Hardy Littlewood maximal function and Lebesgue differentiation theorem: (1)

In this lecture series, we consider two problems related to the reciprocity of differentiation and integration.

1. For f integrable on $[a,b]$, write
$$F(x) = \int_a^x f(y)\,dy.$$
Does it imply that f is diff. (at least a.e. x), and $F' = f$ a.e. x?

We shall see that answer to this question has connection with a broader idea, not limited to dimension one.

2. What conditions on a function F on $[a,b]$ guarantee that $F'(x)$ exists (for a.e. x), F' is integrable and
$$\int_a^b F'(x)\,dx = F(b) - F(a)\ ?$$
The later problem is far difficult than the 1st one.

As an example, we shall see that

If $F : [a,b] \to \mathbb{R}$ is monotone increasing then F is differentiable a.e. $x \in [a,b]$ (2)
and $\int_a^b F'(x)\,dx \leq F(b) - F(a)$.

Notice that even there are continuous monotone increasing functions fail to satisfy this inequality. For example, the Cantor-Lebesgue function F on $[0,1]$ with $F(0)=0$ & $F(1)=1$, $F'(x)=0$ a.e.

In order to prove the above result we need Vitali covering lemma.

Def^n: A collection $\mathcal{G}$ of intervals in $\mathbb{R}$ is said to be Vitali cover of a set $E \subset \mathbb{R}$ if for each $x \in E$, $\forall\, \epsilon > 0$, $\exists\, I \in \mathcal{G}$ s.t. $x \in I$ and $\ell(I) < \epsilon$.

Lemma (Vitali):

Let $m^*(E) < \infty$ and $\mathcal{G}$ is a Vitali's cover of E. Then $\forall\, \epsilon > 0$, $\exists$ a finite subcollection $\{I_1, \cdots, I_N\} \subset \mathcal{G}$ s.t.

$$m^*\left(E \setminus \dot{\bigcup_{n=1}^{N}} I_n\right) < \epsilon.$$

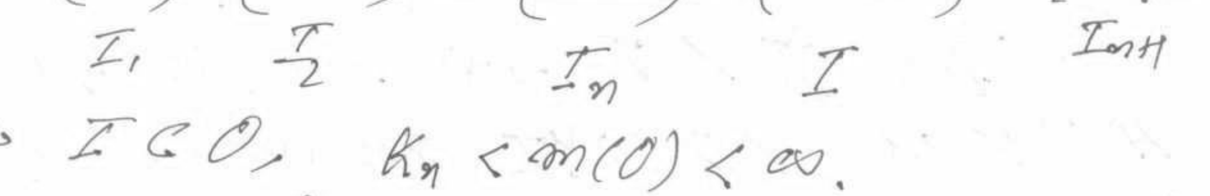
Proof: It is sufficient to prove the lemma while intervals in $\mathcal{G}$ are closed.

Since $m^*(E) < \infty$, we can always find an open set $O \supset E$ and $m^*(E) \leq m(O) < \infty$ (by defn. of m^*). Hence, w.l.g. we can assume that each $I \in \mathcal{G}$ contained in O. ③

Now, we choose a seqn $\{I_n\}$ of disjoint intervals of $\mathcal{G}$ by induction as follows:

Suppose $I_1 \in \mathcal{G}$ and $I_1, I_2, \dots, I_n$ already chosen.

Let $k_n = \sup\{l(I) : I \in \mathcal{G},\ I \cap (\bigcup_{i=1}^{n} I_i) = \phi\}$.



Since $I \subset O$, $k_n < m(O) < \infty$.

Unless, $E \subset \bigcup_{i=1}^{n} I_i$, we can choose $I_{n+1} \in \mathcal{G}$ with $l(I_{n+1}) > \frac{1}{2} k_n$, and $I_{n+1} \cap (\bigcup_{i=1}^{n} I_i) = \Phi$.

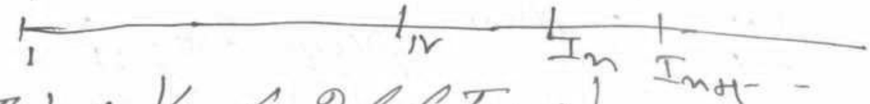
Thus, we have a disjoint seqn $\{I_n\}$ in $\mathcal{G}$ with $\bigcup_{n=1}^{\infty} I_n \subset O$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \leq m(O) < \infty.$$

For $\epsilon > 0$, $\exists$ $N \in \mathbb{N}$ such that $\sum_{N+1}^{\infty} l(I_n) < \frac{\epsilon}{5}$.

Let $R = E \setminus \bigcup_{n=1}^{N} I_n$. Let $x \in R$ be arbitrary. Since $\bigcup_{n=1}^{N} I_n$ is closed and $x \notin \bigcup_{n=1}^{N} I_n$, we can find a small interval $I \in \mathcal{G}$ s.t. $\left(\bigcup_{n=1}^{N} I_n\right) \cap I = \Phi$.

Now, if $I \cap I_i = \phi$ for $i \le n$, then we must have



$$l(I) \le K_n \le 2l(I_{n+1}).$$

As $l(I_{n+1}) \to 0$, I must meet one of the intervals I_n. Let n_0 be the smallest integer s.t. $I \cap I_{n_0} \ne \Phi$. Then $n_0 > N$ and

$$l(I) \le K_{n_0 - 1} \le 2l(I_{n_0})$$

Since $x \in I$, & $I \cap I_{n_0} \ne \phi$, it follows that the distance from x to the mid point of I_{n_0} is at most

$$l(I) + \frac{1}{2} l(I_{n_0}) \le \frac{5}{2} l(I_{n_0}).$$

Thus, $x \in J_{n_0}$ having same mid pt as I_{n_0} with five time length as to I_{n_0}.

Thus, $R \subset \bigcup_{n_0 = N+1}^{\infty} J_{n_0} \Rightarrow$

$$m^*(R) \le \sum_{N+1}^{\infty} l(J_{n0}) = 5\sum_{N+1}^{\infty} l(I_{n0}) < \epsilon.$$ (5)

Consider $f : [a,b] \longrightarrow \mathbb{R}$ and write

$$f'(x) = \lim_{h\to 0} \frac{f(x)-f(x-h)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

when $h>0$.

Note that these limit exist if $\forall\ h_n \to 0$,

$$f'(x) = \lim_{h_n\to 0} \frac{f(x)-f(x-h_n)}{h_n} = \lim_{h_n\to 0} \frac{f(x+h_n)-f(x)}{h_n}.$$

In case f is not differentiable the following four limits could be different.

$$D^+f(x) = \limsup_{h_n\to 0} \frac{f(x+h_n)-f(x)}{h_n}$$

$$D^-f(x) = \limsup_{h_n\to 0} \frac{f(x)-f(x-h_n)}{h_n}$$

$$D_+f(x) = \liminf_{h_n\to 0} \frac{f(x+h_n)-f(x)}{h_n}.$$

$$D_-f(x) = \liminf_{h_n\to 0} \frac{f(x)-f(x+h_n)}{h_n}.$$

Then $D^+f(x) \ge D_+f(x)$ & $D^-f(x) \ge D_-f(x)$

If $D^+f(x) = D_+f(x) = D^-f(x) = D_-f(x) \ne \pm\infty$, we say f is differentiable & common value is called derivative of f at x.

we denote it by $f'(x)$. If $D^+f(x) = D_-f(x)$, f is right diff at x and denote the right derivative by $f'(x+)$. Similarly $f'(x-)$.

(6)

<u>Monotone functions</u>:

If $f : [a,b] \to \mathbb{R}$ is monotone, then f is continuous a.e. x.

Suppose f is $\uparrow$ function, then

$$f(x-) = \lim_{y \uparrow x} f(y) = \sup_{y < x} f(y) \leq f(x) \leq \inf_{z > x} f(z)$$

$$\leq \lim_{z \downarrow x} f(z) = f(x+).$$

Thus both $f(x-)$ & $f(x+)$ exist and

$$f(x-) \leq f(x) \leq f(x+).$$

If $\exists\, x$ s.t. $f(x-) < f(x) < f(x+)$, then such x are countable as

$$(f(x-), f(x+)) \cap (f(x'-), f(x'+)) = \phi.$$

and each interval can be associated with unique rational corresponding to x.

Thus, f is continuous a.e. x.

Now, we state and prove the main result.

Theorem 1: Let $f: [a,b] \to \mathbb{R}$ be a monotone increasing function. Then f' exists a.e. x, f' is measurable and

$$\int_a^b f'(x)\,dx \leq f(b) - f(a).$$

Proof: In order to show that f is differentiable a.e, it is enough to show that the set where any of two derivatives unequal has measure zero. For this we only show that $E = \{x : D^+f(x) > D_-f(x)\}$

$$= \bigcup_{u,v \in \mathbb{Q}} \{x : D^+f(x) > u > v > D_-f(x)\}$$

has measure zero. We claim

$$m^*(E_{u,v}) = m^*\{x : D^+f(x) > u > v > D_-f(x)\} = 0.$$

Note that $m^*(E_{u,v}) \leq \ell[a,b] = b-a < \infty$.

Let $s = m^*(E_{u,v})$. Then for $\epsilon > 0$, $\exists$ open set $O \supset E_{u,v}$ s.t. $m(O) < s + \epsilon$.

For each $x \in E_{u,v}$, $\exists$ small interval $[x-h, x] \subset O$ s.t.

$$f(x) - f(x-h) < hv.$$

Hence $\{[x-h, x] : x \in E_{u,v},\ h > 0,\ h \text{ small}\}$ is a Vitali cover of $E_{u,v}$. By Vitali lemma,

∃ a finite disjoint sub-collection $\{I_n = [x_n - h_n, x_n] : n = 1, 2, \dots, N\}$
whose interiors cover a set $A \subset E$

i.e. $\bigcup_{n=1}^{N} (x_n - h_n, x_n) \supset A = E \cap \left(\bigcup_{n=1}^{N} (x_n - h_n, x_n) \right)$.

Hence, for $O_N = \bigcup_{n=1}^{N} (x_n - h_n, x_n)$,

$$m^*(A) \geq m^*(E) - m^*(E \setminus A)$$
$$\geq m^*(E) - m^*(E \setminus F)$$
$$> \delta - \epsilon.$$

Summing over these intervals, we get

$$\sum_{n=1}^{N} [f(x_n) - f(x_n - h_n)] < v \sum_{n=1}^{N} h_n \qquad F = \bigcup_{n=1}^{N} I_n$$

$$= v \sum \ell([x_n - h_n, x_n]) \quad (\because I_n\text{'s are disjoint})$$

$$< v\, m(O) < v(\delta + \epsilon). \quad \text{——(1)}$$

For each $y \in A$ ($\subset E$), ∃ small $k > 0$ s.t.

$(y, y+k) \subset I_n$ for some $n \leq N$.

and $f(y+k) - f(y) > uk$. ($\because y \in A \subset E$).

Hence, $\{[y, y+k] : y \in A,\ k > 0 \text{ small}\}$ is a Vitali cover of A. By Vitali lemma, ∃ a finite disjoint subcollection ~~of them~~ $\{J_1, \dots, J_M\}$ of such intervals that cover a subset $B \subset A$, where $B = A \cap \left(\bigcup_{i=1}^{M} J_i \right)$

$$m^*(B) \geq m^*(A) - m^*(E \setminus B)$$
$$\geq m^*(A) - m^*\left(A \setminus \bigcup_{i=1}^{M} J_i\right)$$

ie $m^*(B) > \delta - 2\epsilon$. By summing over these intervals, we get (9)

$$\sum_{i=1}^{M}\left(f(y_i+k_i) - f(y_i)\right) > u\sum_{i=1}^{M} \ell([y_i, y_i+k_i])$$
$$\geq u\, m^*(B) \quad (\because B \subset \bigcup_{i=1}^{M} I_i).$$

ie $$\sum_{i=1}^{M}\left(f(y_i+k_i) - f(y_i)\right) > u(\delta - 2\epsilon) \quad \text{--- (2)}$$

Now summing over all those J_i that contained in I_n and use the fact that f is $\uparrow$, it follows that

$$\sum_{J_i \subset I_n} \{f(y_i+k_i) - f(y_i)\} \leq f(x_n) - f(x_n - h_n).$$

Since J_i's are disjoint,

$$\sum_{i=1}^{M}\{f(y_i+k_i) - f(y_i)\} \leq \sum_{n=1}^{N}\left(f(x_n) - f(x_n - h_n)\right).$$

$\Rightarrow u(\delta - 2\epsilon) < v(\delta + \epsilon), \quad \forall \epsilon > 0.$

$\Rightarrow u\delta \leq v\delta$. Since $v < u$, we get $\delta = 0$.

Let $$g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{--- (3)}$$

Then g is defined a.e. and that f is differentiable wherever g is finite. (10)

Let $g_n(x) = n[f(x+\frac{1}{n}) - f(x)]$, and set $f(x) = f(b)$ if $x \geq b$.

Then $g_n(x) \to g(x)$ a.e. x.

So g is msble and by Fatu's lemma

$$\int_a^b g = \int_a^b \lim g_n \leq \underline{\lim} \int_a^b g_n$$

$$= \underline{\lim} \int_a^b n[f(x+\tfrac{1}{n}) - f(x)]\,dx$$

$$= \underline{\lim} \left[n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f - n\int_a^b f \right]$$

$$= \underline{\lim} \left[n\int_b^{b+\frac{1}{n}} f - n\int_a^{a+\frac{1}{n}} f \right]$$

$$= \underline{\lim} \left[f(b) - n\int_a^{a+\frac{1}{n}} f \right]$$

$$\leq [f(b) - f(a)] \quad (\because f \uparrow).$$

$\Rightarrow g \in L^1[a,b]$ and hence finite a.e. Thus, $f' = g$ a.e.

Function of bounded variation:

let $f:[a,b]\longrightarrow \mathbb{R}$ and consider a partition $Q=\{a=x_0<\cdots<x_{i-1}<x_i<\cdots=x_k=b\}$.

let $p=\sum_{i=1}^{k}[f(x_i)-f(x_{i-1})]^+$,

$n=\sum_{i=1}^{k}[f(x_i)-f(x_{i-1})]^-$,

where $g^+(x)=g(x)$ if $g(x)\geq 0$, else $g^+(x)=0$, and $g^-=g^+-g$, $|g|=g^++g^-$.

Denote $t=n+p=\sum_{i=1}^{k}|f(x_i)-f(x_{i-1})|$.

let $P=\sup p$, $N=\sup n$ and $T=\sup t$, where supremum has taken on all possible partitions of $[a,b]$.

If $T=T_a^b(f)<\infty$, then we say that f is of bounded variation, in short (B.V.) on $[a,b]$.

Lemma: If f is of B.V. on $[a,b]$ then $T_a^b=P_a^b+N_a^b$ and

$f(b)-f(a)=P_a^b-N_a^b$.

Proof: For any partition Q of $[a,b]$, (12)
we have
$$p = n + f(b) - f(a)$$
$$\Rightarrow P = N + f(b) - f(a).$$
Also, $t = p + n = p + p - \{f(b) - f(a)\}$
$$\Rightarrow T = 2P - \{f(b) - f(a)\}$$
$$= P + N.$$
Theorem: A function $f : [a,b] \to \mathbb{R}$ is of B.V. iff f is difference of two monotone real-valued functions.

Proof: Suppose $f \in BV[a,b]$, and set $g(x) = P_a^x$ & $h(x) = N_a^x$.

Then g & h are increasing real-valued functions, since
$$0 \leq P_a^x < T_a^x \leq T_a^b < \infty$$
$$\& \; 0 \leq N_a^x \leq T_a^x \leq T_a^b < \infty.$$
By the previous lemma,
$$f(x) = g(x) - h(x) + f(a).$$
On the other hand, if $f = g - h$, where g & h are monotone increasing, then on any partition Q, we get

$$\sum |f(x_i) - f(x_{i-1})| \leq \sum [g(x_i) - g(x_{i-1})] + \sum [h(x_i) - h(x_{i-1})] = g(b) - g(a) + h(b) - h(a).$$ (13)

Hence, $T_a^b(f) \leq g(b) - g(a) + h(b) - h(a) < \infty$.

Cor: If f is of B.V on $[a,b]$, then f' exists a.e. on $[a,b]$.

Differentiation of integrals:

For f is an integrable function on $[a,b]$, set $F(x) = \int_a^x f(y)dy$, does it imply that $F' = f$ a.e.?

Lemma 1: If f is an integrable function on $[a,b]$, then $F(x) = \int_a^x f(y)\,dy$ is continuous and of B.V.

Pf: $F(x+h) - F(x) = \int_x^{x+h} f(y)\,dy = \int_a^b \chi_{[x,x+h]}(y) f(y)\,dy$

Since $f \in L^1([a,b])$ & $\chi_{[x,x+h]} f \to f$ as $h \to 0$, by DCT, it follows that

$F(x+h) - F(x) \to 0$ as $|h| \to 0$.

Now,
$$\sum_{i=1}^{k} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{k} \left| \int_{x_{i-1}}^{x_i} f(y)dy \right|$$

$$\leq \int_a^b |f(y)|\,dy.$$

Hence $T_a^b(f) \leq \int_a^b |f(y)|\,dy < \infty$.

Lemma 2: If f is integrable on $[a,b]$ and $\int_a^x f(t)\,dt = 0$, $\forall\, x \in [a,b]$.

Then $f = 0$ a.e.

Pf: $E = \{x \in [a,b] : f(x) > 0\}$. If $m(E) > 0$, then $\exists$ a closed set $F \subset E$ with $m(F) > 0$. Let $O = (a,b) \setminus F$. Then O is open & $O = \bigcup_{n=1}^{\infty} I_n$, $I_n = (a_n, b_n)$.

Either $\int_a^b f \neq 0$, or

$$0 = \int_a^b f = \int_F f + \int_O f$$

$$\Rightarrow \int_O f = -\int_F f \neq 0 \quad (\because F \subset E)$$

$$\Rightarrow \int_{a_n}^{b_n} f \neq 0, \text{ for some } n.$$

$$\Rightarrow \int_a^{a_n} f - \int_a^{b_n} f \neq 0,$$

$\Rightarrow$ one of them is not zero, which is absurd.

Hence, $m(E) = 0$. Similarly, it can be shown that the set where $f < 0$, be measure zero. This proves lemma.

Lemma 3: Let f be a bounded measurable function on $[a,b]$. Define (15)

$$F(x) = \int_a^x f(y)\,dy + F(a).$$

Then $F'(x) = f(x)$ a.e. on $[a,b]$.

Proof: Since F is of B.V. (by lemma II), it implies that $F'(x)$ exists a.e.

Let $|f(x)| \leq M$, $\forall x \in [a,b]$. Set

$$f_n(x) = n\left\{F\left(x+\tfrac{1}{n}\right) - F(x)\right\}.$$

Then $f_n(x) = n\int_x^{x+\frac{1}{n}} f(t)\,dt$

$$\Rightarrow |f_n| \leq M.$$

Since $f_n(x) \to F'(x)$ pointwise a.e., by BCT it follows that

$$\int_a^c F'(x)\,dx = \lim \int_a^c f_n(x) = \lim n\left[\int_a^c \left(F\left(x+\tfrac{1}{n}\right) - F(x)\right)\right]$$

$$= \lim n\left[\int_c^{c+\frac{1}{n}} F - \int_a^{a+\frac{1}{n}} F\right].$$

Since F is continuous,

$$\int_a^c F'(x)\,dx = F(c) - F(a) = \int_a^c f(x)\,dx.$$

$\Rightarrow \int_a^c \{F'(x) - f(x)\}\,dx = 0, \quad \forall\, c \in [a,b].$

By lemma 2, it follows that $F'(x) = f(x)$ a.e. (16)

Theorem: let f be an integrable function on $[a,b]$, and

$$F(x) = F(a) + \int_a^x f(t)\,dt.$$

Then $F'(x) = f(x)$ a.e.

Proof: without loss of generality, we can assume that $f \geq 0$. For $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n. \end{cases}$$

Then $f - f_n \geq 0$ & $G_n(x) = \int_a^x f - f_n$ is an $\uparrow$ function of x. Then G_n is diff a.e and $G_n' \geq 0$. Now,

$$F(x) = F(a) + \int_a^x (f - f_n) + \int_a^x f_n$$

$$\frac{d}{dx} F(x) = \frac{d}{dx} G_n(x) + \frac{d}{dx} \int_a^x f_n.$$

$$\geq f_n(x) \text{ (by Lemma 3)}.$$
a.e.

$\Rightarrow F'(x) \geq f(x)$ a.e $(\because f_n \to f(x))$.

$$\Rightarrow \int_a^b F'(x)\,dx \geq \int_a^b f(x)\,dx = F(b) - F(a). \quad \text{---(1)}$$

Since $f \geq 0$, F is monotone ↑ and (17)

$$\int_a^b F'(x)\,dx \leq F(b) - F(b). \quad \text{---(2)}.$$

From (1) & (2), we get

$$\int_a^b F'(x)\,dx = \int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b \left(F'(x) - f(x)\right) dx = 0.$$

As $F'(x) - f(x) \geq 0$, we get

$$F'(x) = f(x) \text{ a.e.}$$

<u>Absolute Continuity:</u>

A function $f : [a,b] \to \mathbb{R}$ is said to be absolutely continuous, if for each $\epsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon,$$

whenever a disjoint family of intervals $\{(x_i, x_i')\}_{i=1}^{n}$ satisfies $\sum_{i=1}^{n} |x_i' - x_i| < \delta$.

<u>Lemma:</u> Every absolutely continuous function on $[a,b]$ is of B.V. on $[a,b]$.

Proof: Given that f is absolutely continuous, for $\epsilon=1$, $\exists\, \delta>0$ s.t.

$$\sum_{i=1}^{n} |f(y_i)-f(x_i)|<1,$$

whenever $\{(x_i,y_i)\}_{i=1}^{n}$ of disjoint intervals satisfies $\sum_{i=1}^{n} |y_i-x_i|<\delta$.

Let N be the smallest +ve integer s.t. $N>\frac{b-a}{\delta}$. Let $a_j = a+\frac{j(b-a)}{N}$, then

$$a_j-a_{j-1}=\frac{b-a}{N}<\delta.$$

Then any partition P of $[a_{j-1},a_j]$,

$\|P\|$ = maxi of length of subintervals $<\delta$.

It follows that $T_{a_{j-1}}^{a_j}(f)<1$; $j=1,2,\dots,N$.

Hence, $T_a^b(f) \le \sum T_{a_{j-1}}^{a_j}(f) < N$.

Cor: If f is absolutely continuous, then f is diff. a.e.

Lemma: If f is absolutely continuous on $[a,b]$ and $f'(x)=0$ a.e. then f = const.

Proof: We prove that $f(a)=f(c)$ for any $c\in[a,b]$.

Let $E=\{x\in(a,c): f'(x)=0\}$. Then $m(E)=c-a$. ($\because f'=0$ a.e.).

Let ϵ & η be arbitrary +ve numbers. For each $x\in E$, $\exists$ a small interval $[x,x+h]$ s.t

$$|f(x+h)-f(x)|<\eta h.$$

By Vitali Covering Lemma, $\exists$ a disjoint finite sub collection $\{[x_k,y_k]\}_{k=1}^{n}$ that cover all of E except a set A of measure less than $\delta>0$, where δ corresponds to ϵ in the def^n of absolute continuity of f.

If label $x_k<x_{k+1}$, then

$$y_0=a\le x_1<y_1<x_2<\cdots<y_n\le c=x_{n+1},$$

and $\displaystyle\sum_{k=0}^{n}|x_{k+1}-y_k|<\delta.$

Now, $\displaystyle\sum_{k=1}^{n}|f(y_k)-f(x_k)|\le\eta\sum_{k=1}^{n}(y_k-x_k)<\eta(c-a),$

by the way the intervals $\{[x_k,y_k]\}$ constructed, and

$$\sum_{k=1}^{n} |f(x_{k+1}) - f(y_k)| < \epsilon,$$ by the absolute continuity of f. Thus,

(20)

$$|f(c) - f(a)| = \Big| \sum_{k=1}^{n} [f(x_{k+1}) - f(y_k)] + \sum_{k=1}^{n} [f(y_k) - f(x_k)] \Big|$$

$$< \epsilon + \eta (c-b), \quad \forall \epsilon, \eta > 0$$

Hence, $f(c) = f(a)$, $\forall\, c \in [a,b]$.

<u>Theorem</u>: A function F is an indefinite integral iff it is absolutely continuous.

Pf: Let $F(x) = \int_a^x f(t)\,dt$. Then F is absolutely continuous.

Suppose F is absolutely continuous. Then F is of B.V., and

$$F(x) = F_1(x) - F_2(x),$$ where F_i are $\uparrow$ functions. Hence $F'(x)$ exists a.e. and

$$|F'(x)| \leq F_1'(x) + F_2'(x).$$

This implies that

$\int |F'(x)|dx \leq F_1(b)+F_2(b)-F_1(a)-F_2(a) < \infty.$

$\Rightarrow F'$ is integrable. Let

$$G(x) = \int_a^x F'(t)\,dt.$$

Then G is absolutely continuous and so is $f = F - G$. Hence,

$$f'(x) = F'(x) - G'(x) \quad \text{a.e.}$$
$$= F'(x) - F'(x) = 0 \text{ a.e.}$$

So by Lemma 8, f is constant.

Hence $F(x) = \int_a^x F'(t)\,dt + F(a)$.

Cor: Every absolutely continuous function is the indefinite integral of its derivative.

Remark: $$\int_a^b F'(t)\,dt = F(b) - F(a).$$

Differentiation of integrals in $\mathbb{R}^n$

Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Write

$$F(x) = \int_a^x f(y)\,dy, \text{ for } x \in [a,b].$$

Then

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(y)\,dy.$$

Let $I = (x, x+h)$, $|I| =$ length of I. Letting $|I| \to 0$, does it imply

$$\lim_{|I|\to 0} \frac{1}{|I|}\int_I f(y)\,dy = f(x) \quad \text{a.e. } x \,?$$

We can reformulate this problem by integrating function f on those intervals containing x. That is, does

$$\lim_{\substack{|I|\to 0 \\ x\in I}} \frac{1}{|I|}\int_I f(y)\,dy = f(x) \quad \text{a.e.}\,?$$

In an analogous way, similar question can be posed in the higher dim. spaces, for instance in $\mathbb{R}^n$ ($n \geq 2$).

Let $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$. Then $m(B_r(x)) = r^n m(B_1(0))$, since

Lebesgue outer measure m^* is translation and dilation invariant. If we denote

$$V_n = m(B_1(0)), \text{ then}$$

$$m(B_r(x)) = V_n r^n.$$

Suppose f is integrable on $\mathbb{R}^n$, and B denote a ball containing x.

Does

$$(*) \quad \lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x) \quad ?$$

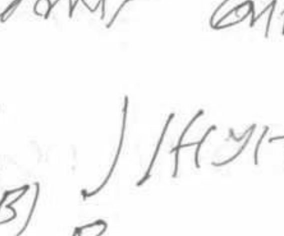
As an example, if f is continuous at $x\in\mathbb{R}^n$, then the above limit converges to $f(x)$.

$$\left| \frac{1}{m(B)} \int_B (f(y)-f(x))\,dy \right| \le \frac{1}{m(B)} \int_B |f(y)-f(x)|\,dy.$$

For $\epsilon>0$, $\exists\, \delta>0$ st

$$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\epsilon$$

If B is a ball of radius less than $\delta/2$, and containing x, then



$$\frac{1}{m(B)} \int_B |f(y)-f(x)|\,dy < \epsilon,$$

which is desire.

From (*) we can make an observation that the limit there could be result of taking supremum of a seqn of shrinking balls! This will give a way to define maximal function for $|f|$. (24)

Hardy-Littlewood maximal function:

For $f \in L^1(\mathbb{R}^n)$, define

$$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$

Here f^* is known as Hardy-Littlewood maximal function of $|f|$.

Theorem: Let $f \in L^1(\mathbb{R}^n)$, then

(i) f^* is measurable

(ii) $f^*(x) < \infty$ a.e. x

(iii) $m\{x \in \mathbb{R}^n : f^*(x) > \alpha\} \leq \frac{A}{\alpha} \|f\|_1$, where $A = 3^n$.

Before proving this result, we shall little focus on conclusion (iii). We shall see later that $f^*(x) \geq |f(x)|$ for a.e. x, however (iii) suggest that f^* is not much larger than $|f|$. And we may expect f^* is integrable, but that is not the case, as f^* of a non-zero function $f \in L^1(\mathbb{R}^n)$ decays too slowly at infinity. For this, let $\alpha > 0$, and $r = |x| > \alpha$. Then $B_\alpha(0) \subset B_{2r}(x)$, and (25)

$$f^*(x) \geq \frac{1}{m(B_{2r}(x))} \int_{B_{2r}(x)} |f(y)|\,dy$$

$$= \frac{C}{|x|^n} \int_{B_{2r}} |f(y)|\,dy \geq \frac{C}{|x|^n} \int_{B_\alpha(0)} |f(y)|\,dy.$$

But $\frac{1}{|x|^n}$ is not integrable on $\mathbb{R}^n \setminus B_\alpha(0)$. Hence, if $f^* \in L^1(\mathbb{R}^n)$, then $\int_{B_\alpha(0)} |f(y)| = 0$, $\forall \alpha$

$$\Rightarrow |f| = 0, \text{ a.e.}$$

Exercise: If $f(x) = \frac{1}{x(\log x)^2} \chi_{(0,\frac{1}{2})}(x)$, then $f \in L^1(\mathbb{R})$, but $f^* \notin L^1_{loc}(\mathbb{R})$.

For $0 < x < \frac{1}{2}$,

$$f^*(x) \geq \frac{1}{2x} \int_0^{2x} |f(y)|\,dy \geq \frac{1}{2x} \int_0^{x} |f(y)|\,dy$$

$$= \frac{1}{2x} \int_0^{x} \frac{1}{y(\log y)^2}\,dy \geq \frac{1}{2x\,|\log x|} \notin L^1_{loc}(\mathbb{R}).$$

Considering $F(x) = f(|x|)$, we can construct $F \in L^1(\mathbb{R}^n)$ with above properties via polar decomposition. 26

The inequality in (iii) is called weak inequality as it is weaker than corresponding inequality in L^1-norm, (due to Chebychev inequality).

$$m\{x : |f(x)| > \alpha\} = \frac{1}{\alpha}\int_{\mathbb{R}^n} \alpha \, \chi_{\{x : |f(x)| > \alpha\}}$$

$$\le \frac{1}{\alpha}\int_{\mathbb{R}^n} |f(x)| \, \chi_{\{x : |f(x)| > \alpha\}}$$

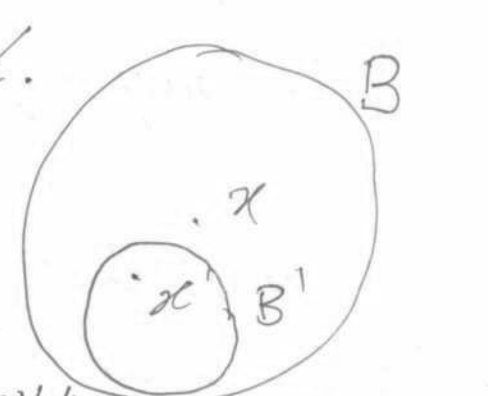
$$\le \frac{1}{\alpha} \|f\|_1 .$$

(i) f^* is a measurable function. To show this it is enough to show that the set

$$E_\alpha = \{x \in \mathbb{R}^n : f^*(x) > \alpha\} \text{ is open.}$$

For any $x \in E_\alpha$, $\exists$ an open ball B such that $x \in B$ and

$$\frac{1}{m(B)}\int_B |f(y)|\,dy > \alpha .$$



If x' is any point close enough to x, then $x' \in B$. Hence,

$$\sup_{x' \in B'} \frac{1}{m(B')}\int_{B'} |f(y)|\,dy \ge \frac{1}{m(B)}\int_B |f(y)|\,dy > \alpha, \text{ since}$$

B belongs to the family on which sup. is taken. It means, $x' \in E_\alpha$. That is, (27)
small ball surrounding x is contained in E_α.

(ii) If we assume (iii) for the time being, then $\{x : f^*(x) = \infty\} \subset \{x : f^*(x) > \alpha\}$, $\forall \alpha > 0$.

Hence, $m\{x : f^*(x) = \infty\} \leq \frac{1}{\alpha} \|f\|_1 \to 0$.

Finally proof of (iii) will be followed by the following lemma.

<u>Covering lemma:</u>

Let $\mathcal{B} = \{B_1, B_2, \ldots, B_N\}$ be a finite collection of balls in $\mathbb{R}^n$. Then $\exists$ a disjoint subcollection $\{B_{i_1}, B_{i_2}, \ldots, B_{i_K}\}$ of $\mathcal{B}$ s.t.

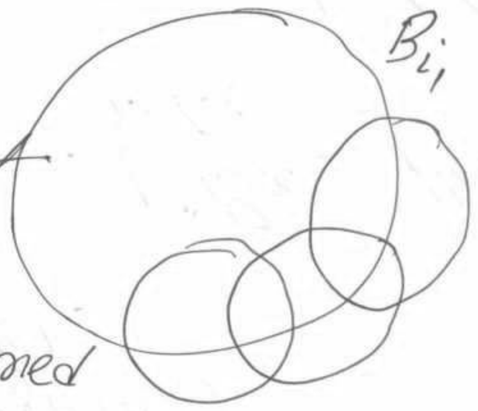
$$m\left(\bigcup_{l=1}^{N} B_l\right) \leq 3^n \sum_{j=1}^{K} m(B_{i_j}).$$

Proof: If all balls in $\mathcal{B}$ are disjoint, then okay. If not, let B & B' be two balls in $\mathcal{B}$ that intersect, and radius $(B) \geq$ radius (B').

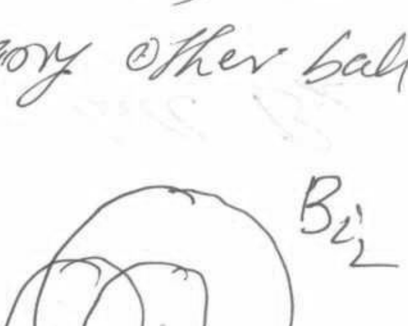
Then $B' \subset 3B = \tilde{B}$ (say), where

$\tilde{B}$ is the ball with centre same to B and radius 3 times than to B.

First, we pick a ball B_{i_1} in $\mathcal{B}$ with largest radius. Then delete B_{i_1} from $\mathcal{B}$ and any other ball that intersect B_{i_1}. Then all the deleted balls are contained in $\tilde{B}_{i_1}$.



The remaining balls yield a new collection say $\mathcal{B}'$, for which we repeat the procedure. We pick up ball B_{i_2} in $\mathcal{B}'$ with largest radius, and delete the ball B_{i_2} and any other ball intersecting B_{i_2}.



Continuing this way we find after atmost N steps, a collection of disjoint balls $B_{i_1}, B_{i_2}, \ldots B_{i_K}$. Let $\tilde{B}_{i_j} = 3 B_{i_j}$. Since any ball $B \in \mathcal{B}$ must intersect some of B_{i_j}, and here B has equal or small radius than B_{i_j}.

Therefore, we must have $B \subset \breve{B}_{ij}$

That is, $\bigcup_{\ell=1}^{N} B_\ell \subset \bigcup_{j=1}^{K} \breve{B}_{ij}$. (21)

$$\Rightarrow m\left(\bigcup_{\ell=1}^{N} B_\ell\right) \le m\left(\bigcup_{j=1}^{K} \breve{B}_{ij}\right)$$

$$\le 3^n \sum_{j=1}^{K} m(B_{ij})$$

$$= 3^n\, m\left(\bigcup_{j=1}^{n} B_{ij}\right).$$

<u>Proof of (iii):</u>

If $x \in E_\alpha$, then $\exists$ a ball B_x containing x s.t

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|\,dy > \alpha$$

$$\Rightarrow m(B_x) < \frac{1}{\alpha} \int_{B_x} |f(y)|\,dy$$

Since m is inner regular,

$$m(E_\alpha) = \sup_{K \subset E_\alpha} m(K).$$

Let $K \subset E_\alpha$. Then $K \subset \bigcup_{x \in E_\alpha} B_x$. Hence,

$$K \subset \bigcup_{\ell=1}^{N} B_\ell .$$

By covering lemma, $\exists$ disjoint balls $B_{i_1}, \ldots, B_{i_K}$ s.t

$$m(K) \le m\left(\bigcup_{\ell=1}^{N} B_\ell\right) \le 3^n \sum_{j=1}^{K} m(B_{ij})$$

$$\leq \frac{3^n}{\alpha} \sum_{j=1}^{K} \int_{B_{ij}} |f(y)|\,dy = \frac{3^n}{\alpha} \int_{\cup B_{ij}} |f(y)|\,dy$$

i.e. $$m(K) \leq \frac{3^n}{\alpha} \|f\|_1 .$$ 30

<u>Lebesgue differentiation theorem</u>:

If $f \in L^1(\mathbb{R}^n)$, then

$$\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x) \quad \text{a.e. } x.$$

proof: It is enough to show that for each $\alpha > 0$, the set

$$N_\alpha = \left\{ x : \varlimsup_{\substack{m(B)\to 0 \\ x\in B}} \left| \frac{1}{m(B)} \int f(y)\,dy - f(x) \right| > 2\alpha \right\}$$

has measure zero. Since $f \in L^1(\mathbb{R}^n)$, for each $\epsilon > 0$, $\exists\, g \in C_c(\mathbb{R}^n)$ s.t

$$\|f - g\|_1 < \epsilon .$$

Since g is continuous, for each $x \in \mathbb{R}^n$, we have

$$\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B g(y)\,dy = f(x).$$

Now,

$$\frac{1}{m(B)} \int_B f(y)\,dy - f(x) = \frac{1}{m(B)} \int (f(y) - g(y)) + \frac{1}{m(B)} \int_B g(y)\,dy - g(x) + g(x) - f(x).$$

Hence,

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \left|\frac{1}{m(B)}\int_B f(y)\,dy - f(x)\right| \leq (f-g)^*(x) + |f(x)-g(x)|. \quad \text{---}(*)$$

let $G_\alpha = \{x : |f(x)-g(x)| > \alpha\}$ and (31)

$F_\alpha = \{x : (f-g)^*(x) > \alpha\}$.

Then $N_\alpha \subset F_\alpha \cup G_\alpha$. (from $(*)$).

By Chebyshev inequality,

$$m(G_\alpha) \leq \frac{1}{\alpha}\|f-g\|_1,$$

and by weak inequality

$$m(F_\alpha) \leq \frac{A}{\alpha}\|f-g\|_1$$

$$\Rightarrow m(E_\alpha) < \frac{A}{\alpha}\epsilon + \frac{1}{\alpha}\epsilon, \quad \forall \epsilon > 0$$

Thus, $m(E_\alpha) = 0$.

Applying the above result to $|f|$,

$$f^*(x) = \sup_{x\in B} \frac{1}{m(B)}\int_B |f(y)|\,dy \geq \lim_{\substack{x\in B\\ m(B)\to 0}} \frac{1}{m(B)}\int |f(y)|\,dy = |f(x)|.$$

Hence, $f^*(x) \geq |f(x)|$, a.e. x.

Since differentiation is a local notion, and we are considering behaviour of function f on balls which shrink to a point x, it enough for function f be locally integrable. $f \in L^1_{loc}(\mathbb{R}^n)$ if f is integrable over each compact subset of $\mathbb{R}^n$! that is, $f\chi_K \in L^1(\mathbb{R}^n)$, for $K \in \mathbb{R}^n$!

32

Cor: If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)} \int_B f(y)\,dy = f(x) \quad \text{a.e. } x.$$

(Hint: Consider LDT for $\chi_K f \in L^1(\mathbb{R}^n)$.)

If E is a measurable set, then by the above Corollary for $\chi_E \in L^1_{loc}(\mathbb{R}^n)$,

$$\lim_{\substack{m(B)\\ x\in B}} \frac{m(B\cap E)}{m(B)} = 1, \quad \text{a.e. } x$$

That is, small balls around x are almost entirely covered by E. Moreover, if $0<\epsilon<1$, then $\exists$ ball B containing x s.t

$$m(B\cap E) > (1-\epsilon)\, m(B).$$

That is, E covers at least $1-\epsilon$ part of B.