DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA4128: Measure Theory	MidSem
Instructor: Rajesh Srivastava	September 18, 2026
Duration: Two hours	Maximum Marks: 30

N.B. Answers without proper justification will receive no credit.

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1. (a) Do there exist infinitely many uncountable nowhere dense subsets of $\mathbb{R}$ whose union has Lebesgue measure zero? $\boxed{1}$

   (b) Does there exist a nonconstant continuous function $f : \mathbb{R} \to \mathbb{R}$ which zero everywhere except on a set of Lebesgue measure zero? $\boxed{1}$

   (c) Let C be the Cantor set in $[0, 1]$. Is every function $f : C \to \mathbb{R}$ necessarily Borel measurable? $\boxed{1}$

   (d) For $A, B \subseteq \mathbb{R}$, define $A - B = \{x - y : (x, y) \in A \times B\}$. If $m^*(A - B) > 0$, does there exist $(x, y) \in A \times B$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$? $\boxed{1}$

   (e) Is the continuous image of a closed subset of $\mathbb{R}$ necessarily a Borel set? $\boxed{1}$

2. Let $E = \mathbb{N} \times \{0\}$ and, for each $n \in \mathbb{N}$, let $O_n = \left\{x \in \mathbb{R}^2 : d(x, E) < \frac{1}{n}\right\}$. Show that $\lim\limits_{n\to\infty} m(O_n) = \infty$ and $m\left(\bigcap\limits_{n=1}^{\infty} O_n\right) = 0$. $\boxed{3}$

3. Suppose (f_n) is a sequence of Lebesgue measurable functions $f_n : \mathbb{R} \to \mathbb{R}$. Show that the set $\{x \in \mathbb{R} : \liminf\limits_{n\to\infty} f_n(x) \text{ exists}\}$ is Lebesgue measurable. $\boxed{3}$

4. Let $E \subset \mathbb{R}$ satisfy $0 < m(E) < \infty$. For $x \in \mathbb{R}$, define $f(x) = m\big(E \cap (-\infty, x)\big)$. Show that f is uniformly continuous and that, for every α with $0 < \alpha < m(E)$, there exists $x \in \mathbb{R}$ such that $f(x) = \alpha$. $\boxed{4}$

5. Let (X, S, μ) be a finite measure space and let $f : X \to \mathbb{R}$ be measurable. Prove that, for every $\epsilon > 0$, there exists $M > 0$ such that $\mu\big(\{x \in X : |f(x)| > M\}\big) < \epsilon$. $\boxed{4}$

6. Let $A \subset \mathbb{R}$ satisfy $m^*(A) > 0$. Prove that there exists a bounded interval I such that $0 < m^*(A \cap I) < \infty$. Further, show that there exists an interval J such that $m^*(A \cap J) \geq \dfrac{3}{4} m(J)$. $\boxed{4}$

7. Let $E \subset \mathbb{R}$ be Lebesgue measurable and suppose that $m\big(E \cap (x - 1, x + 1)\big) = 1$ for every $x \in \mathbb{R}$. Show that $m(E) = \infty$. $\boxed{3}$

8. Let $f : [0, 1] \to [0, 1]$ be a Lebesgue measurable bijection, and assume that f^{-1} is also Lebesgue measurable. Determine whether it necessarily follows that $m(f(E)) = m(E)$ for every Lebesgue measurable set $E \subseteq [0, 1]$. $\boxed{4}$

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