

**DEPARTMENT OF MATHEMATICS**  
**INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI**

**Course:** MA224: Real Analysis  
**Instructor:** Rajesh Srivastava  
**Duration:** 03 hours

**End-Semester Examination**  
May 03, 2026  
**Maximum Marks:** 50

**Note:** Answers that are not supported by adequate justification may receive little or no credit.

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1. (a) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  satisfy  $\|f(x_1, x_2)\|_2 \leq |x_1| + |x_2|$ . Is  $f$  differentiable at  $(0, 0)$ ? **1**
- (b) Can the equation  $x^2 + ye^x - \sin(xy) = 0$  be solved for  $x$  as a differentiable function of  $y$  in a neighbourhood of  $(0, 0)$ ? **1**
- (c) Does there exist an unbounded subset  $A$  of the real line such that  $m^*(A) = 0$  and  $m(\bar{A}) = 1$ ? **1**
- (d) Let  $E \in \mathcal{M}(\mathbb{R})$  with  $m(E) < \infty$ . Is it possible that  $m(E \cap (n, n + 1)) \geq \frac{1}{2}$  for every  $n \in \mathbb{Z}$ ? **1**
- (e) Does there exist an open dense set  $D \subset \mathbb{R}$  such that  $0 < m(D) < 1$ ? **1**
- (f) Does every continuous function on  $\mathbb{R}$  map every closed set to a Lebesgue measurable set? **1**

2. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by  $f(x, y) = \begin{cases} 1 & \text{if } y^2 < x < 2y^2, \\ 0 & \text{otherwise.} \end{cases}$   
Determine all  $v \in S^1$  for which  $D_v f(0, 0)$  exists. Is  $f$  continuous at  $(0, 0)$ ? **3**

3. Show that the system of equations

$$\begin{aligned} u^2 + v + x &= 1, \\ u + v^2 + y &= 0, \end{aligned}$$

can be solved for  $u$  and  $v$  as differentiable functions of  $x$  and  $y$  in a neighbourhood of  $(u, v, x, y) = (0, 0, 1, 0)$ . **3**

4. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a  $C^1$  map satisfying  $f(0, 0) = 0$  and  $f_x(0, 0) = 1$ . For  $(x, y) \in \mathbb{R}^2$ , define  $g(x, y) = (f(x, y), y)$ . Show that  $g$  is injective in some small neighbourhood of  $(0, 0)$ . **2**
5. Let  $A \in GL_n(\mathbb{R})$ . Show that the set  $O = \{B \in L_n(\mathbb{R}) : \|B - A\| < \frac{1}{3\|A^{-1}\|}\}$  is open in  $GL_n(\mathbb{R})$ . **4**
6. Let  $F$  be a closed subset of  $\mathbb{R}$  such that  $m(F) > 0$ . Show that there exists a compact set  $E \subseteq F$  such that  $m(E) > 0$ . **3**

**P.T.O.**

7. Let  $E$  be a Lebesgue measurable subset of  $\mathbb{R}$ . Show that  $A = \bigcup_{x \in E} [x - 1, x + 1)$  is Lebesgue measurable. 4
8. Let  $A_n = \{x \in \mathbb{R} : d(x, \mathbb{N}) < \frac{1}{n}\}$ . Show that  $\lim_{n \rightarrow \infty} m(A_n) = \infty$  and  $m(\bigcap_{n=1}^{\infty} A_n) = 0$ . 4
9. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a surjective Borel measurable function, and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. Show that  $g \circ f$  is Borel measurable. 2
10. Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) = \sup\{|x + y| : y \in [0, 1]\}$ . Show that  $f$  is Lebesgue measurable on  $\mathbb{R}$ . 4
11. Let  $E \in \mathcal{M}(\mathbb{R})$ , and let  $O$  be a bounded open subset of  $\mathbb{R}$ . For  $x \in \mathbb{R}$ , define  $f(x) = m(E \cap (O + x))$ . Show that  $f$  can be represented as a series of bounded uniformly continuous functions on  $\mathbb{R}$ . Is  $f$  bounded on  $\mathbb{R}$ ? 5
12. Show that there exists a sequence  $f_n \in L^1(\mathbb{R}, \mathcal{M}, m)$  such that  $\lim_{n \rightarrow \infty} \|f_n\|_1 = 0$ , but  $\sum_{n=1}^{\infty} f_n$  diverges in  $L^1(\mathbb{R}, \mathcal{M}, m)$ . 2
13. Let  $f \in L^1(\mathbb{R}, \mathcal{M}, m)$ . For  $E \in \mathcal{M}$ , define  $\mu(E) = \int_E |f| dm$ . Show that  $\mu$  is a finite measure on  $(\mathbb{R}, \mathcal{M})$ . 4
14. Evaluate  $\lim_{n \rightarrow \infty} \int_0^2 \frac{x^n + 2}{x^n + 1} dm(x)$ , where  $m$  denotes the restriction of Lebesgue measure to  $[0, 2]$ . 4

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