^\perp$, then φ vanishes on a dense set. By continuity, $\varphi = 0$ on all of X, so $M^\perp = \{0\}$.

If M is a proper closed subspace and $x_0 \notin M$, then by the Hahn–Banach separation theorem there exists $\varphi \in X^*$ such that $\varphi|_M = 0$ and $\varphi(x_0) \neq 0$. Hence $\varphi \in M^\perp \setminus \{0\}$. □

Theorem 5.7 (Dual of a quotient)**.** *Let M be a closed subspace of a normed space X. Then*

$$(X/M)^* \cong M^\perp$$

isometrically, via

$$\Lambda \in (X/M)^* \longmapsto \varphi_\Lambda := \Lambda \circ q,$$

where $q : X \to X/M$ is the quotient map.

Proof. Let $\Lambda \in (X/M)^*$ and define $\varphi_\Lambda = \Lambda \circ q$. Then φ_Λ is bounded and vanishes on M, so $\varphi_\Lambda \in M^\perp$.

Conversely, if $\varphi \in M^\perp$, define

$$\widetilde{\varphi}(x + M) := \varphi(x).$$

This is well-defined: if $x + M = y + M$, then $x - y \in M$ and hence $\varphi(x - y) = 0$, so $\varphi(x) = \varphi(y)$. The map $\widetilde{\varphi}$ is linear and bounded, and $\varphi = \widetilde{\varphi} \circ q$.

Thus the correspondence $\Lambda \mapsto \Lambda \circ q$ is bijective between $(X/M)^*$ and $M^\perp$. For norm preservation, note first that

$$\|\varphi_\Lambda\| \le \|\Lambda\| \, \|q\| = \|\Lambda\|,$$

since $\|q\| = 1$. Conversely, if $\|x + M\| < 1$, choose $m \in M$ with $\|x + m\| < 1 + \varepsilon$. Then

$$|\Lambda(x + M)| = |\varphi_\Lambda(x + m)| \le \|\varphi_\Lambda\|(1 + \varepsilon).$$

Taking the supremum over $\|x + M\| < 1$ and letting $\varepsilon \downarrow 0$ gives $\|\Lambda\| \le \|\varphi_\Lambda\|$. Hence the correspondence is an isometry. □

Theorem 5.8 (Dual of a subspace)**.** *Let M be a closed subspace of a normed space X. Then*

$$X^*/M^\perp \cong M^*$$

isometrically.

Proof. Define

$$R : X^* \to M^*, \qquad R(\varphi) = \varphi|_M.$$

This is a bounded linear map with $\|R\| \le 1$, and $\ker R = M^\perp$. By the Hahn–Banach theorem, every $g \in M^*$ extends to some $\varphi \in X^*$ with $\|\varphi\| = \|g\|$, so R is surjective. Therefore the induced map

$$\widetilde{R} : X^*/M^\perp \to M^*, \qquad \widetilde{R}(\varphi + M^\perp) = \varphi|_M,$$

is a linear bijection.

For $\varphi \in X^*$ we have

$$\|\widetilde{R}(\varphi + M^\perp)\| = \|\varphi|_M\| \le \inf_{\psi \in M^\perp} \|\varphi + \psi\| = \|\varphi + M^\perp\|.$$

Therefore $\|\widetilde{R}\| \le 1$. Conversely, given $g \in M^*$, choose by Hahn–Banach an extension $\varphi \in X^*$ with $\|\varphi\| = \|g\|$. Then

$$\|\varphi + M^\perp\| \le \|\varphi\| = \|g\| = \|\widetilde{R}(\varphi + M^\perp)\|.$$

Thus $\widetilde{R}$ is an isometry. □

5.3 Weak and weak* topologies

Norm convergence is often too strong for compactness arguments in infinite-dimensional spaces. Weak and weak* topologies remedy this by retaining exactly the continuity of linear functionals, thereby providing weaker but far more flexible notions of convergence.

The two basic weak topologies are defined as follows.

Definition 5.9 (Weak topology)**.** The *weak topology* on a normed space X is the smallest topology for which every functional in X^* is continuous. It is denoted by $\sigma(X, X^*)$. A net (x_α) converges to x in $\sigma(X, X^*)$ if and only if

$$\varphi(x_\alpha) \to \varphi(x) \qquad \text{for every } \varphi \in X^*.$$

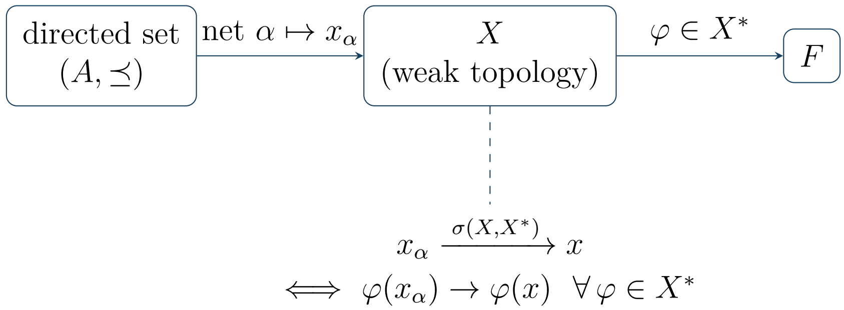

Figure 5.1: Weak convergence is expressed by testing against all continuous linear functionals.

Definition 5.10 (Weak* topology)**.** The *weak* topology* on X^* is the smallest topology for which all evaluation maps

$$e_x : X^* \to F, \qquad e_x(\varphi) = \varphi(x),$$

are continuous. It is denoted by $\sigma(X^*, X)$. A net (φ_α) converges to φ in $\sigma(X^*, X)$ if and only if

$$\varphi_\alpha(x) \to \varphi(x) \qquad \text{for every } x \in X.$$

Remark 5.11 (Comparison with norm convergence). Norm convergence implies weak convergence, and norm convergence in X^* implies weak* convergence. The converse implications fail in infinite dimensions.

Proposition 5.12. *Every weakly convergent sequence in X is norm bounded. More generally, every weakly Cauchy sequence in X is norm bounded.*

Proof. Let (x_n) be weakly Cauchy. Then for each $\varphi \in X^*$ the scalar sequence $(\varphi(x_n))$ is Cauchy, hence bounded. Define

$$F_n : X^* \to F, \qquad F_n(\varphi) := \varphi(x_n).$$

Each F_n is a bounded linear functional on X^*, and

$$\|F_n\| = \sup_{\|\varphi\| \le 1} |\varphi(x_n)| = \|x_n\|$$

by the Hahn–Banach theorem. Since $(F_n(\varphi))$ is bounded for each $\varphi \in X^*$, the uniform boundedness principle yields

$$\sup_n \|F_n\| < \infty.$$

Therefore $\sup_n \|x_n\| < \infty$. $\square$

Proposition 5.13. *Let M be a norm-dense subspace of X^*, and let (x_n) be a norm-bounded sequence in X. Suppose there exists $x \in X$ such that*

$$f(x_n) \to f(x) \qquad \text{for every } f \in M.$$

Then $x_n \to x$ weakly in X.

Proof. Set $C := \sup_n \|x_n\| + \|x\| < \infty$. Fix $\varphi \in X^*$ and $\varepsilon > 0$. Choose $f \in M$ such that

$$\|\varphi - f\| < \frac{\varepsilon}{3C}.$$

Then for all n,

$$|\varphi(x_n) - \varphi(x)| \le |\varphi(x_n) - f(x_n)| + |f(x_n) - f(x)| + |f(x) - \varphi(x)| < \frac{\varepsilon}{3} + |f(x_n) - f(x)| + \frac{\varepsilon}{3}.$$

Since $f(x_n) \to f(x)$, the middle term is $< \varepsilon/3$ for all large n. Hence $\varphi(x_n) \to \varphi(x)$ for every $\varphi \in X^*$. $\square$

5.4 Banach–Alaoglu theorem

In infinite-dimensional normed spaces the closed unit ball is almost never compact in the norm topology. The Banach–Alaoglu theorem identifies the correct replacement: compactness is recovered in the weak* topology, and this fact becomes a basic compactness principle throughout the subject.

The central compactness statement is the following.

Theorem 5.14 (Banach–Alaoglu)**.** *For a normed space X, the closed unit ball*

$$B_{X^*} = \{\varphi \in X^* : \|\varphi\| \le 1\}$$

is compact in the weak topology.*

Proof. For each $x \in X$ define the compact set

$$D_x := \{z \in F : |z| \le \|x\|\}.$$

Consider the product space

$$D := \prod_{x \in X} D_x,$$

equipped with the product topology. By Tychonoff's theorem, D is compact.

Define

$$\Phi : B_{X^*} \to D, \qquad \Phi(\varphi) := (\varphi(x))_{x \in X}.$$

If $\varphi \in B_{X^*}$, then $|\varphi(x)| \le \|\varphi\| \, \|x\| \le \|x\|$, so $\Phi(\varphi) \in D$. The map Φ is injective, because if $\Phi(\varphi) = \Phi(\psi)$ then $\varphi(x) = \psi(x)$ for all $x \in X$, hence $\varphi = \psi$.

The product topology on D is precisely the topology of coordinatewise convergence. Therefore, by definition of the weak* topology, Φ is a homeomorphism from B_{X^*} (with its weak* topology) onto its range $\Phi(B_{X^*})$.

It remains to show that $\Phi(B_{X^*})$ is closed in D. Let $\xi = (\xi_x)_{x \in X}$ belong to the closure of $\Phi(B_{X^*})$. Then there exists a net $(\varphi_\alpha) \subset B_{X^*}$ such that

$$\Phi(\varphi_\alpha) \to \xi \quad \text{in } D.$$

Coordinatewise convergence means

$$\varphi_\alpha(x) \to \xi_x \qquad \text{for every } x \in X.$$

Define $\varphi : X \to F$ by $\varphi(x) := \xi_x$. Since pointwise limits of linear maps are linear, φ is linear. Moreover,

$$|\varphi(x)| = \lim_\alpha |\varphi_\alpha(x)| \le \|x\| \qquad (x \in X),$$

so $\varphi \in B_{X^*}$. Finally,

$$\Phi(\varphi) = (\varphi(x))_{x \in X} = (\xi_x)_{x \in X} = \xi.$$

Therefore $\xi \in \Phi(B_{X^*})$, so the image is closed in the compact space D. Therefore $\Phi(B_{X^*})$ is compact, and since Φ is a homeomorphism onto its image, B_{X^*} is weak* compact. □

Corollary 5.15 (Banach–Mazur embedding). *Every normed space X is isometrically isomorphic to a subspace of $C(K)$ for some compact Hausdorff space K. If X is Banach, the image is closed.*

Proof. Take

$$K := B_{X^*}$$

with the weak* topology. By theorem 5.14, K is compact Hausdorff.

For each $x \in X$ define $\widehat{x} \in C(K)$ by

$$\widehat{x}(\varphi) := \varphi(x) \qquad (\varphi \in K).$$

This function is continuous because weak* convergence is exactly pointwise convergence on X. The map

$$J : X \to C(K), \qquad J(x) = \widehat{x},$$

is linear. Also,

$$\|J(x)\|_\infty = \sup_{\varphi\in K} |\varphi(x)| \le \|x\|.$$

By the Hahn–Banach theorem there exists, for each $x \neq 0$, some $\varphi \in X^*$ with $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$. Thus

$$\|J(x)\|_\infty \ge |\varphi(x)| = \|x\|,$$

so $\|J(x)\|_\infty = \|x\|$ for every $x \in X$. Hence J is an isometry.

If X is Banach, then $J(X)$ is complete in the supremum norm and therefore closed in the Banach space $C(K)$. □

Theorem 5.16 (Metrization of the weak* topology on the dual ball)**.** *Assume X is separable, and let $\{x_k\}_{k=1}^\infty$ be a countable dense subset of the unit sphere of X. Define, for $\varphi, \psi \in B_{X^*}$,*

$$d(\varphi, \psi) := \sum_{k=1}^\infty \frac{1}{2^k} |\varphi(x_k) - \psi(x_k)|.$$

Then d is a metric on B_{X^*}, and the metric topology coincides with the weak topology on B_{X^*}.*

Proof. Because $\|\varphi\|, \|\psi\| \le 1$ and $\|x_k\| = 1$, we have

$$|\varphi(x_k) - \psi(x_k)| \le 2,$$

so the series defining d converges absolutely. The triangle inequality is immediate. If $d(\varphi, \psi) = 0$, then $\varphi(x_k) = \psi(x_k)$ for all k. Since the $\{x_k\}$ are dense in the unit sphere and $\varphi - \psi$ is continuous and homogeneous, it follows that $\varphi = \psi$ on all of X. Thus d is a metric.

First suppose that $d(\varphi_n, \varphi) \to 0$. Then for each fixed k,

$$|\varphi_n(x_k) - \varphi(x_k)| \to 0.$$

Let $x \in X$ and $\varepsilon > 0$. Choose k such that $\|x/\|x\| - x_k\| < \varepsilon$ when $x \neq 0$. Since $\|\varphi_n\|, \|\varphi\| \le 1$, we obtain

$$|\varphi_n(x) - \varphi(x)| \le \|x\|\, |\varphi_n(x_k) - \varphi(x_k)| + 2\|x\|\, \varepsilon.$$

Letting $n \to \infty$ and then $\varepsilon \downarrow 0$ shows $\varphi_n(x) \to \varphi(x)$ for every $x \in X$, i.e. $\varphi_n \to \varphi$ weak*.

Conversely, assume $\varphi_n \to \varphi$ weak*. Then $\varphi_n(x_k) \to \varphi(x_k)$ for each k. Fix $\varepsilon > 0$ and choose N such that

$$\sum_{k>N} 2^{-k+1} < \varepsilon/2.$$

For the finitely many indices $1 \le k \le N$, choose n_0 such that

$$|\varphi_n(x_k) - \varphi(x_k)| < \frac{\varepsilon}{2N} \qquad (1 \le k \le N,\ n \ge n_0).$$

Then for $n \ge n_0$,

$$d(\varphi_n, \varphi) \le \sum_{k=1}^{N} \frac{1}{2^k} \frac{\varepsilon}{2N} + \sum_{k>N} \frac{1}{2^k} \cdot 2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $d(\varphi_n, \varphi) \to 0$. □

Corollary 5.17. *If X is separable, then every bounded sequence in X^* has a weak* convergent subsequence.*

Proof. Let (φ_n) be bounded in X^*, and choose $M > 0$ such that $\|\varphi_n\| \le M$ for all n. Then the sequence $(M^{-1}\varphi_n)$ is contained in the weak* compact set B_{X^*}. By theorem 5.16, the weak* topology on B_{X^*} is metrizable; hence compactness implies sequential compactness. Therefore $(M^{-1}\varphi_n)$ has a weak* convergent subsequence. Multiplying that subsequence by M gives a weak* convergent subsequence of (φ_n). □

5.5 Reflexive spaces

A space is reflexive when duality does not create genuinely new directions beyond those already present in the original space. Reflexivity occupies a central position because it ties together weak compactness, structural finiteness properties, and the geometry of Banach spaces.

The definition is expressed through the canonical embedding into the bidual.

Definition 5.18 (Canonical embedding and reflexivity)**.** For $x \in X$, define $Jx \in X^{**}$ by

$$(Jx)(\varphi) = \varphi(x) \qquad (\varphi \in X^*).$$

The map $J : X \to X^{**}$ is called the *canonical embedding*. The space X is said to be *reflexive* if J is onto.

Proposition 5.19. *The canonical embedding $J : X \to X^{**}$ is linear and isometric.*

Proof. Linearity is immediate. For $\|\varphi\| \le 1$ we have

$$|(Jx)(\varphi)| = |\varphi(x)| \le \|x\|,$$

so $\|Jx\| \le \|x\|$. Conversely, if $x \ne 0$, the Hahn–Banach theorem yields $\varphi \in X^*$ with $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$. Then

$$\|Jx\| \ge |(Jx)(\varphi)| = |\varphi(x)| = \|x\|.$$

Therefore $\|Jx\| = \|x\|$. □

Remark 5.20 (Basic examples and consequences). Every reflexive space is Banach, because it is isometric to the Banach space X^{**}. Finite-dimensional spaces are reflexive. Moreover, ℓ^p and L^p are reflexive for $1 < p < \infty$, whereas ℓ^1, ℓ^∞, c_0, and L^1 are not reflexive.

Theorem 5.21 (Reflexive spaces have weakly compact unit balls)**.** *If X is reflexive, then its closed unit ball B_X is compact in the weak topology of X.*

Proof. Under the canonical embedding $J : X \to X^{**}$, the set $J(B_X)$ is the closed unit ball of $J(X)$. If X is reflexive, then $J(X) = X^{**}$, so $J(B_X) = B_{X^{**}}$. By Banach–Alaoglu, $B_{X^{**}}$ is compact in the weak* topology $\sigma(X^{**}, X^*)$. The weak topology on X is exactly the topology transported from this weak* topology via J, because

$$(Jx_\alpha)(f) = f(x_\alpha) \qquad (f \in X^*).$$

Therefore B_X is weakly compact. □

Theorem 5.22 (Goldstine's theorem)**.** *For every normed space X, the canonical image $J(B_X)$ is weak* dense in $B_{X^{**}}$.*

Remark. The proof may be obtained from the Hahn–Banach separation theorem. We record the result here because it is the standard bridge from Banach–Alaoglu compactness to the characterization of reflexive spaces. □

Theorem 5.23 (Kakutani characterization of reflexivity)**.** *A Banach space X is reflexive if and only if its closed unit ball B_X is weakly compact.*

Proof. One implication is theorem 5.21. Conversely, assume B_X is weakly compact. Let $x^{**} \in X^{**}$ with $\|x^{**}\| \le 1$. By Goldstine's theorem, $J(B_X)$ is weak* dense in $B_{X^{**}}$. Hence there exists a net $(x_\alpha) \subset B_X$ such that $Jx_\alpha \to x^{**}$ in the weak* topology of X^{**}. Since B_X is weakly compact, the net has a subnet (x_β) converging weakly in X to some $x \in B_X$. For every $f \in X^*$ we then have

$$x^{**}(f) = \lim_\beta (Jx_\beta)(f) = \lim_\beta f(x_\beta) = f(x) = (Jx)(f).$$

Thus $x^{**} = Jx$. Hence every element of the unit ball of X^{**} lies in $J(X)$, so J is onto and X is reflexive. □

Corollary 5.24. *Let X be reflexive and let M be a closed subspace of X. Then both M and X/M are reflexive.*

Proof. The closed unit ball of M is

$$B_M = B_X \cap M.$$

Since M is norm closed, it is also weakly closed; therefore B_M is weakly closed in the weakly compact set B_X. Hence B_M is weakly compact. By theorem 5.23, M is reflexive.

For the quotient, let $q : X \to X/M$ be the quotient map. The map q is weak-to-weak continuous, and

$$q(B_X) = B_{X/M}.$$

Therefore the closed unit ball of X/M is weakly compact as the continuous image of a weakly compact set. Another application of theorem 5.23 yields that X/M is reflexive. □

Chapter 6

Spectra of Bounded Linear Operators

This chapter begins the spectral theory of bounded linear operators on complex Banach spaces. It introduces the spectrum and resolvent, explains their basic algebraic meaning, and shows that spectral questions are governed by analytic properties of the resolvent map. The fundamental facts that the spectrum is nonempty, compact, and controlled by the operator norm are developed with full rigor, culminating in the spectral radius formula. In this way the chapter establishes the conceptual and technical framework required for the later study of compact and self-adjoint operators.

6.1 Resolvent and spectrum

In finite dimensions the behaviour of an operator is strongly influenced by its eigenvalues. For general bounded operators on Banach spaces, the correct replacement is the spectrum, defined through invertibility of $\lambda I - T$ and encoded analytically by the resolvent.

Let X be a complex Banach space and $T \in \mathcal{B}(X)$. The *resolvent set* is

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is bijective and } (\lambda I - T)^{-1} \in \mathcal{B}(X)\},$$

and the *spectrum* is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Proposition 6.1 (Large scalars lie in the resolvent set)**.** *If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$. Consequently,*

$$\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \le \|T\|\},$$

so the spectrum is compact.

Proof. Write

$$\lambda I - T = \lambda\Big(I - \lambda^{-1}T\Big).$$

If $|\lambda| > \|T\|$, then $\|\lambda^{-1}T\| < 1$, so the Neumann series gives the invertibility of $I - \lambda^{-1}T$, and hence of $\lambda I - T$. Since theorem 6.5 below shows that $\rho(T)$ is open, the complement $\sigma(T)$ is closed; combined with the displayed bound, this proves compactness. $\square$

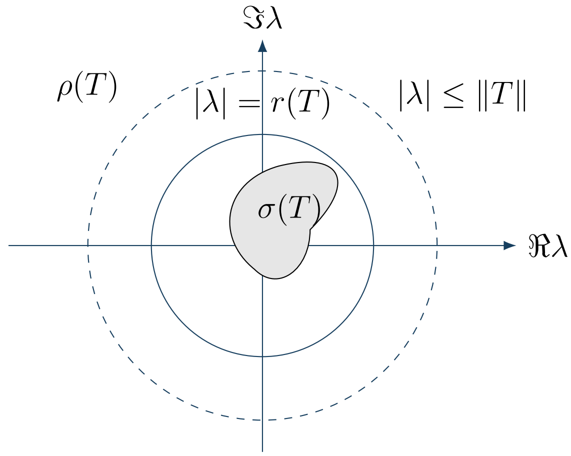

Figure 6.1: Schematic location of the spectrum: $\sigma(T)$ is compact and contained in the closed disk $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$, while the resolvent set occupies the complement.

Definition 6.2 (Point, continuous, and residual spectra)**.** Let $T \in \mathcal{B}(X)$. The *point spectrum* $\sigma_p(T)$ consists of the eigenvalues of T, i.e.

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \neq \{0\}\}.$$

The *continuous spectrum* $\sigma_c(T)$ consists of those λ for which $\lambda I - T$ is injective with dense, proper range. The *residual spectrum* $\sigma_r(T)$ consists of those λ for which $\lambda I - T$ is injective but its range is not dense. Thus

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T),$$

with the three parts pairwise disjoint.

Remark 6.3 (Finite-dimensional versus infinite-dimensional spectra)*.* The decomposition above is useful for concrete examples. In finite-dimensional spaces the continuous and residual spectra are empty, so the spectrum reduces to the set of eigenvalues. In infinite dimensions, however, all three phenomena can occur.

Lemma 6.4 (Neumann series)**.** *If $\|S\| < 1$ for some $S \in \mathcal{B}(X)$, then $I - S$ is invertible and*

$$(I - S)^{-1} = \sum_{n=0}^{\infty} S^n,$$

with convergence in operator norm.

Proof. Since $\|S\| < 1$, the series $\sum_{n=0}^{\infty} \|S^n\|$ converges, so the operator series

$$A := \sum_{n=0}^{\infty} S^n$$

converges in $\mathcal{B}(X)$. For each N,

$$(I - S) \sum_{n=0}^{N} S^n = I - S^{N+1} = \sum_{n=0}^{N} S^n (I - S).$$

Letting $N \to \infty$ and using $\|S^{N+1}\| \leq \|S\|^{N+1} \to 0$, we obtain

$$(I - S)A = A(I - S) = I.$$

Thus $A = (I - S)^{-1}$. $\square$

Proposition 6.5. *The resolvent set $\rho(T)$ is open and the resolvent map*

$$R_T(\lambda) := (\lambda I - T)^{-1}$$

is holomorphic on $\rho(T)$. Consequently, $\sigma(T)$ is closed.

Proof. Let $\lambda_0 \in \rho(T)$. Then

$$\lambda I - T = (\lambda_0 I - T)\Big(I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}\Big).$$

If

$$|\lambda - \lambda_0| < \|(\lambda_0 I - T)^{-1}\|^{-1},$$

then

$$\|(\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}\| < 1.$$

By theorem 6.4, the factor in parentheses is invertible, so $\lambda \in \rho(T)$. Hence $\rho(T)$ is open.

Moreover, in that same neighborhood,

$$R_T(\lambda) = \Big(I - (\lambda_0 - \lambda) R_T(\lambda_0)\Big)^{-1} R_T(\lambda_0) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R_T(\lambda_0)^{n+1},$$

an operator-norm convergent power series in λ. Thus R_T is holomorphic on $\rho(T)$. Since $\sigma(T) = \mathbb{C} \setminus \rho(T)$, the spectrum is closed. $\square$

Theorem 6.6 (Non-emptiness of the spectrum)**.** *If $X \neq \{0\}$ is complex and $T \in \mathcal{B}(X)$, then $\sigma(T) \neq \emptyset$.*

Proof. Assume, for contradiction, that $\sigma(T) = \emptyset$. Then $\rho(T) = \mathbb{C}$, so the resolvent

$$R_T(\lambda) = (\lambda I - T)^{-1}$$

is an entire operator-valued function.

Fix $x \in X$ and $f \in X^*$, and define the scalar entire function

$$g(\lambda) := f(R_T(\lambda)x).$$

For $|\lambda| > \|T\|$, we may write

$$\lambda I - T = \lambda\Big(I - \lambda^{-1}T\Big),$$

hence by the Neumann series,

$$R_T(\lambda) = \lambda^{-1}\sum_{n=0}^{\infty} \lambda^{-n}T^n.$$

Therefore

$$\|R_T(\lambda)\| \leq \frac{1}{|\lambda|}\sum_{n=0}^{\infty}\Big(\frac{\|T\|}{|\lambda|}\Big)^n = \frac{1}{|\lambda| - \|T\|} \qquad (|\lambda| > \|T\|),$$

so g is bounded outside a large disk. Being entire, g is bounded on the whole plane and hence constant by Liouville's theorem. Since $R_T(\lambda) \to 0$ in norm as $|\lambda| \to \infty$, we have $g(\lambda) \to 0$, so $g \equiv 0$.

Because this holds for every $f \in X^*$, the Hahn–Banach theorem implies $R_T(\lambda)x = 0$ for every λ and every $x \in X$, a contradiction to invertibility. Hence $\sigma(T) \neq \emptyset$. $\square$

Proposition 6.7 (Resolvent identity)**.** *For $\lambda, \mu \in \rho(T)$ one has*

$$R_T(\lambda) - R_T(\mu) = (\mu - \lambda)R_T(\lambda)R_T(\mu).$$

Proof. Since $(\lambda I - T)R_T(\lambda) = I = (\mu I - T)R_T(\mu)$, we obtain

$$\begin{aligned} R_T(\lambda) - R_T(\mu) &= R_T(\lambda)\Big((\mu I - T) - (\lambda I - T)\Big)R_T(\mu) \\ &= (\mu - \lambda)R_T(\lambda)R_T(\mu). \end{aligned}$$

□

6.2 Spectral radius

The spectrum contains qualitative information about an operator, but one often needs a single numerical quantity that measures its spectral size. The spectral radius serves this purpose and, through Gelfand's formula, links spectral information with the asymptotic growth of operator powers.

The *spectral radius* is

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Theorem 6.8 (Spectral radius formula)**.**

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\geq 1} \|T^n\|^{1/n}.$$

Proof. Set

$$a_n := \|T^n\|^{1/n} \qquad (n \geq 1).$$

Since $\|T^{m+n}\| \leq \|T^m\|\,\|T^n\|$, the sequence $b_n := \log\|T^n\|$ is subadditive. By Fekete's lemma,

$$\lim_{n\to\infty} \frac{b_n}{n} = \inf_{n\geq 1} \frac{b_n}{n},$$

which is equivalent to

$$\lim_{n\to\infty} a_n = \inf_{n\geq 1} a_n =: L.$$

We first prove $r(T) \leq L$. If $|\lambda| > L$, choose n such that $\|T^n\| < |\lambda|^n$. Then

$$\|T^n/\lambda^n\| < 1,$$

so by the Neumann series,

$$I - \lambda^{-n}T^n$$

is invertible. Since

$$I - \lambda^{-n}T^n = \prod_{k=0}^{n-1}\Big(I - \omega_k\lambda^{-1}T\Big),$$

where $\omega_k = e^{2\pi ik/n}$, each factor must be invertible. In particular, $I - \lambda^{-1}T$ is invertible, i.e. $\lambda \in \rho(T)$. Thus $|\lambda| > r(T)$, proving $r(T) \leq L$.

For the reverse inequality, let $|\lambda| > r(T)$. Then $\lambda \in \rho(T)$ and

$$R_T(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n$$

converges in operator norm (as the Taylor expansion of the resolvent at infinity). Hence

$$\sup_n \|\lambda^{-n} T^n\| < \infty,$$

which implies

$$\|T^n\|^{1/n} \le |\lambda| \, C^{1/n}$$

for some constant $C > 0$. Letting $n \to \infty$ yields $L \le |\lambda|$. Since this holds for every $|\lambda| > r(T)$, we obtain $L \le r(T)$.

Combining the two inequalities gives $L = r(T)$. $\square$

Chapter 7

Compact Operators and the Fredholm Alternative

Compact operators form the first major class of infinite-dimensional operators whose behaviour still resembles finite-dimensional linear algebra. This chapter develops their basic approximation and continuity properties, emphasizing the extent to which compactness restores discreteness to spectral phenomena. The central result is the Fredholm alternative, from which one deduces that every nonzero spectral value of a compact operator is an eigenvalue of finite multiplicity. The chapter therefore serves as a bridge between abstract Banach-space operator theory and the more geometric spectral theory that emerges fully in Hilbert spaces.

7.1 Compact operators

Compact operators are the infinite-dimensional analogue of matrices with finite-dimensional range, and many of their properties reflect this near-finite-dimensional character. They form one of the most tractable and useful classes of operators in analysis, especially in spectral theory and integral equations.

Let X and Y be normed spaces. A bounded linear operator $K : X \to Y$ is *compact* if $K(B_X)$ is relatively compact in Y, where $B_X = \{x \in X : \|x\| \le 1\}$.

Proposition 7.1. *Finite-rank operators are compact. The collection $\mathcal{K}(X,Y)$ of compact operators is a closed subspace of $\mathcal{B}(X,Y)$, and if $K \in \mathcal{K}(X,Y)$ and $S \in \mathcal{B}(Y,Z)$, $R \in \mathcal{B}(W,X)$, then $SKR \in \mathcal{K}(W,Z)$.*

Proof. If $F : X \to Y$ has finite-dimensional range, then $F(B_X)$ is bounded in a finite-dimensional subspace of Y. In finite dimensions, bounded sets are relatively compact, so F is compact.

If K_1, K_2 are compact and $\alpha, \beta \in F$, then

$$(\alpha K_1 + \beta K_2)(B_X) \subset \alpha K_1(B_X) + \beta K_2(B_X).$$

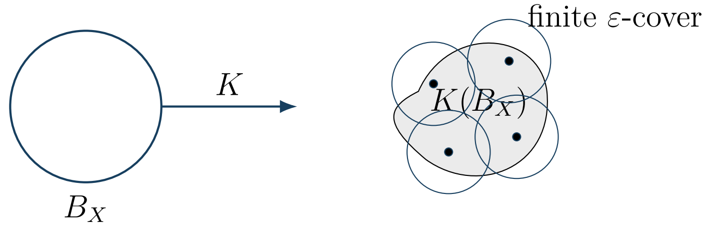

Figure 7.1: A compact operator carries the unit ball to a relatively compact set; equivalently, $K(B_X)$ is totally bounded in the target space.

The right-hand side is relatively compact because sums and scalar multiples of relatively compact sets are relatively compact. Hence $\alpha K_1 + \beta K_2$ is compact.

Now let $K_n \to K$ in operator norm with each K_n compact. Fix $\varepsilon > 0$. Choose n so large that

$$\|K - K_n\| < \varepsilon/3.$$

Since $K_n(B_X)$ is relatively compact, it can be covered by finitely many balls of radius $\varepsilon/3$. Then $K(B_X)$ is covered by the same centers with radius ε. Thus $K(B_X)$ is totally bounded, hence relatively compact; so K is compact.

Finally, if K is compact, then $R(B_W) \subset \|R\|\, B_X$, so $K(R(B_W))$ is relatively compact in Y, and applying the bounded map S preserves relative compactness. Therefore SKR is compact. □

7.2 Fredholm alternative for compact operators

For compact perturbations of the identity, injectivity and surjectivity are no longer independent phenomena. The Fredholm alternative makes this interaction precise and gives a remarkably clean criterion for solvability of equations of the form $(\lambda I - K)x = y$.

We first isolate the estimate that drives the argument.

Lemma 7.2. *Let $K \in \mathcal{K}(X)$ and let $\lambda \neq 0$. If $\lambda I - K$ is injective, then there exists $c > 0$ such that*

$$\|(\lambda I - K)x\| \geq c\|x\| \qquad (x \in X).$$

In particular, $\operatorname{Ran}(\lambda I - K)$ *is closed.*

Proof. Assume no such c exists. Then there is a sequence (x_n) with $\|x_n\| = 1$ and

$$(\lambda I - K)x_n \to 0.$$

Since K is compact, passing to a subsequence if necessary, we may assume $Kx_n \to y$ in norm. Then

$$\lambda x_n = Kx_n + (\lambda I - K)x_n \to y,$$

so $x_n \to x := \lambda^{-1} y$ in norm. By continuity of $\lambda I - K$,

$$(\lambda I - K)x = \lim_{n\to\infty} (\lambda I - K)x_n = 0.$$

Because $\lambda I - K$ is injective, $x = 0$, contradicting $\|x_n\| = 1$ and $x_n \to 0$. Hence the stated inequality holds.

If $(\lambda I - K)x_n \to z$, then the inequality shows that (x_n) is Cauchy, so $x_n \to x$ for some $x \in X$. Continuity yields $(\lambda I - K)x = z$, proving that the range is closed. $\square$

Theorem 7.3 (Fredholm alternative)**.** *Let $K \in \mathcal{K}(X)$ and let $\lambda \neq 0$. Then either λ is an eigenvalue of K, or $\lambda I - K$ is invertible. Equivalently,*

$$\sigma(K) \setminus \{0\} = \{\lambda \neq 0 : \ker(\lambda I - K) \neq \{0\}\}.$$

Proof. Scaling by λ^{-1}, it suffices to prove the statement for $\lambda = 1$. Thus let $S := K$ and assume $I - S$ is injective. We must show that $I - S$ is surjective.

By theorem 7.2, $M_1 := (I - S)(X)$ is closed. Define inductively

$$M_n := (I - S)^n(X) \qquad (n \geq 1).$$

Since S commutes with $I - S$, each M_n is S-invariant. Also, $I - S$ is injective on each M_n, so by the previous lemma applied to the restriction $S|_{M_n}$, each $M_{n+1} = (I - S)(M_n)$ is closed in M_n, hence in X.

Suppose, for contradiction, that $I - S$ is not surjective, so $M_1 \neq X$. We now show that then every inclusion

$$M_{n+1} \subsetneq M_n$$

is proper. Indeed, if $M_{n+1} = M_n$ for some $n \geq 1$, then for $y \in M_{n-1}$ we have $(I - S)y \in M_n = M_{n+1}$. Hence $(I - S)y = (I - S)z$ for some $z \in M_n$. Since $I - S$ is injective, $y = z \in M_n$. Thus $M_{n-1} \subset M_n$, and therefore $M_{n-1} = M_n$. Repeating this argument backward gives $X = M_1$, contradiction.

Therefore each M_{n+1} is a proper closed subspace of M_n. By Riesz's lemma, for each n we can choose $x_n \in M_n$ with

$$\|x_n\| = 1 \qquad \text{and} \qquad \operatorname{dist}(x_n, M_{n+1}) > \frac{1}{2}.$$

If $m > n$, then $x_m \in M_m \subset M_{n+1}$. Also,

$$x_n - Sx_n = (I - S)x_n \in M_{n+1}.$$

Therefore

$$\|Sx_n - Sx_m\| \geq \operatorname{dist}(Sx_n, M_{n+1}) = \operatorname{dist}(x_n, M_{n+1}) > \frac{1}{2}.$$

Thus the sequence (Sx_n) has no Cauchy subsequence, contradicting compactness of S.

This contradiction shows that $I - S$ must be surjective. Since it is already injective and bounded, the inverse mapping theorem implies that $I - S$ is invertible. Restoring λ gives the result. □

Corollary 7.4. *Let $K \in \mathcal{K}(X)$ and let $\lambda \neq 0$. Then the eigenspace*

$$E_\lambda := \ker(\lambda I - K)$$

is finite-dimensional.

Proof. If E_λ were infinite-dimensional, then the restriction of K to E_λ would be

$$K|_{E_\lambda} = \lambda I_{E_\lambda}.$$

Since $K|_{E_\lambda}$ is compact and $\lambda \neq 0$, it would follow that the identity operator I_{E_λ} is compact. This is impossible on an infinite-dimensional normed space. Therefore E_λ must be finite-dimensional. □

Theorem 7.5 (Schauder theorem)**.** *Let X and Y be Banach spaces and let $T \in \mathcal{B}(X, Y)$. Then T is compact if and only if $T^* : Y^* \to X^*$ is compact.*

Proof. First assume that T is compact, and let (g_n) be a bounded sequence in Y^*. Set

$$K := \overline{T(B_X)} \subset Y,$$

which is compact. The restrictions $g_n|_K$ form an equibounded and equicontinuous family in $C(K)$, because

$$|g_n(y) - g_n(z)| \le \|g_n\| \, \|y - z\| \qquad (y, z \in K).$$

By the Arzelà–Ascoli theorem, there exists a subsequence (g_{n_j}) such that $g_{n_j}|_K$ converges uniformly on K. Hence, for every j, k and every $x \in B_X$,

$$|(T^* g_{n_j} - T^* g_{n_k})(x)| = |g_{n_j}(Tx) - g_{n_k}(Tx)| \le \sup_{y \in K} |g_{n_j}(y) - g_{n_k}(y)|.$$

Taking the supremum over $x \in B_X$ shows that $(T^* g_{n_j})$ is Cauchy in X^*, hence convergent because X^* is Banach. Thus every bounded sequence in Y^* has a norm-convergent image subsequence under T^*, so T^* is compact.

Conversely, assume that T^* is compact. Then $T^{**} : X^{**} \to Y^{**}$ is compact by the implication already proved, applied to T^*. Let $J_X : X \to X^{**}$ and $J_Y : Y \to Y^{**}$ be the canonical embeddings. Since

$$T^{**} J_X = J_Y T,$$

and $J_X(B_X)$ is bounded, the set $J_Y(T(B_X)) = T^{**}(J_X(B_X))$ is relatively compact in Y^{**}. Because J_Y is an isometric embedding, this implies that $T(B_X)$ is relatively compact in Y. Hence T is compact. □

Theorem 7.6 (Riesz–Schauder theorem)**.** *Let $K \in \mathcal{K}(X)$, where X is an infinite-dimensional complex Banach space. Then the nonzero spectrum of K is at most countable, every nonzero spectral value is an isolated eigenvalue, each corresponding eigenspace is finite-dimensional, and 0 is the only possible accumulation point of $\sigma(K)$.*

Proof sketch. By theorem 7.3, every $\lambda \in \sigma(K) \setminus \{0\}$ is an eigenvalue, and by theorem 7.4 its eigenspace is finite-dimensional.

Fix $\lambda_0 \in \sigma(K) \setminus \{0\}$ and write

$$A_0 := \lambda_0 I - K.$$

Then $\ker A_0$ is finite-dimensional and, by the compact Fredholm alternative, A_0 has closed range. Choose a closed complement

$$X = \ker A_0 \oplus Z.$$

The restriction $A_0|_Z : Z \to A_0(Z)$ is bijective and bounded below, hence has a bounded inverse onto its range. A standard finite-dimensional reduction of

$$A_\lambda := \lambda I - K = A_0 + (\lambda - \lambda_0)I$$

now shows that, for λ near λ_0, invertibility of A_λ is equivalent to invertibility of a linear operator acting on the finite-dimensional space $\ker A_0$. Consequently, the nonzero spectral points near λ_0 are exactly the zeros of a finite-dimensional determinant, so they form a discrete set. Hence λ_0 is isolated.

It follows that every nonzero spectral value is isolated, and the compactness of $\sigma(K)$ implies that 0 is the only possible accumulation point. Finally, for each $m \in \mathbb{N}$ the set

$$\{\lambda \in \sigma(K) : |\lambda| \geq 1/m\}$$

is finite, because it is a compact discrete subset of $\mathbb{C}$. Therefore

$$\sigma(K) \setminus \{0\} = \bigcup_{m=1}^{\infty} \{\lambda \in \sigma(K) : |\lambda| \geq 1/m\}$$

is at most countable. $\square$

Chapter 8

Hilbert Space Theory

This chapter develops the distinctive geometric structure of Hilbert spaces, where the presence of an inner product makes orthogonality, projection, and expansion theory available in a precise and powerful form. After establishing the basic theory of orthonormal systems, Parseval's identity, and the Gram–Schmidt process, the chapter turns to bounded operators on Hilbert spaces and to the special role of adjoints. The principal operator classes—self-adjoint, unitary, and normal operators—are introduced in a manner that highlights their geometric meaning and their first spectral consequences. The chapter thus prepares the ground for the compact self-adjoint spectral theorem proved next.

8.1 Inner product spaces, orthogonality, and projections

Hilbert spaces refine Banach-space theory by supplying a notion of angle and orthogonality. This extra geometric structure leads to projection theorems, best approximation results, and a far more rigid operator theory than is available in general normed spaces.

Hilbert spaces are the setting in which one can speak of angles, orthogonality, and "best approximation" by closed subspaces in complete generality. The theory begins with the notion of an inner product.

Definition 8.1 (Inner product)**.** Let X be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ is an *inner product* if, for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$,

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry);

(iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ (linearity in the first variable).

The pair $(X, \langle \cdot, \cdot \rangle)$ is called an *inner product space.*

Definition 8.2 (Induced norm and Hilbert space)**.** On an inner product space define

$$\|x\| := \langle x, x \rangle^{1/2}, \qquad x \in X.$$

A *Hilbert space* is an inner product space that is complete with respect to the metric induced by this norm.

Proposition 8.3 (Cauchy–Schwarz and triangle inequality)**.** *Let* $(X, \langle\cdot,\cdot\rangle)$ *be an inner product space with induced norm* $\|\cdot\|$*. Then for all* $x, y \in X$*,*

$$|\langle x, y\rangle| \le \|x\|\,\|y\| \qquad \textit{(Cauchy–Schwarz)},$$

and consequently $\|x+y\| \le \|x\| + \|y\|$*. In particular,* $\|\cdot\|$ *is a norm on* X*.*

Proof. If $y = 0$, the claim is immediate. For $y \neq 0$ and $t \in \mathbb{F}$ we have $0 \le \|x - ty\|^2 = \langle x - ty, x - ty\rangle$, hence

$$0 \le \|x\|^2 - t\,\langle y, x\rangle - \bar{t}\,\langle x, y\rangle + |t|^2\|y\|^2.$$

Choosing $t = \langle x, y\rangle / \|y\|^2$ gives $0 \le \|x\|^2 - |\langle x, y\rangle|^2/\|y\|^2$, i.e. $|\langle x, y\rangle| \le \|x\|\,\|y\|$. Then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\Re\langle x, y\rangle \le (\|x\| + \|y\|)^2,$$

which yields the triangle inequality. □

Proposition 8.4 (Parallelogram law and polarization)**.** *In every inner product space,*

$$\|x+y\|^2 + \|x-y\|^2 = 2\Big(\|x\|^2 + \|y\|^2\Big) \qquad (x, y \in X). \tag{8.1}$$

Conversely, if a normed space $(X, \|\cdot\|)$ *satisfies* (8.1)*, then the inner product is recovered from the norm by the polarization identity:*

$$\langle x, y\rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) \quad (\mathbb{F} = \mathbb{R}),$$

and

$$\langle x, y\rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2 + i\Big(\|x+iy\|^2 - \|x-iy\|^2\Big)\Big) \quad (\mathbb{F} = \mathbb{C}).$$

Proof. If $\|\cdot\|$ is induced by an inner product, then expanding $\langle x \pm y, x \pm y\rangle$ yields (8.1). Conversely, assume that (8.1) holds. Define $\langle x, y\rangle$ by the corresponding polarization identity. A direct verification based on (8.1) shows that $\langle\cdot,\cdot\rangle$ is sesquilinear, conjugate symmetric, and positive definite, and that

$$\|x\|^2 = \langle x, x\rangle.$$

Hence the given norm is induced by this inner product. □

Definition 8.5 (Orthogonality and orthogonal complement)**.** Let X be an inner product space. Vectors $x, y \in X$ are *orthogonal*, written $x \perp y$, if $\langle x, y\rangle = 0$. For a subset $S \subset X$ define

$$S^{\perp} := \{x \in X : \langle x, s\rangle = 0 \text{ for all } s \in S\}.$$

If M is a linear subspace, then $M^{\perp}$ is a linear subspace, and it is always closed (in the norm topology).

Proposition 8.6. *Let X be an inner product space and $M \subset X$ a linear subspace. Then:*

(a) $M \subset (M^{\perp})^{\perp}$.

(b) If X is complete (i.e. a Hilbert space), then $(M^{\perp})^{\perp} = \overline{M}$.

Proof. (a) If $m \in M$ and $x \in M^{\perp}$, then $\langle x, m\rangle = 0$, hence $\langle m, x\rangle = 0$ by conjugate symmetry; thus $m \in (M^{\perp})^{\perp}$. (b) The inclusion $\overline{M} \subset (M^{\perp})^{\perp}$ follows from continuity of $x \mapsto \langle x, y\rangle$ for fixed y. For the reverse inclusion, let $x \in (M^{\perp})^{\perp}$ and consider the closed subspace $\overline{M}$. By Theorem 8.7 below, there is a decomposition $x = u + v$ with $u \in \overline{M}$ and $v \in \overline{M}^{\perp} = M^{\perp}$. But $x \in (M^{\perp})^{\perp}$ implies $0 = \langle x, v\rangle = \langle u, v\rangle + \langle v, v\rangle = \|v\|^2$, hence $v = 0$ and $x = u \in \overline{M}$. □

Theorem 8.7 (Orthogonal decomposition and the projection theorem)**.** *Let H be a Hilbert space and let $M \subset H$ be a closed linear subspace. Then for every $x \in H$ there exist unique elements $u \in M$ and $v \in M^{\perp}$ such that*

$$x = u + v.$$

Equivalently, $H = M \oplus M^{\perp}$. Moreover:

(a) The element u is the unique minimizer of $y \mapsto \|x - y\|$ over $y \in M$, and $\operatorname{dist}(x, M) = \|x - u\|$.

(b) The map $P_M : H \to M$ given by $P_M x := u$ is a bounded linear projection (the orthogonal projection *onto M). If $M \neq \{0\}$, then $\|P_M\| = 1$; also $P_M^2 = P_M$ and $P_M^* = P_M$.*

(c) Writing $Q_M := I - P_M$, one has $Q_M(H) \subset M^{\perp}$, $Q_M^2 = Q_M$, and $\|Q_M\| = 1$ if $M \neq H$.

Proof. Fix $x \in H$ and set $d := \inf_{y \in M} \|x - y\|$. Choose $(y_n) \subset M$ with $\|x - y_n\| \to d$. By the parallelogram law (8.1) and the fact that $(y_n + y_m)/2 \in M$, one obtains

$$\|y_n - y_m\|^2 \leq 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4d^2 \longrightarrow 0 \quad (n, m \to \infty),$$

so (y_n) is Cauchy. Since M is closed in H, there exists $u \in M$ with $y_n \to u$. Then $\|x - u\| = d$ by continuity, so u is a best approximation to x from M.

Let $v := x - u$. We now show $v \in M^{\perp}$. For any $m \in M$ and $t \in \mathbb{R}$, the point $u + tm \in M$ and minimality of u gives

$$\|v\|^2 = \|x - u\|^2 \le \|x - (u + tm)\|^2 = \|v - tm\|^2 = \|v\|^2 - 2t\,\Re\langle v, m\rangle + t^2\|m\|^2.$$

Thus $\Re\langle v, m\rangle = 0$ for all $m \in M$. Replacing m by im (in the complex case) yields also $\Im\langle v, m\rangle = 0$, hence $\langle v, m\rangle = 0$ and $v \in M^{\perp}$. Therefore $x = u + v$ with $u \in M$ and $v \in M^{\perp}$.

Uniqueness follows: if $x = u_1 + v_1 = u_2 + v_2$ with $u_j \in M$ and $v_j \in M^{\perp}$, then $u_1 - u_2 = v_2 - v_1 \in M \cap M^{\perp} = \{0\}$, so $u_1 = u_2$ and $v_1 = v_2$.

Define $P_M x := u$. Linearity follows from uniqueness of the decomposition applied to $x_1 + x_2$ and αx. Moreover, since $x = P_M x + Q_M x$ with orthogonality $\langle P_M x, Q_M x\rangle = 0$, we have

$$\|P_M x\| \le \|P_M x + Q_M x\| = \|x\|,$$

so $\|P_M\| \le 1$. If $M \ne \{0\}$ and $0 \ne m \in M$, then $P_M m = m$, hence $\|P_M\| \ge 1$ and $\|P_M\| = 1$. The identities $P_M^2 = P_M$ and $Q_M^2 = Q_M$ are immediate; self-adjointness $P_M^* = P_M$ follows from $\langle x - P_M x, m\rangle = 0$ for all $m \in M$. □

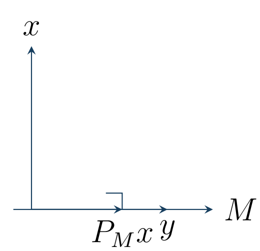

Figure 8.1: Orthogonal projection: $P_M x \in M$ is the unique point of M closest to x.

8.2 Riesz representation theorem

One of the decisive advantages of Hilbert spaces is that every continuous linear functional is represented by an inner product against a unique vector. The Riesz representation theorem converts duality into geometry and is the mechanism behind adjoints, orthogonal projections, and many variational arguments.

The next theorem identifies the continuous dual of a Hilbert space with the Hilbert space itself.

Theorem 8.8 (Riesz representation)**.** *Let H be a Hilbert space. For each $y \in H$ define $f_y : H \to \mathbb{F}$ by*

$$f_y(x) = \langle x, y \rangle, \qquad x \in H,$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then $f_y \in H^*$ and $\|f_y\| = \|y\|$. Moreover, for every $f \in H^*$ there exists a unique $y \in H$ such that $f = f_y$.

Proof. For fixed $y \in H$, the Cauchy–Schwarz inequality gives

$$|f_y(x)| = |\langle x, y \rangle| \leq \|x\| \, \|y\|,$$

so f_y is continuous and $\|f_y\| \leq \|y\|$. If $y \neq 0$, then taking $x = y/\|y\|$ yields $|f_y(x)| = \|y\|$, hence $\|f_y\| = \|y\|$.

Conversely, let $0 \neq f \in H^*$ and set $M = \ker f$, a closed subspace of H. Choose $x_0 \in H$ with $f(x_0) = 1$ and write the orthogonal decomposition

$$x_0 = m_0 + z, \qquad m_0 \in M, \; z \in M^\perp.$$

Then $f(z) = f(x_0 - m_0) = 1$. For arbitrary $x \in H$ we have $x - f(x)z \in \ker f = M$, hence $x - f(x)z \perp z$. Taking inner products with z gives

$$0 = \langle x - f(x)z, \, z \rangle = \langle x, z \rangle - f(x)\langle z, z \rangle,$$

so

$$f(x) = \frac{\langle x, z \rangle}{\|z\|^2} = \langle x, \, y \rangle \quad \text{with} \quad y = \frac{z}{\|z\|^2}.$$

Uniqueness follows because $\langle x, y_1 - y_2 \rangle = 0$ for all x implies $y_1 = y_2$. □

Corollary 8.9. *If M is a closed subspace of H, then*

$$M = (M^\perp)^\perp = \bigcap_{y \in M^\perp} \ker f_y.$$

In particular, M is the kernel of a single *continuous linear functional if and only if* codim $M = 1$.

Proof. If $x \in M$, then $\langle x, y \rangle = 0$ for every $y \in M^\perp$, and therefore $x \in (M^\perp)^\perp$. Thus $M \subseteq (M^\perp)^\perp$. Conversely, let $x \in (M^\perp)^\perp$. Since $H = M \oplus M^\perp$, we may write $x = m + n$ with $m \in M$ and $n \in M^\perp$. Because $x \perp M^\perp$, we have

$$0 = \langle x, n \rangle = \langle m, n \rangle + \langle n, n \rangle = \|n\|^2,$$

so $n = 0$ and hence $x = m \in M$. This proves that $(M^\perp)^\perp = M$. The identity

$$M = \bigcap_{y \in M^\perp} \ker\Big(x \mapsto \langle x, y\rangle\Big)$$

is an equivalent reformulation of the condition $x \perp M^\perp$. $\square$

Definition 8.10 (Orthonormal sets and orthonormal bases)**.** A subset $E \subset H$ is called *orthonormal* if every $e \in E$ satisfies $\|e\| = 1$ and

$$\langle e, f\rangle = 0 \qquad (e, f \in E,\ e \neq f).$$

An orthonormal set E is called an *orthonormal basis* of H if

$$\overline{\operatorname{span} E} = H.$$

Equivalently, E is an orthonormal basis if and only if $x \perp E$ implies $x = 0$.

Proposition 8.11 (Existence of orthonormal bases)**.** *Every Hilbert space admits an orthonormal basis.*

Proof. Let $\mathcal{F}$ be the family of all orthonormal subsets of H, ordered by inclusion. If $\mathcal{C} \subset \mathcal{F}$ is a chain, then $\bigcup_{E \in \mathcal{C}} E$ is again orthonormal, so it is an upper bound for $\mathcal{C}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element, say E_0.

We now show that E_0 is complete in the sense that its linear span is dense in H. If not, then there exists $0 \neq x \in H$ with $x \perp E_0$. After normalization, the set

$$E_0 \cup \left\{ \frac{x}{\|x\|} \right\}$$

is orthonormal, contradicting maximality. Hence $\overline{\operatorname{span} E_0} = H$, so E_0 is an orthonormal basis. $\square$

Lemma 8.12 (Finite orthogonal expansion and finite Bessel inequality)**.** *Let* $\{e_1, \ldots, e_n\}$ *be an orthonormal set in* H*, and define*

$$S_n x := \sum_{j=1}^{n} \langle x, e_j\rangle e_j.$$

Then, for every $x \in H$*,*

$$x - S_n x \perp e_k \qquad (k = 1, \ldots, n), \tag{8.2}$$

$$\|S_n x\|^2 = \sum_{j=1}^{n} |\langle x, e_j\rangle|^2, \tag{8.3}$$

and

$$\|x - S_n x\|^2 = \|x\|^2 - \sum_{j=1}^{n} |\langle x, e_j\rangle|^2. \tag{8.4}$$

In particular,

$$\sum_{j=1}^{n} |\langle x, e_j\rangle|^2 \le \|x\|^2.$$

Proof. For each $k \in \{1, \dots, n\}$,

$$\langle x - S_n x, e_k\rangle = \langle x, e_k\rangle - \sum_{j=1}^{n} \langle x, e_j\rangle\langle e_j, e_k\rangle = \langle x, e_k\rangle - \langle x, e_k\rangle = 0,$$

which proves (8.2). Next,

$$\|S_n x\|^2 = \langle S_n x, S_n x\rangle = \sum_{j,k=1}^{n} \langle x, e_j\rangle\overline{\langle x, e_k\rangle}\langle e_j, e_k\rangle = \sum_{j=1}^{n} |\langle x, e_j\rangle|^2,$$

so (8.3) holds. Finally, the orthogonality of $x - S_n x$ and $S_n x$ gives the Pythagorean identity

$$\|x\|^2 = \|x - S_n x\|^2 + \|S_n x\|^2,$$

which, together with (8.3), yields (8.4). □

Lemma 8.13. *Let E be an orthonormal set in a Hilbert space H. Then for each $x \in H$, the set*

$$E_x := \{e \in E : \langle x, e\rangle \neq 0\}$$

is countable.

Proof. For each $m \in \mathbb{N}$ let

$$E_m := \left\{ e \in E : |\langle x, e\rangle|^2 > \frac{1}{m} \right\}.$$

If $e_1, \dots, e_r \in E_m$ are distinct, then Lemma 8.12 gives

$$\frac{r}{m} < \sum_{j=1}^{r} |\langle x, e_j\rangle|^2 \le \|x\|^2,$$

so $r < m\|x\|^2$. Hence each E_m is finite. Since every $e \in E_x$ satisfies $|\langle x, e\rangle|^2 > 1/m$ for some m, we obtain

$$E_x = \bigcup_{m=1}^{\infty} E_m,$$

which is a countable union of finite sets. □

Theorem 8.14 (Bessel inequality). *Let $E = \{e_\alpha\}_{\alpha\in I}$ be an orthonormal set in H. Then for every $x \in H$,*

$$\sum_{\alpha\in I} |\langle x, e_\alpha\rangle|^2 \le \|x\|^2. \tag{8.5}$$

The sum is well defined because, by Lemma 8.13, at most countably many terms are nonzero.

Proof. Enumerate the countable set $E_x = \{e \in E : \langle x, e\rangle \neq 0\}$ as $\{e_n\}_{n\geq 1}$. For each $k \in \mathbb{N}$, Lemma 8.12 yields

$$\sum_{n=1}^{k} |\langle x, e_n\rangle|^2 \le \|x\|^2.$$

Letting $k \to \infty$ proves (8.5). □

8.3 Parseval's identity

An orthonormal basis allows a Hilbert-space vector to be analyzed through its scalar coordinates, just as in Euclidean space. Parseval's identity shows that this coordinate expansion preserves norm exactly, thereby providing the correct infinite-dimensional analogue of the Pythagorean theorem.

The norm identity associated with an orthonormal basis is the following.

Theorem 8.15 (Parseval's identity). *Let $E = \{e_\alpha\}_{\alpha\in I}$ be an orthonormal basis of a Hilbert space H. Then for every $x \in H$,*

$$x = \sum_{\alpha\in I} \langle x, e_\alpha\rangle e_\alpha, \tag{8.6}$$

and

$$\|x\|^2 = \sum_{\alpha\in I} |\langle x, e_\alpha\rangle|^2. \tag{8.7}$$

Again, only countably many terms in these expressions are nonzero.

Proof. Fix $x \in H$ and enumerate the countable set $E_x = \{e_\alpha : \langle x, e_\alpha \rangle \neq 0\}$ as $\{e_n\}_{n \geq 1}$. Define

$$y_n := \sum_{j=1}^{n} \langle x, e_j \rangle e_j.$$

If $m > n$, then orthonormality gives

$$\|y_m - y_n\|^2 = \sum_{j=n+1}^{m} |\langle x, e_j \rangle|^2,$$

which tends to 0 as $m, n \to \infty$ by Bessel's inequality. Hence (y_n) is a Cauchy sequence, so $y_n \to y \in H$.

For every $\alpha \in I$, continuity of the inner product gives

$$\langle y, e_\alpha \rangle = \lim_{n \to \infty} \langle y_n, e_\alpha \rangle = \langle x, e_\alpha \rangle.$$

Therefore $\langle x - y, e_\alpha \rangle = 0$ for all $\alpha \in I$. Since E is complete, this forces $x - y = 0$, and (8.6) follows. Finally,

$$\|x\|^2 = \lim_{n \to \infty} \|y_n\|^2 = \lim_{n \to \infty} \sum_{j=1}^{n} |\langle x, e_j \rangle|^2 = \sum_{\alpha \in I} |\langle x, e_\alpha \rangle|^2,$$

which is (8.7). □

8.4 Gram–Schmidt process

Orthogonality is most useful when one can produce orthonormal systems from arbitrary linearly independent families. The Gram–Schmidt process is the basic constructive device that achieves this and thereby connects algebraic bases with the geometric apparatus of Hilbert spaces.

The constructive orthonormalization procedure may be stated succinctly as follows.

Theorem 8.16 (Gram–Schmidt orthonormalization)**.** *Let $x_1, \ldots, x_n$ be linearly independent vectors in a Hilbert space H. Define*

$$\nu_1 := x_1, \qquad \nu_k := x_k - \sum_{j=1}^{k-1} \langle x_k, g_j \rangle g_j \quad (k \geq 2),$$

and set

$$g_k := \frac{\nu_k}{\|\nu_k\|} \qquad (k = 1, \ldots, n).$$

Then $\{g_1, \dots, g_n\}$ is an orthonormal set and, for every $k \le n$,

$$\operatorname{span}\{x_1, \dots, x_k\} = \operatorname{span}\{g_1, \dots, g_k\}.$$

Proof. We argue by induction on k. Since $x_1 \neq 0$, the vector $g_1 = x_1/\|x_1\|$ is well defined and has norm one. Assume that $g_1, \dots, g_{k-1}$ have already been constructed and are orthonormal. Because $x_k \notin \operatorname{span}\{x_1, \dots, x_{k-1}\} = \operatorname{span}\{g_1, \dots, g_{k-1}\}$, the vector ν_k is nonzero. By construction,

$$\langle \nu_k, g_j \rangle = \langle x_k, g_j \rangle - \sum_{i=1}^{k-1} \langle x_k, g_i \rangle \langle g_i, g_j \rangle = 0 \qquad (j = 1, \dots, k-1),$$

so $g_k = \nu_k/\|\nu_k\|$ is orthogonal to $g_1, \dots, g_{k-1}$ and has norm one. Moreover,

$$x_k = \nu_k + \sum_{j=1}^{k-1} \langle x_k, g_j \rangle g_j \in \operatorname{span}\{g_1, \dots, g_k\},$$

which shows that $\operatorname{span}\{x_1, \dots, x_k\} \subseteq \operatorname{span}\{g_1, \dots, g_k\}$. The reverse inclusion is immediate from the definition of the g_j. $\square$

Example 8.17 (Hermite functions)**.** Applying the Gram–Schmidt process to the sequence $1, t, t^2, \dots$ in the Gaussian-weighted setting leads to the Hermite functions

$$h_n(t) := c_n H_n(t) e^{-t^2/2}, \qquad n \ge 0,$$

where H_n denotes the nth Hermite polynomial and c_n is the normalizing constant. The family $(h_n)_{n \ge 0}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Sketch of completeness. Suppose $g \in L^2(\mathbb{R})$ is orthogonal to every h_n. Equivalently,

$$\int_{\mathbb{R}} g(t) e^{-t^2/2} t^n \, dt = 0 \qquad (n \ge 0).$$

Define

$$G(z) := \int_{\mathbb{R}} g(t) e^{-t^2/2} e^{itz} \, dt, \qquad z \in \mathbb{C}.$$

Then G is entire, and all derivatives of G at 0 vanish because the moments above are zero. Hence $G \equiv 0$. By uniqueness of the Fourier transform, this implies $g = 0$. Therefore the Hermite functions are complete. $\square$

Theorem 8.18. *Every infinite-dimensional separable Hilbert space is isometrically isomorphic to ℓ^2.*

Proof. Let $\{e_n\}_{n\geq 1}$ be a countable orthonormal basis for H. Define

$$F : H \to \ell^2, \qquad F(x) := (\langle x, e_1\rangle, \langle x, e_2\rangle, \dots).$$

By Bessel's inequality, $F(x) \in \ell^2$ for every $x \in H$, and by Parseval's identity,

$$\|F(x)\|_{\ell^2}^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = \|x\|^2.$$

Thus F is an isometry. To see that F is onto, let $y = (y_n) \in \ell^2$. Then the series $\sum_{n=1}^{\infty} y_n e_n$ converges in H because

$$\left\|\sum_{n=m}^{N} y_n e_n\right\|^2 = \sum_{n=m}^{N} |y_n|^2 \xrightarrow[m,N\to\infty]{} 0.$$

If we set $x := \sum_{n=1}^{\infty} y_n e_n$, then

$$F(x) = (y_1, y_2, \dots) = y.$$

Therefore F is a surjective linear isometry. □

8.5 Adjoint operators and basic spectral constraints

On a Hilbert space the adjoint packages the interaction between an operator and the inner product. This additional symmetry makes it possible to distinguish special classes of operators—such as self-adjoint, unitary, and normal operators—whose spectral behaviour is far more rigid than in the general Banach-space setting.

The adjoint in a Hilbert space is the Hilbert-space counterpart of the transpose map on general normed spaces. Its existence is a direct application of the Riesz representation theorem.

Theorem 8.19 (Existence and uniqueness of the adjoint)**.** *Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. Then there exists a unique operator $T^* \in \mathcal{B}(H)$ such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad (x, y \in H).$$

Moreover, $\|T^*\| = \|T\|$, $(S+T)^* = S^* + T^*$, $(\alpha T)^* = \overline{\alpha} T^*$, and $(ST)^* = T^*S^*$.

Proof. Fix $y \in H$. The map $x \mapsto \langle Tx, y\rangle$ is a bounded linear functional on H, with

$$|\langle Tx, y\rangle| \leq \|T\|\,\|x\|\,\|y\|.$$

By the Riesz representation theorem, there exists a unique vector $z_y \in H$ such that

$$\langle Tx, y\rangle = \langle x, z_y\rangle \qquad (x \in H).$$

Define $T^*y := z_y$. Linearity of T^* follows from the uniqueness clause in Riesz representation, and the bound above gives $\|T^*y\| \leq \|T\|\,\|y\|$. Hence $T^* \in \mathcal{B}(H)$ and $\|T^*\| \leq \|T\|$. Applying the same argument to T^* yields $\|T\| \leq \|T^*\|$. Thus $\|T^*\| = \|T\|$. The remaining identities are immediate from the defining relation. □

Proposition 8.20 (Basic norm identities)**.** *For every $T \in \mathcal{B}(H)$,*

$$\|T\|^2 = \|T^*T\| = \|TT^*\|.$$

If T is normal, then $\|Tx\| = \|T^*x\|$ for every $x \in H$. If U is unitary, then $\|Ux\| = \|x\|$ for every $x \in H$.

Proof. For $x \in H$,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*T\|\,\|x\|^2,$$

so $\|T\|^2 \leq \|T^*T\|$. The reverse inequality follows from submultiplicativity: $\|T^*T\| \leq \|T^*\|\,\|T\| = \|T\|^2$. Replacing T by T^* gives $\|TT^*\| = \|T\|^2$. If T is normal, then

$$\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \|T^*x\|^2.$$

If U is unitary, then $U^*U = I$, so $\|Ux\|^2 = \langle U^*Ux, x\rangle = \|x\|^2$. □

An operator T is *self-adjoint* if $T = T^*$, *unitary* if $T^*T = TT^* = I$, and *normal* if $T^*T = TT^*$.

Proposition 8.21 (Characterizations of the basic classes)**.** *Let $T \in \mathcal{B}(H)$.*

(a) T is self-adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for every $x \in H$.

(b) U is unitary if and only if U is surjective and norm preserving.

(c) If T is normal, then eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. (a) If $T = T^*$, then $\langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$, so the scalar is real. Conversely, assume $\langle Tx, x\rangle \in \mathbb{R}$ for every x. Polarization applied to the sesquilinear form $b(x, y) = \langle Tx, y\rangle - \langle x, Ty\rangle$ shows that $b(x, x) = 0$ for every x forces $b \equiv 0$, hence $T = T^*$.

(b) Every unitary operator is surjective and norm preserving by definition and the previous proposition. Conversely, if U is surjective and $\|Ux\| = \|x\|$ for every x, then the polarization identity shows that $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all x, y. Therefore, for every $y = Uz$,

$$\langle Ux, y\rangle = \langle Ux, Uz\rangle = \langle x, z\rangle,$$

which implies $U^*y = z = U^{-1}y$. Hence $U^*U = UU^* = I$.

(c) Suppose $Tx = \lambda x$ and $Ty = \mu y$ with $\lambda \neq \mu$. Then

$$\lambda\langle x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle.$$

Since $Ty = \mu y$, we have $(T - \mu I)y = 0$. The operator $T - \mu I$ is normal, so by theorem 8.20,

$$0 = \|(T - \mu I)y\| = \|(T^* - \overline{\mu} I)y\|,$$

which gives $T^*y = \overline{\mu}y$. Therefore

$$\lambda\langle x, y\rangle = \langle x, T^*y\rangle = \langle x, \overline{\mu}y\rangle = \mu\langle x, y\rangle.$$

Therefore $(\lambda - \mu)\langle x, y\rangle = 0$, and since $\lambda \neq \mu$ we conclude that $\langle x, y\rangle = 0$. □

Proposition 8.22. *If T is self-adjoint then $\sigma(T) \subset \mathbb{R}$. If U is unitary then $\sigma(U) \subset \{z \in \mathbb{C} : |z| = 1\}$.*

Proof. First assume that $T = T^*$ and let $\lambda = a + ib \in \mathbb{C}$ with $b \neq 0$. For any $x \in H$,

$$\Im\langle(\lambda I - T)x, x\rangle = \Im\Big(\lambda\|x\|^2 - \langle Tx, x\rangle\Big) = b\,\|x\|^2,$$

because $\langle Tx, x\rangle \in \mathbb{R}$ for self-adjoint T. Hence

$$|b|\,\|x\|^2 \leq |\langle(\lambda I - T)x, x\rangle| \leq \|(\lambda I - T)x\|\,\|x\|.$$

Therefore

$$\|(\lambda I - T)x\| \geq |b|\,\|x\| \qquad (x \in H).$$

So $\lambda I - T$ is injective with closed range. If its range were proper, choose $0 \neq y$ orthogonal to the range. Then

$$0 = \langle(\lambda I - T)x, y\rangle = \langle x, (\overline{\lambda} I - T)y\rangle \qquad (x \in H),$$

which implies $(\overline{\lambda}I - T)y = 0$. Applying the inequality above to $\overline{\lambda}$ gives $y = 0$, a contradiction. Thus $\lambda I - T$ is surjective, hence invertible. Therefore $\lambda \in \rho(T)$, and so $\sigma(T) \subset \mathbb{R}$.

Now let U be unitary. If $|\lambda| \neq 1$, then

$$\|Ux - \lambda x\|^2 = \|Ux\|^2 + \|\lambda x\|^2 - 2\Re\langle Ux, \lambda x\rangle = \|x\|^2 + |\lambda|^2\|x\|^2 - 2\Re\Big(\lambda\langle Ux, x\rangle\Big).$$

Since $|\langle Ux, x\rangle| \leq \|Ux\|\,\|x\| = \|x\|^2$, we get

$$\|Ux - \lambda x\|^2 \geq (1 - |\lambda|)^2\|x\|^2.$$

Therefore $U - \lambda I$ is injective with closed range. If its range were proper, choose $0 \neq y$ orthogonal to the range. Then

$$0 = \langle (U - \lambda I)x, y\rangle = \langle x, (U^* - \overline{\lambda}I)y\rangle \qquad (x \in H),$$

so $(U^* - \overline{\lambda}I)y = 0$. But U^* is also unitary, and the same lower bound shows $y = 0$, a contradiction. Therefore $U - \lambda I$ is surjective and hence invertible. Thus $\lambda \in \rho(U)$ whenever $|\lambda| \neq 1$, which proves $\sigma(U) \subset \{z \in \mathbb{C} : |z| = 1\}$. $\square$

Chapter 9

The Spectral Theorem for Compact Self-Adjoint Operators

This chapter culminates the Hilbert-space portion of the course with the spectral theorem for compact self-adjoint operators. The theorem shows that such operators admit a complete orthonormal set of eigenvectors and therefore possess a convergent diagonal representation relative to a Hilbert basis. Beyond its immediate consequences, the result provides the clearest model of how geometry, compactness, and self-adjointness combine to produce an exact spectral decomposition. It is both a definitive theorem in its own right and a prototype for the broader spectral theory developed in more advanced settings.

9.1 Spectral theorem

The spectral theorem is the culmination of the compact self-adjoint theory. It shows that such operators admit an orthonormal basis of eigenvectors and can therefore be diagonalized in a manner closely analogous to Hermitian matrices, but now in an infinite-dimensional Hilbert space.

We can now state the central structural theorem of the chapter.

Theorem 9.1 (Compact self-adjoint spectral theorem)**.** *Let H be a Hilbert space over $\mathbb{C}$ (or $\mathbb{R}$) and let $K \in \mathcal{B}(H)$ be compact and self-adjoint. Then there exist an orthonormal family $(e_n)_{n\geq 1} \subset H$ and real numbers $(\lambda_n)_{n\geq 1}$ such that*

$$Ke_n = \lambda_n e_n \qquad (n \geq 1),$$

$\lambda_n \neq 0$ for all n, and $\lambda_n \to 0$ if the family is infinite. If $M = \overline{\operatorname{span}\{e_n : n \geq 1\}}$, then $K|_{M^\perp} = 0$ and for every $x \in H$,

$$Kx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \, e_n,$$

with convergence in norm. In particular, H admits an orthonormal basis consisting of eigenvectors of K (with eigenvalue 0 on $\ker K$).

Proof. We use two standard facts:

(F1) Weak compactness of the unit ball. The closed unit ball of H is compact in the weak topology.

(F2) Compact operators are weak-to-strong continuous. If K is compact and $x_\alpha \to x$ weakly with $\sup_\alpha \|x_\alpha\| < \infty$, then $Kx_\alpha \to Kx$ in norm.

For completeness, we justify (F2): since $\{Kx_\alpha\}$ is relatively compact, every subnet has a further subnet converging in norm to some y. For any $z \in H$,

$$\langle y, z\rangle = \lim\langle Kx_\alpha, z\rangle = \lim\langle x_\alpha, Kz\rangle = \langle x, Kz\rangle = \langle Kx, z\rangle,$$

using weak convergence of x_α and self-adjointness of K in the middle identity. Hence $y = Kx$. Therefore every subnet of Kx_α has a further subnet converging to Kx, which forces $Kx_\alpha \to Kx$ in norm.

If $K = 0$ there is nothing to prove. Assume $K \neq 0$ and consider the continuous functional

$$\Phi(x) := \langle Kx, x\rangle \qquad (\|x\| = 1).$$

Let $\alpha = \sup\{\Phi(x) : \|x\| = 1\}$. Choose a net (x_β) on the unit sphere with $\Phi(x_\beta) \to \alpha$. By (F1) there is a subnet (still denoted x_β) converging weakly to some $x \in H$ with $\|x\| \leq 1$. By (F2), $Kx_\beta \to Kx$ in norm, and since $x_\beta \to x$ weakly we obtain

$$\Phi(x_\beta) = \langle Kx_\beta, x_\beta\rangle \longrightarrow \langle Kx, x\rangle = \Phi(x).$$

Thus $\Phi(x) = \alpha$. In particular $\alpha \neq 0$ implies $x \neq 0$. Replacing x by $x/\|x\|$ (which does not change Φ because $\Phi(tx) = t^2\Phi(x)$) we may assume $\|x\| = 1$.

Claim. The maximizer x is an eigenvector of K.

Let $v \in H$ with $\langle v, x\rangle = 0$ and consider $f(t) = \Phi\Big(\frac{x+tv}{\|x+tv\|}\Big)$ for real t near 0. Since $t = 0$ is a maximum of f, we have $f'(0) = 0$. A direct computation (using $\langle v, x\rangle = 0$) gives

$$0 = f'(0) = 2\,\Re\langle Kx, v\rangle,$$

and replacing v by iv in the complex case shows $\langle Kx, v\rangle = 0$ for all $v \perp x$. Hence $Kx \in \operatorname{span}\{x\}$, so $Kx = \lambda x$ for some $\lambda \in \mathbb{R}$ (real because K is self-adjoint). Setting $e_1 := x$ and $\lambda_1 := \lambda$, we obtain $Ke_1 = \lambda_1 e_1$.

Now let $H_1 = e_1^\perp$ and consider the restriction $K_1 := K|_{H_1} : H_1 \to H_1$. Then K_1 is compact and self-adjoint. If $K_1 \neq 0$ we repeat the argument to obtain an eigenvector $e_2 \in H_1$ with eigenvalue $\lambda_2 \in \mathbb{R}$. Proceeding inductively yields an orthonormal family (e_n) with $Ke_n = \lambda_n e_n$, where at each step $\lambda_n \neq 0$ and $|\lambda_{n+1}| \leq |\lambda_n|$ (choose at each stage an eigenvalue of maximal modulus on the orthogonal complement; apply the above argument to either K or $-K$ on that complement to realize the extremum).

If the construction stops, we have $K = 0$ on the remaining orthogonal complement and we are done. If it does not stop, compactness forces $\lambda_n \to 0$: otherwise there exists $\varepsilon > 0$ and infinitely many n with $|\lambda_n| \geq \varepsilon$, but then the sequence $Ke_n = \lambda_n e_n$ has no norm-convergent subsequence (since (e_n) is orthonormal), contradicting that $K(B_H)$ is relatively compact.

Let $M = \overline{\operatorname{span}\{e_n : n \geq 1\}}$. By construction, M is K-invariant and K is diagonal on $\operatorname{span}\{e_n\}$. If $x \in M^\perp$ and $Kx \neq 0$, then $K|_{M^\perp}$ is a nonzero compact self-adjoint operator and the above argument would produce an eigenvector orthogonal to all e_n, contradicting maximality of the family. Hence $K|_{M^\perp} = 0$.

Finally, for arbitrary $x \in H$ write $x = x_M + x_\perp$ with $x_M \in M$ and $x_\perp \in M^\perp$. Then $Kx_\perp = 0$ and, for partial sums $S_N x := \sum_{n=1}^N \lambda_n \langle x, e_n \rangle e_n$,

$$\|Kx - S_N x\| = \|\sum_{n>N} \lambda_n \langle x, e_n \rangle e_n\| \leq \sup_{n>N} |\lambda_n| \left(\sum_{n>N} |\langle x, e_n \rangle|^2 \right)^{1/2} \xrightarrow[N\to\infty]{} 0,$$

since $\sup_{n>N} |\lambda_n| \to 0$ and $(\langle x, e_n \rangle) \in \ell^2$ by Bessel's inequality. This establishes the expansion and norm convergence. $\square$

Problem Sets

These problem sets are intended to reinforce the principal concepts, examples, and proof techniques developed in the main chapters. They are arranged with a deliberate progression: early problems test command of definitions and examples, the middle range emphasizes standard theorem-proof mechanisms, and the later problems encourage the reader to synthesize ideas drawn from several chapters at once. Some exercises are routine, some are conceptual, and a number are deliberately phrased so as to train the reader in reusable abstract arguments rather than one-off computations.

Problem Set 1

Focus: norms, completeness, density, quotient spaces, separability, and the classical sequence and function spaces.

1. State whether each assertion is true or false, giving a complete justification in each case.
   (a) $\{x \in \ell^1 : \|x\|_2 \leq 1\}$ is a bounded set in the Banach space $(\ell^1, \|\cdot\|_1)$.
   (b) $\{x \in \ell^1 : \|x\|_1 < 1\}$ is an open set in the normed linear space $(\ell^1, \|\cdot\|_2)$.
   (c) $\{x \in C[0,1] : \|x\|_\infty < 1\}$ is an open set in the normed linear space $(C[0,1], \|\cdot\|_1)$.
   (d) If (x_n) is a sequence in a Banach space X such that $\sum\limits_{n=1}^{\infty} n^2\|x_n\|^2 < \infty$, then the series $\sum\limits_{n=1}^{\infty} x_n$ must be convergent in X.
   (e) Is it possible that the quotient space ℓ^∞/c_0 contains a Schauder basis?
   (f) Is the set $\{x \in \ell^\infty : \|x\|_1 < 1\}$ separable in $(\ell^\infty, \|\cdot\|_\infty)$?
   (g) Suppose (x_n) is a Cauchy sequence in a normed linear space X and $\|x_n\| \to 1$. Does it follow that $\left(\frac{x_n}{\|x_n\|}\right)$ is a Cauchy sequence in X?

2. Determine whether $\|\cdot\|$ defines a norm on $\mathbb{R}^n$, where
   (a) $\|(x_1, \ldots, x_n)\| = \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ ($n \geq 2$ and $0 < p < 1$).
   (b) $\|(x_1, \ldots, x_n)\| = (x_1^2 + \frac{1}{2}x_2^2 + \cdots + \frac{1}{n}x_n^2)^{\frac{1}{2}}$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

3. Determine whether $\|\cdot\|$ defines a norm on $C[0,1]$, where $\|x\| = \min\{\|x\|_\infty, 2\|x\|_1\}$ for all $x \in C[0,1]$.

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function which vanishes at most at one point in $\mathbb{R}^n$ and satisfies $f(\alpha x) = |\alpha| f(x)$, for each $(\alpha, x) \in \mathbb{R} \times \mathbb{R}^n$. Show that f is a norm on $\mathbb{R}^n$.

5. Let X be a normed linear space and let $x \in X$. Show that $\|x\| = \inf \left\{ \frac{1}{|\alpha|} : \alpha \in \mathbb{K} \setminus \{0\}, \ \|\alpha x\| \leq 1 \right\}$.

6. Let M be a proper closed subspace of a normed linear space X. Define a map $f : X \to \mathbb{R}$ by $f(x) = \inf_{m \in M} \|x + m\|$. Show that f is a uniformly continuous function on X.

7. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $1 \leq p < q \leq \infty$, prove that

$$\|x\|_q \leq \|x\|_p \leq n^{\left(\frac{1}{p} - \frac{1}{q}\right)} \|x\|_q.$$

8. Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$ be an enumeration of the rational numbers. Define a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \cap [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Show that (f_n) is a Cauchy sequence in $(\mathcal{R}[0, 1], \| \cdot \|_1)$ but it does not converge to a function in $\mathcal{R}[0, 1]$. Does (f_n) converge in $L^1[0, 1]$?

9. Let $C^1[0, 1]$ denote the space of all continuously differentiable functions on $[0, 1]$. For $f \in C^1[0, 1]$, define $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that the space $(C^1[0, 1], \| \cdot \|)$ is a Banach space.

10. Let $X = \{f \in C^1[0, 1] : \ f(0) = 0\}$. For $f \in X$, define $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$. Prove that $\|f\|_1 \leq 2\|f'\|_\infty$.

11. Let $X = \{f \in C^1[0, 1] : \ f(0) = 0\}$. Then $\|f\| = \left(\int_0^1 |f'|^2 \right)^{\frac{1}{2}}$ defines a norm on $C^1[0, 1]$. Is $(X, \| \cdot \|)$ a Banach space?

12. Suppose $\alpha > 0$. For $f \in L^\infty[0, 1]$, write $\|f\| = \min\{\|f\|_\infty, \alpha \|f\|_1\}$. Then $\| \cdot \|$ is a norm on $L^\infty[0, 1]$ if and only if $\alpha \leq 1$.

13. Show that $X = \{(x_n) \in \ell^1 : \sum_{n=1}^\infty n|x_n| < \infty\}$ is a proper dense subspace of ℓ^1.

14. Show that $L^p[0, 1]$ is a proper dense subspace of $L^1[0, 1]$, whenever $1 < p < \infty$.

15. Let C_0 denote the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(C_0, \ \|\cdot\|_\infty)$ is a Banach space.

16. Let $C_c(\mathbb{R}^n)$ denote the space of all compactly supported continuous functions on $\mathbb{R}^n$.
    (a) Prove that $C_c(\mathbb{R})$ is a proper dense subspace of $L^p(\mathbb{R})$, whenever $1 \le p < \infty$.
    (b) Is $C_c(\mathbb{R})$ dense in $L^\infty(\mathbb{R})$?
    (c) Prove that $C_c(\mathbb{R})$ is a dense subspace of $(C_0, \ \|\cdot\|_\infty)$.
    (d) Prove that $L^1 \cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, whenever $1 < p < \infty$.

17. Let $1 \le p < q < \infty$. Prove that $L^q[0,1]$ is a dense proper subspace of $L^p[0,1]$.

18. Let $S(\mathbb{R})$ be the space of simple functions on $\mathbb{R}$. Prove that $S(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, for $1 \le p < \infty$. Why is $S(\mathbb{R})$ not dense in $L^\infty(\mathbb{R})$?

19. Let (x_n) be a sequence in a normed linear space X which converges to a non-zero vector $x \in X$. Show that $\frac{x_1+\cdots+x_n}{n^\alpha} \to x$ if and only if $\alpha = 1$. If the sequence $x_n \to 0$, prove that $\frac{x_1+\cdots+x_n}{n^\alpha} \to 0$, for all $\alpha \ge 1$.

20. Let M be a subspace of a normed linear space X. Show that M is closed if and only if $\{y \in M : \ \|y\| \le 1\}$ is closed in X.

21. Let $D = \{z \in \mathbb{C} : \ |z| < 1\}$. Let X be the class of all functions f which are analytic on D and continuous on $\bar{D}$. Define $\|f\|_\infty = \sup\{|f(e^{it})| : \ 0 \le t \le 2\pi\}$. Prove that $(X, \ \|\cdot\|_\infty)$ is a Banach space.

22. Let M be a closed subspace of a normed linear space X. Prove that projection $\pi : \ X \to X/M$ defined by $\pi(x) = \tilde{x}$ is a continuous map.

23. Let c be the space of all convergence sequences on $\mathbb{C}$. Prove that the quotient norm on c/c_0 is given by $\|\widetilde{(x_n)}\| = \lim\limits_{n\to\infty} |x_n|$. Deduce further that $c/c_0 \cong \mathbb{C}$.

24. Prove that $L^p(\mathbb{R})$ is separable for $1 \le p < \infty$ but $L^\infty(\mathbb{R})$ is not separable.

25. Let M be a closed subspace of a normed linear space X. Then show that X is separable if and only if M and X/M both are separable.

26. For each $n \in \mathbb{N}$, let $x_n = 1 + \frac{(-1)^n}{n}$. Determine all p with $1 \le p \le \infty$ for which $(x_n) \in \ell^p$.

27. Let $1 \le p < q \le \infty$. Show that
    (a) $\ell^p \subsetneq c_0$

(b) $\ell^p \subsetneq \ell^q$

(c) $\|x\|_q \leq \|x\|_p$ for all $x \in \ell^p$

28. Can you find an element in c_0 which does not belong to ℓ^p for any $1 \leq p < \infty$?

29. Prove the following:
    (a) If $x \in \mathbb{R}^n$, then $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.
    (b) If $x \in \ell^q$ for some $1 \leq q < \infty$, then $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.
    (c) If $x \in C[a,b]$, then $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

30. Prove that c_{00} is dense in $(c_0, \|\cdot\|_\infty)$ and also in $(\ell^p, \|\cdot\|_p)$ for $1 \leq p < \infty$.

31. Show that $\left\{(x_n) \in c_{00} : \sum\limits_{n=1}^{\infty} x_n = 0\right\}$ is dense in $(c_0, \|\cdot\|_\infty)$.

32. Show that $\{(x_n) \in \ell^2 : |x_n| \leq \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$ is a compact convex subset of ℓ^2 with empty interior.

33. Let X be a metric space. Prove that the normed linear space $(C_b(X), \|\cdot\|_\infty)$ is finite dimensional if and only if X is finite.

34. Let X be a nonzero normed linear space, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_\varepsilon[x] = B_\delta[y]$, then show that $x = y$ and $\varepsilon = \delta$. Does the result remain true if X is assumed to be only a metric space?

35. Let Y be a subspace of a normed linear space X. Show that
    (a) $Y^0 \neq \emptyset$ if and only if $Y = X$.
    (b) Y is nowhere dense in X if and only if Y is not dense in X.

36. Let X be a nonzero normed linear space. Show that $\{x \in X : 1 < \|x\| \leq 2\}$ is neither an open set nor a closed set in X.

37. Let A be a subset of a normed linear space. Show that $\overline{A} = \bigcap\limits_{r>0} (A + B_r(0))$.

38. Let A be a nonempty subset of a nonzero normed linear space X. Show that $\overline{A + B_X} = \{x \in X : d(x, A) \leq 1\}$.

39. Show that $\|(\alpha_n) + c_0\| = \limsup\limits_{n\to\infty} |\alpha_n|$ for each $(\alpha_n) \in \ell^\infty$.

40. If K is a compact set in the Banach space $(\ell^\infty, \|\cdot\|_\infty)$, then show that $\{x + c_0 : x \in K\}$ is a compact set in the Banach space ℓ^∞/c_0 with the quotient norm.

41. Show that a normed linear space X is separable if and only if S_X is separable.

42. Let $(X, \|\cdot\|)$ be an infinite-dimensional separable Banach space. Show that there is a norm $\|\cdot\|_0$ on X such that $(X, \|\cdot\|_0)$ is nonseparable.

43. Let (x_n) be a sequence in a Banach space X. Which of the following conditions ensure(s) that (x_n) is convergent in X?
    (a) $\|x_n - x_{n+1}\| \to 0$.
    (b) $\sum\limits_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty$.

44. Let Y be a subspace of a normed linear space X. If for every sequence (y_n) in Y and for every $x \in X$ satisfying $\sum\limits_{n=1}^{\infty} y_n = x$, we have $x \in Y$, then show that Y is closed in X.

45. If a normed linear space X has a complete subspace Y such that X/Y is complete, then show that X is a Banach space.

46. If $1 \le p < q \le \infty$, then examine whether $(\ell^p, \|\cdot\|_q)$ is a Banach space.

47. Show that $(c_0, \|\cdot\|)$ is not a Banach space, where $\|(x_n)\| = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}|x_n|$ for all $(x_n) \in c_0$.

48. Consider the normed linear space $(\ell^\infty, \|\cdot\|)$, where $\|(x_n)\| = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}|x_n|$ for all $(x_n) \in \ell^\infty$. Examine whether $(\ell^\infty, \|\cdot\|)$ is a Banach space.

49. If $\|(x_n)\| = \sup\limits_{n\in\mathbb{N}} \left| \sum\limits_{i=1}^{n} x_i \right|$ for all $(x_n) \in \ell^1$, then show that $\|\cdot\|$ is a norm on ℓ^1.

    Also, examine whether the normed linear space $(\ell^1, \|\cdot\|)$ is a Banach space.

50. Let $X = \left\{ x \in C[0,1] : \int\limits_0^1 x(t)\,dt = 0 \right\}$. Prove that $(X, \|\cdot\|_\infty)$ is an infinite-dimensional Banach space.

51. Consider the normed linear space $(C[0,1], \|\cdot\|)$, where $\|x\| = \sup\{t|x(t)| : t \in [0,1]\}$. Show that $(C[0,1], \|\cdot\|)$ is not a Banach space.

52. Let $(X, \|\cdot\|)$ be a normed linear space and let p be a seminorm on X. Show that $p : (X, \|\cdot\|) \to \mathbb{R}$ is continuous if and only if there exists $\alpha > 0$ such that $p(x) \leq \alpha\|x\|$ for all $x \in X$.

53. Let $(X, \|\cdot\|)$ be a normed linear space and let p be a seminorm on X. Show that $p : (X, \|\cdot\|) \to \mathbb{R}$ is continuous if and only if $\{x \in X : p(x) = 1\}$ is closed in $(X, \|\cdot\|)$.

Problem Set 2

Focus: bounded operators, Banach-space structure, Baire-category methods, and the three fundamental principles.

1. State whether each assertion is true or false, giving a complete justification in each case.

   (a) Let $\mathbb{P}(\mathbb{R})$ be the space of all polynomials with coefficients in $\mathbb{R}$. Does there exist a norm $\|\cdot\|$ on $\mathbb{P}(\mathbb{R})$ such that $(\mathbb{P}(\mathbb{R}), \|\cdot\|)$ is a Banach space?

   (b) Let $C_c(\mathbb{R})$ denote the class of all compactly supported continuous functions on $\mathbb{R}$. Is the linear functional given by $T(f) = \int_{-\infty}^{\infty} f(t)\,dt$ continuous in $(C_c(\mathbb{R}), \|\cdot\|_\infty)$? Is T continuous in $(C_c(\mathbb{R}), \|\cdot\|_1)$?

   (c) Let X and Y be normed linear spaces. If $T : X \to Y$ is a linear map such that $\inf\{\|Tx\| : \|x\| = 1\} > 0$, does it follow that T is injective?

   (d) Let X and Y be normed linear spaces. Suppose $T \in B(X, Y)$ is open and injective. Does it follow that T is invertible?

   (e) Is the identity map $I : (\ell^1, \|\cdot\|_1) \to (\ell^1, \|\cdot\|_\infty)$ a closed map?

2. Show that every infinite-dimensional separable normed linear space contains a countable linearly independent dense subset.

3. Let X be a separable Banach space. Prove that there exists a closed subspace M of ℓ^1 such that X is isomorphic to ℓ^1/M.

4. Prove that an infinite-dimensional Banach space X cannot be expressed as the countable union of compact subsets of X.

5. Let $f \in C^\infty(\mathbb{R})$ be such that for each $t \in \mathbb{R}$, there exists $n_t \in \mathbb{N}$ satisfying $f^{(n_t)}(t) = 0$. Show that there exist an interval (a, b) and polynomial $p(x)$ such that $f(t) = p(t)$ for all $t \in (a, b)$.

6. Let X, Y be normed linear spaces and let $T : X \to Y$ and $S : X \to Y$ be linear maps. Let $x_0 \in X$ and $r > 0$ such that $Tx = Sx$ for all $x \in B_r(x_0)$. Show that $T = S$.

7. Let X, Y be normed linear spaces and let $T : X \to Y$ be linear such that $\{Tx : x \in X, \|x\| < 1\}$ is an open set in Y. If G is an open set in X, then show that $T(G)$ is an open set in Y.

8. Let X and Y be normed linear spaces, and let $T : X \to Y$ be a linear map. Show that the following statements are equivalent.
   (a) T is bounded.
   (b) For every Cauchy sequence (x_n) in X, (Tx_n) is a Cauchy sequence in Y.
   (c) For every nonempty bounded open subset G of X, $T(G)$ is a bounded subset of Y.

9. Let X, Y be normed linear spaces and let $T : X \to Y$ be a linear map. If $T(G)$ is bounded in Y for some nonempty open subset G of X, show that T is continuous.

10. Let X, Y be normed linear spaces and let $T : X \to Y$ be a linear map such that $\{x \in X : \|Tx\| < 1\}^0 \neq \emptyset$. Show that T is continuous.

11. Let X, Y be normed linear spaces and let $T : X \to Y$ be linear. If $T(K)$ is a bounded set in Y for every compact set K in X, then show that T is continuous.

12. Let $T_n : \ell^1 \to \ell^1$ be a sequence of linear transformations such that for each $x = (x_n) \in \ell^1$, we have $T_n(x_1, x_2, \ldots) = (x_{n+1}, x_{n+2}, \ldots)$. Show that $\|T_n(x)\| \to 0$ but $\|T_n\| = 1$.

13. For $f \in C^1[0, 1]$, define its norm by $\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Show that the linear map $T : (C^1[0, 1], \|.\|) \to (C[0, 1], \|.\|_\infty)$ given by $T(f) = f'$ is continuous and $\|T\| = 1$.

14. Let X, Y be normed linear spaces and let $T : X \to Y$ be a linear map. If for every absolutely convergent series $\sum\limits_{n=1}^{\infty} x_n$ in X, $\sum\limits_{n=1}^{\infty} Tx_n$ is a convergent series in Y, then show that T is bounded.

15. For $1 \leq p \leq \infty$, define a linear map on $\ell^p(\mathbb{N})$ by $T(x) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$. Find $\|T\|$.

16. Let $X = (C[0,1], \|\cdot\|_\infty)$. For $f \in C[0,1]$ define $K(f)(t) = \int\limits_0^t f(s)ds$. Show that
    (a) K is one-to-one but not onto.
    (b) For each $n \in \mathbb{N}$, the power of operator K satisfies $\|K^n\| = \frac{1}{n!}$.
    (c) Operator $T = I + K$ is invertible.

17. Let $X = (C[0,\pi], \|\cdot\|_\infty)$. For $f \in C[0,\pi]$ define $T(f)(x) = \int_0^x \sin(x+y)f(y)dy$. Find $\|T\|$.

18. Let $T : (L^1[0,1], \|\cdot\|_1) \to (C[0,1], \|\cdot\|_\infty)$ be the linear transformation defined by $T(f)(x) = \int\limits_0^x e^{-t^2} f(t)dt$. Show that T satisfies the condition $\frac{1}{3} \leq \|T\| \leq 1$.

19. Let X be a Banach space. If the sequence $T_n \in B(X,X)$ converges to T such that T_n^{-1} exists and bounded in $B(X,X)$. Then show that T is invertible.

20. Let X and Y be two Banach spaces and $T : X \to Y$ is a continuous linear bijection. Then show that $\inf\{\|x\| : x \in X,\ \|Tx\| = 1\} \leq \|T^{-1}\|$. Does equality hold?

21. Let $\phi \in L^\infty(\mathbb{R})$. For $1 \leq p < \infty$, define an operator on $L^p(\mathbb{R}^n)$ by $M_\phi(f) = \phi f$. Show that $\|M_\phi\| = \|\phi\|_\infty$. Is the conclusion still true for $p = \infty$?

22. Let M be a closed subspace of a normed linear space X. Show that the projection $\pi :\ X \to X/M$ defined by $\pi(x) = x + M$ is a continuous linear surjective open map with $\|\pi\| < 1$. If $M \subsetneq X$ then $\|\pi\| = 1$.

23. Let $X = \{f \in C[0,1] :\ f(0) = 0\}$ and $M = \left\{f \in X :\ \int_0^1 f(t)dt = 0\right\}$. Show that
    (a) M is an infinite-dimensional closed subspace of X.
    (b) There does not exist any $f \in X$ with $\|f\|_\infty = 1$ such that $\text{dist}(f, M) = 1$.

24. Let X and Y be two Banach spaces and $T \in B(X,Y)$. Show that the following are equivalent:
    (a) T is injective and has closed range.
    (b) There is $k \geq 0$ such that $\|x\| \leq k\|T(x)\|$ for all $x \in X$.

25. Let X and Y be two normed linear spaces and $T : X \to Y$ such that $\dim T(X) < \infty$ and $\ker T$ is closed. Then show that T is bounded.

26. Suppose X can be made into a Banach space with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If there exists $m > 0$ such that $\|x\|_1 \leq m\|x\|_2$ for all $x \in X$. Then both norms are equivalent.

27. Let X be a Banach space and Y be a normed linear space. Suppose $T_n \in B(X,Y)$ such that $\lim\limits_{n\to\infty} T_n(x)$ exists for each $x \in X$. Write $T(x) = \lim\limits_{n\to\infty} T_n(x)$. Show that T is bounded.

28. Let (a_n) be a sequence of real numbers such that for each $x = (x_n) \in \ell^2$, the sequence $(a_n x_n)$ belongs to $\ell^2(\mathbb{N})$. Define an operator T on ℓ^2 by $T(x_1, x_2, \ldots) = (a_1 x_1, a_2 x_2, \ldots)$. Show that T is bounded.

29. Let X and Y be two normed linear spaces and $T_n, T \in B(X,Y)$ such that $T_n \to T$. Suppose $x_n \to x$. Then show that $T_n x_n \to Tx$ in Y.

30. Let X be a Banach space and let $T \in B(X)$ with $\|T\| < 1$. Then show that $I - T$ is invertible and $(I-T)^{-1} = \sum\limits_{n=0}^{\infty} T^n \in B(X)$.

31. Let (T_n) be a sequence of bounded linear operator on a Banach space X such that $\|T_n - T\| \to 0$. If T_n^{-1} exists, $\forall\, n \in \mathbb{N}$ and $\|T_n^{-1}\| < 1$, then prove that $T^{-1} \in B(X)$.

32. Let X and Y be two normed linear spaces and $T \in B(X,Y)$ sends each open subset in X to an open subset in Y. Prove that T is onto.

33. Let $T : (C^1[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ be a linear transformation that defined by $Tf(s) = f'(s) + \int\limits_0^s f(t)dt$. Show that the graph of T is closed but T is not continuous.

Problem Set 3

Focus: Hahn–Banach extension, separation, duality, weak topologies, and reflexivity.

1. State whether each assertion is true or false, giving a complete justification in each case.

   (a) If $f((x_n)) = \sum\limits_{n=1}^{\infty} \frac{x_n}{\sqrt{n}}$ for all $(x_n) \in \ell^1$, then the linear functional $f : (\ell^1, \|\cdot\|_2) \to \mathbb{K}$ is continuous.

   (b) If X is a normed linear space and $f \in X^*$, then $\{x \in X : f(x) \neq 1\}$ must be dense in X.

(c) Let X be a Banach space and $f_n \to 0$ in the weak* topology of X^*. Is it necessary that f_n is bounded in X^*?

(d) Let M be a proper dense subspace of normed linear space X. Does it imply that every continuous linear functional on M has a unique Hahn–Banach extension to X?

(e) Suppose $(x_n) \in \ell^2$. Does it imply the sequence $\left(\frac{x_n}{\sqrt{n}}\right) \in \ell^1$?

2. Let f be a linear functional on a normed linear space X. Then f is bounded if and only if $\ker f$ is closed.

3. Let X^* denote the dual space of a normed linear space X. For $x \in X$, show that $\|x\| = \sup\{|f(x)| : \ f \in X^* \text{ and } \|f\| = 1\}$.

4. Let $1 \leq p < \infty$. Define a linear map $T : \ell^p \to \ell^p$ by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Find the adjoint operator T^* of T.

5. Show that the linear map $T : (C^1[0,1], \|\cdot\|) \to (C[0,1], \|\cdot\|)$ defined by $(Tf)(t) = f'(t)$ does not have a bounded adjoint.

6. Let X and Y be two normed linear spaces. Suppose $T \in B(X,Y)$. Show that $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$.

7. Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L^p(\mathbb{R})$, prove that
$$\|f\|_p = \sup\left\{\left|\int_{\mathbb{R}} f(x)g(x)dx\right| : g \in L^q(\mathbb{R}) \text{ and } \|g\|_q = 1\right\}.$$

8. Define a family of linear functionals $f_n : c_0(\mathbb{N}) \to \mathbb{C}$ by $f_n(x) = \frac{1}{n}\sum_{j=1}^{n} x_j$. Show that $\lim_{n\to\infty} f_n(x) = 0$ but $\|f_n\| = 1$.

9. Let $M = \{(x_1, x_2, \ldots) \in \ell^1 : x_1 + x_2 = 0\}$. Define a linear functional f on M by $f(x_1, x_2, \ldots) = 2x_1$. Find a norm preserving extension of f to ℓ^1.

10. Let $M = \{(x, y) \in \mathbb{R}^2 : \ 2x - 3y = 0\}$. Define a linear functional f on M by $f(x,y) = x$. Find all possible Hahn–Banach extensions of f to $(\mathbb{R}^2, \|\cdot\|_2)$.

11. Let C be an open subset of a normed linear space X and $0 \in C$. For $x \in X$, define $p(x) = \inf\{t > 0 : t^{-1}x \in C\}$. Show that there exists $M > 0$ such that $p(x) \leq M\|x\|$ for all $x \in X$.

12. Let c_0 be the space of all sequences on $\mathbb{C}$ that converges to 0. Show that the dual of $(c_0, \|\cdot\|_\infty)$ is isomorphic to $(\ell^1, \|\cdot\|_1)$

13. Let Y be a a proper dense subspace of a normed linear space X. Show that the identity operator on Y cannot be extended as a continuous operator from Y to X.

14. Let M be a proper subspace of a normed linear space X. Suppose $\text{dist}(x_o, M) = \delta > 0$ for some $x_o \notin M$. Prove that there exists $f \in X^*$ such that $\|f\| = 1,\ f(x_o) = \delta$ and $f(x) = 0$ for all $x \in M$. Does such f exist uniquely?

15. Let $T : \ell^2 \to \ell^2$ be a linear map such that $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. Find the adjoint T^* of T.

16. Let $M = \{(y_1, y_2, \ldots) \in \ell^2 : 2y_1 - y_2 = 0\}$ and $x_o = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. Find $y_o \in M$ such that $\text{dist}(x_o, M) = \|x_o - y_o\|_2$.

17. Let $\{e_1, e_2, \ldots, e_n\}$ be a linearly independent set in an infinite-dimensional normed linear space X. For $(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$, prove that there exists $f \in X^*$ such that $f(e_j) = a_j$, for $j = 1, 2, \ldots, n$.

18. Suppose the sequence $g_n \in L^2[0, 1]$ is defined by

$$g_n(t) = \begin{cases} \sqrt{n} & \text{if } 0 \leq t < 1/n, \\ 0 & \text{if } 1/n \leq t \leq 1. \end{cases}$$

    Show that $\|g_n\|_2 = 1$ and g_n converges weakly to 0.

19. For $f \in L^2[-\pi, \pi]$, define a sequence (φ_n) of linear functionals by $\varphi_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$. Show that $\|\varphi_n\| = 1$ and $\varphi_n(f) \to 0$.

20. Let c_0 be the space of all sequences converging to zero. Show that $(c_0)^* = \ell^1$ and $(c_0)^{**} = \ell^\infty$. Further, show that the map $x \mapsto \sum_1^\infty x_n$ is weakly continuous on c_0 but not weak* continuous.

21. Let X and Y be two Banach spaces and $T_n, T \in B(X, Y)$. If $T_n \to T$ weakly, show that $\sup \|T_n\| < \infty$.

22. Let $X = (C[0, 1], \|\cdot\|_\infty)$. Define a map $T : X \to \mathbb{C}$ by $T(f) = \int_0^1 tf(t)dt,$ for all $f \in X$. Find a vector $f \in X$ such that $T(f) = \|T\|$.

23. Suppose X and Y be two Banach spaces and $T : X \to Y$ such that $f \circ T \in X^*$ for all $f \in Y^*$. Show that T is continuous.

24. Let X and Y be two normed linear spaces. For $T \in B(X, Y)$, define $T^* : Y^* \to X^*$ by $T^*(f) = f \circ T$, for all $f \in Y^*$. Show that
    (a) $\ker T^* = (\mathrm{Im} T)^\perp$.
    (b) T is bijective then T^* is bijective.

25. Let X and Y be two Banach spaces. Suppose $S : X \to Y$ and $T : Y^* \to X^*$ be linear maps satisfying $f \circ S = T(f)$ for all $f \in Y^*$. Show that S is continuous. (Hint: use the closed graph theorem.)

26. Let X and Y be two Banach spaces and $T \in B(X, Y)$ be such that the range $\mathcal{R}(T)$ is closed. Prove that $\mathcal{R}(T^*) = (\ker T)^\perp$, where $M^\perp = \{f \in X^* : \ f(x) = 0, \ \forall x \in M\}$, for $M \subseteq X$.

27. Let X be a reflexive Banach space and $f \in X^*$. Show that there exists $x \in \overline{B(0, 1)}$ such that $f(x) = \|f\|$.

28. Let K be a closed bounded convex subset of a reflexive Banach space X. Prove that K is weakly compact.

29. Suppose M is a subspace of a Banach space X. Then $M^\perp$ is a weak*-closed subspace of X^*.

30. Let X be a normed linear space and let $f \in X^*$ with $f \neq 0$. If $x_0 \in X$ and if $\alpha \in \mathbb{K}$, then show that $d(x_0, \{x \in X : f(x) = \alpha\}) = \frac{|f(x_0) - \alpha|}{\|f\|}$.

31. Let $Y = \{(x, y) \in \mathbb{R}^2 : 2x - y = 0\}$ and let $g(x, y) = x$ for all $(x, y) \in Y$. Determine all the Hahn-Banach extensions of g to $(\mathbb{R}^2, \| \cdot \|_2)$.

32. Let (x_n) be a sequence in a Banach space X and let (f_n) be a sequence in X^*. Prove the following:
    (a) If $x_n \xrightarrow{w} x \in X$, then (x_n) is bounded and $\|x\| \leq \liminf\limits_{n \to \infty} \|x_n\|$.
    (b) If $f_n \xrightarrow{w^*} f \in X^*$, then (f_n) is bounded and $\|f\| \leq \liminf\limits_{n \to \infty} \|f_n\|$.

33. Let X be a Banach space, let $x \in X$ and let $f \in X^*$. If (x_n) is a sequence in X such that $x_n \to x$ and if (f_n) is a sequence in X^* such that $f_n \xrightarrow{w^*} f$, then show that $f_n(x_n) \to f(x)$.

Problem Set 4

Focus: spectrum, compact operators, Hilbert spaces, adjoints, and the spectral theorem.

1. State whether each assertion is true or false, giving a complete justification in each case.
   (a) Every orthonormal set in a Hilbert space H must be closed in H.
   (b) In any infinite-dimensional Hilbert space, there exists a convergent series which is not absolutely convergent.
   (c) If (x_n) is a sequence in a Hilbert space H such that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$, then the series $\sum_{n=1}^{\infty} x_n$ must converge in H.
   (d) If (u_n) is an orthonormal sequence in a Hilbert space H and if $x \in H$, then the series $\sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$ must converge in H but not necessarily to x.
   (e) If $\{u_n : n \in \mathbb{N}\}$ is a countably infinite orthonormal basis of a Hilbert space H and if $x \in H$, then the series $\sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$ in H must be absolutely convergent.
   (f) If, in a Hilbert space H, every weakly convergent sequence is norm convergent, then H must be separable.

2. Let $(X, \langle ., . \rangle)$ be an inner product space. Prove the following generalized parallelogram law.

$$\sum_{\epsilon_k = \pm 1} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|^2 = 2^n \sum_{k=1}^{n} \|x_k\|^2 .$$

3. Let ω be a primitive nth root of unity and $n > 2$. Then Show that for x, y in an inner product space X following holds.

$$\langle x, y \rangle = \frac{1}{n} \sum \omega^p \|x + \omega^p y\|^2 .$$

4. Let X be an inner product space, let $x \in X$ and let (x_n) be a sequence in X such that $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$. Show that $x_n \to x$ in X.

5. Let X be an inner product space and (x_n) and (y_n) be sequences in $\overline{B(0,1)}$ such that $\|\frac{1}{2}(x_n + y_n)\| \to 1$ as $n \to \infty$. Show that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

6. Let X be an inner product space and let $y, z \in X$. If $Tx = \langle x, y \rangle z$ for all $x \in X$, then show that $\|T\| = \|y\|\|z\|$.

7. Consider $C_{\mathbb{R}}[0,1]$ with the usual inner product. Let $S = \{p_n : n = 0, 1, 2, \ldots\}$, where $p_n(t) = t^n$ for all $t \in [0,1]$ and for $n = 0, 1, 2, \ldots$. Prove that the orthogonal complement of S in $C_{\mathbb{R}}[0,1]$ is $\{0\}$.

8. Let M be a closed subspace of a Hilbert space H. If $x \in M$ and if (x_n) is a sequence in M, then show that $x_n \xrightarrow{w} x$ in H if and only if $x_n \xrightarrow{w} x$ in M.

9. Using Riesz representation theorem, show that $\{(x_n) \in \ell^2 : \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}} x_n = 0\}$ is not a closed subset of the Hilbert space ℓ^2.

10. Determine $\|f\|$ for the linear functional $f : (\ell^2, \|\cdot\|_2) \to \mathbb{C}$, defined by $f(x) = \sum\limits_{n=1}^{\infty} \frac{x_n}{\sqrt{n(n+1)}}$, for all $x = (x_1, x_2, \ldots) \in \ell^2$.

11. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space H. If $f(x) = \sum\limits_{n=1}^{\infty} \frac{1}{3^n} \langle x, e_n \rangle$ for all $x \in H$, then determine $\|f\|$.

12. Let H be a Hilbert space and let (T_n) be a sequence in $B(H, H)$ such that for each $x, y \in H$, $\lim\limits_{n \to \infty} \langle T_n x, y \rangle$ exists. Show that $\sup\{\|T_n\| : n \in \mathbb{N}\} < \infty$.

13. Let $(X, \|\cdot\|)$ be a separable Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. If $\|x\|_0 = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} |\langle x, e_n \rangle|$ for all $x \in X$, then show that $\|\cdot\|_0$ is a norm on X which is not equivalent to $\|\cdot\|$.

14. Let $T : L^2[0,1] \to L^2[0,1]$ be a linear map which is defined by $(Tf)(x) = \int\limits_0^x f(t)dt$. Define $\langle Tf, g \rangle = \langle f, T^* g \rangle$. Find the adjoint operator T^* of T.

15. Let T be linear operator on a Hilbert space H such that $\langle Tx, y \rangle = \langle x, Ty \rangle$. Show that T is continuous.

16. Let H be a Hilbert space and let $T : H \to H$ be linear. If $\langle T^2 x, x \rangle \geq 0$ and $\langle Tx, x \rangle = 0$ for all $x \in H$, then show that $T = 0$.

17. If A is a subset of an inner product space such that $A^0 \neq \emptyset$, then show that $A^{\perp} = \{0\}$.

18. If A is a dense subset of an inner product space, then show that $A^{\perp} = \{0\}$.

19. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for a separable Hilbert space H. Define a linear map $T : H \to H$ by $T(e_n) = a_n e_n,\ n = 1, 2, \ldots$. Show that T is bounded if and only if sequence $\{a_n\}$ is bounded.

20. Let T be a normal operator on a Hilbert space H. Show that $\|T^2\| = \|T\|^2$.

21. Let M be a closed subspace of a Hilbert space H and let $x \in H \setminus M$. Prove that $d(x, M) = \sup\{|\langle x, y\rangle| : y \in M^\perp,\ \|y\| \leq 1\}$.

22. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H and $(\alpha_1, \alpha_2, \ldots) \in \ell^\infty$. Define a linear map $T : H \to H$ by $T(x) = \sum\limits_{n=1}^{\infty} \langle x, e_n\rangle \alpha_n e_n$. Prove that T is continuous and $\|T\| = \sup |\alpha_n|$.

23. Let φ be a bounded function on $\mathbb{R}$. Define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $T(f)(t) = \varphi(t)f(t)$. Show that T is a bounded operator. Find the adjoint operator T^* of T.

24. Suppose T is a bounded linear operator on a complex Hilbert space H such that $\langle Tx, x\rangle = 0$, for all $x \in H$. Show that $T = 0$.

25. Let T be a bounded and self-adjoint operator on a Hilbert space H. Suppose there exists $k > 0$ such that $\|Tx\| \geq k\|x\|$, for each $x \in H$. Prove that the equation $Tx = y$ has a unique solution for each $y \in H$.

26. Let $E = \{e_\alpha : \alpha \in I\}$ be an orthonormal basis of a Hilbert space H. For each $x \in H$, show that the set $\left\{e_\alpha \in E : |\langle x, e_\alpha\rangle|^2 > \left(\frac{2}{n} - e^{-n}\right)\|x\|^2\right\}$ is a finite set for each fixed $n \in \mathbb{N}$.

27. Let M be a proper closed subspace of a Hilbert space H. Define $\pi : H \to H/M$ by $\pi(x) = x + M$. Show that $\|\pi\| = 1$. Further, if there exists $0 \neq x \in H$ such that $\|\pi(x)\| = \|x\|$, then there exists a unique $0 \neq z \in M$ such that $\|z\|^2 = 2\mathrm{Re}\langle x, z\rangle$.

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