REPRESENTATIONS OF FINITE GROUPS

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by

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to the

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CERTIFICATE

This is to certify that the work contained in this project report entitled "**Representations of finite groups**" submitted by **Tathagat Lokhande** (**Roll No.: 120123024**) to the Department of Mathematics, Indian Institute of Technology, Guwahati, towards the requirement of the course **MA498, Project-II** has been carried out by him under my supervision.

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ABSTRACT

The aim of this project is to study the representations of a finite group. The idea is to know an expansion of a function defined on a group G in terms of elementary function on the group G. These elementary functions are obtained on the basis of the fact that how the group G act on a vector space V. These elementary functions are orthogonal to themselves. Essentially, when a finite group acts on a vector space V, it acts only on a finite subspace of V. The minimal subspace which is stable under the action of G is called irreducible representation of G. Each irreducible representation of G gives rise to an elementary function on G. Therefore, it is natural to study only irreducible representations. Schur's lemma is the main source of getting all irreducible representations of G.

Contents

Chapter 1

Introduction

In this chapter, we will set up a few basic notations, definitions and some preliminary results for study the representations of a finite group.

1.1 Representation of a finite group

We will first illustrate the idea of representing a group by matrices through finite group. Let $G = \{1, \omega, \omega^2\}$ and $V = \mathbb{C}$. Can the group G act on the linear space V? Consider $1 \mapsto 1.z = z$, $\omega \mapsto \omega.z = \omega z$ and $\omega^2 \mapsto \omega^2.z = \omega^2 z$. Then the map $\pi : G \to GL(\mathbb{C}) \cong \mathbb{C}^*$ is a group homomorphism. That is, $\pi(gh) = \pi(g).\pi(h)$.

Let G be a group which act on a linear space V. That is, $G.V \subseteq V$. A homorphism $\pi : G \to GL(V)$ such that $\pi(gh) = \pi(g)\pi(h)$ is called a representation of group G.

(a) Since, $\pi(e) = \pi(e.e) = \pi(e).\pi(e)$, it implies that $\pi(e)(I - \pi(e)) = 0$. Hence, $\lambda - \lambda^2 = 0$. If $\lambda = 0$, then $\pi = 0$, which is a contradiction. Hence $\pi(e) = I$.

(b) $\pi(s^{-1}) = [\pi(s)]^{-1}$ for all $s \in G$.

Remark 1.1.1. Using the forthcoming schur's lemma, it is enough to consider finite vector space for any worthwhile representation of a finite group. If G is finite group and V is a finite dimensional space then for homomorphism $\pi : G \to GL(V)$, the degree of π = dimension of V.

Example 1.1.2. Let π be an 1-d representation of finite group G of order k. Then $\pi : G \to GL(\mathbb{C}) \cong \mathbb{C}^*$. That is, $\pi(g^k) = \pi(e) = |\pi(g)| = 1, \ \forall g \in G$. This implies that G can have at most k many 1-dim representations.

Definition 1.1.3. A subspace $W \subseteq V$ is called stable (or invariant) under π if $\pi(G)W \subset W$. Eventually, this is a process that enable to cut the size of representation space only to acted vectors in V.

Definition 1.1.4. Let W be an invariant space for representation (π, G). Then (π_W, G) is called a sub-representation of (π, G) if $\pi_W(gh) = \pi_W(g)\pi_W(h)$, where $\pi_W(g) = \pi(g)|_W$.

Theorem 1.1.5. (Maschke's theorem) Let $\pi : G \to GL(V)$ be a representation of a finite group G and W be an π-invariant subspace of V. Then, there exists a π-invariant subspace $W_0 \subseteq V$ such that $V = W \oplus W_0$.

Proof. Let W' be a complementary subspace of W in V and $P : V \to W$ be a projection. Define $P_0 = \frac{1}{k} \sum\limits_{t \in G} \pi(t) p \, \pi^{-1}(t)$. Then for $x \in V$, we have $P_0 x = \frac{1}{k} \sum\limits_{t \in G} \pi(t) p \, \pi^{-1}(t) x \in W$. Thus, $P_0 x$ is a projection of V onto W. That is, P_0 is a projection of V onto W corresponding to some complement W_0 of W. Now, we have

$$\begin{aligned}
\pi(s) P_0 \pi^{-1}(t) &= \frac{1}{k} \sum_{t \in G} \pi(s)\pi(t) p \pi^{-1}(t) \pi^{-1}(s) \\
&= \frac{1}{k} \sum_{t \in G} \pi(st) p \pi^{-1}(st) \\
&= P_0.
\end{aligned}$$

If $x \in W_0$, then $P_0 x = 0$, which in turn implies that $P_0(\pi(s)x) = \pi(s)P_0 x = \pi(s)(0) = 0$. Hence, $\pi(s)x \in W_0, \forall\, s \in G$. Thus, W_0 is a π-invariant subspace of V and $W \oplus W_0 = V$. Notice that the linear complement W_0 is not unique. □

Definition 1.1.6. A representation $\pi : G \to GL(V)$ is called irreducible if the π-invariant subspace of V are $\{0\}$ and V. Let $\pi : G \to GL(V_n)$ and $\pi' : G \to GL(V'_m)$ be two representation of g. Then,

$$(\pi \oplus \pi')(g) = \pi(g) \oplus \pi'(g) \text{ and } (\pi \oplus \pi')(g)(V + V') = (\pi(g)(V), \pi(g)(V')).$$

That is, $(\pi \oplus \pi')(g) = \begin{bmatrix} \pi(g) & 0 \\ 0 & \pi'(g) \end{bmatrix}$. Thus $g \mapsto \begin{bmatrix} \pi(g) & 0 \\ 0 & \pi'(g) \end{bmatrix}$.

Now, question that whether a representation be the direct sum of irreducible representations? Suppose G is a finite group, then we will see that any finite dimension representation of G can be decomposed as the finite direct sum of irreducible representations of G.

Definition 1.1.7. A representation is said to be completely reducible if it is the direct sum of irreducible representations.

Theorem 1.1.8. *Let G be a finite group. Then every finite dimension representation of G is the direct sum of irreducible representations.*

Proof. Let $\pi : G \to GL(V)$ be a finite dimensional representation of G. If $V = 0$, then π is trivially irreducible. Suppose $\dim V \geq 1$. Since every one dimension representation is irreducible, therefore, we can assume that the result is true for $\dim V = n - 1$. By Maschke's theorem, $V = V_1 \oplus V_2$, where $\pi(G)(V_i) \subseteq V_i$ and therefore, $\dim V_i \leq n - 1$, for $i = 1, 2$. Hence, $V = V_1 \oplus V_2$. □

Example 1.1.9. Let $G = \mathbb{Z}$ and $V = \{(a_1, a_2, \ldots) : a_i \in \mathbb{R}\}$ be the sequence space. Define, $\pi(n)(a_1, a_2, \ldots) = (0, 0, \ldots, 0, a_1, a_2, \ldots)$. Then π has no invariant subspace. Hence Maschke's theorem fails in this case.

Example 1.1.10. Let $G = \mathbb{R}$ and $V = \mathbb{R}^2$. Define $\pi : G \to GL(\mathbb{R}^2)$ by

$$\pi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

Then, invariant subspaces of V are 0 and span $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus π is not completely reducible.

Definition 1.1.11. Let $\pi_i : G \to GL(V_i),\ i = 1, 2$ be two representations of G and $T : V_1 \to V_2$ be a linear map such that $T \circ \pi_1(g) = \pi_2(g) \circ T, \forall g \in G$. Then, T is said to be intertwining map. The set of all intertwining map is denoted by $\mathrm{Hom}_G(\pi_1, \pi_2)$. Suppose π_1 and $\pi_2 \in \hat{G}$ (set of all irreducible representations up to an isomorphism), then any $T \in \mathrm{Hom}_G(\pi_1, \pi_2)$ is either 0 or isomorphism.

Lemma 1.1.12. (Schur's lemma) *Let* $\pi_1, \pi_2 \in \hat{G}$ *and* $T \in Hom_G(\pi_1, \pi_2)$. *Then,*

(a) $T = 0$ *or* T *is an isomorphism, and*

(b) if $\pi_1 \circ T = T \circ \pi_1,\ \forall\, t \in G$. *Then,* $T = \lambda I$ *for some* $\lambda \in \mathbb{C}$.

Proof. Suppose $T \neq 0$, then write $W_i = \{x \in V_i : Tx = 0\},\ i = 1, 2$. For $x \in W_1,\ T \circ \pi_1(t)x = \pi_2(t) \circ Tx = 0$ this means $\pi_1(t)x \in W_i,\ \forall x \in W_i, t \in G$ this implies $\pi_1(G)W_1 \subseteq W_1$. Since π_1 is irreducible either $W_1 = 0$ or V_1 and $\ker T = \{0\}$ or V_1.

For the proof of second part, let λ be an eigenvector of T and denote $T' = T - \lambda I$. Then $\ker T' \neq \{0\}$. It is easy to see that $T' \circ \pi_1(t) = \pi_2(t) \circ T'$. Thus, from (1), it follows that $T' = 0$, making $T = \lambda I$. □

Corollary 1.1.13. *Any irreducible representation of an abelian group G (need not be finite) is 1-dimensional.*

Proof. Let G be a abelian group. Then, it follows that $\pi(gh) = \pi(hg)$ and hence $\pi(g)\pi(h) = \pi(h)\pi(g)$. For fixed h, we have $\pi(h) \in \mathrm{Hom}_G(\pi, \pi)$. Theofore, by Schur's lemma, we obtain $\pi(h) = \lambda I$. That is, π leaves invariant every 1-dimensional subspace of V. Since, π is irreducible, it implies that $\dim V = 1$. □

Theorem 1.1.14. *Let G be a finite group. Then every irreducible representation of G is 1-dimensional if and only if G is abelian.*

Proof. Suppose all irreducible representations of G is 1-dimensional. Consider the left regular representation $L : G \to GL(V)$, where $V = \mathbb{C}([G])$ is the linear space whose basis element are the members of G. Since $L(g)(h) = gh$, by Maschke's theorem, it follows that L is completely reducible and that $V = \bigoplus_{i=1}^{m} V_i$, where $V_i's$ are irreducible. By hypothesis, $\dim V_i = 1$, therefore, $L(g)$ is a diagonal matrix for all $g \in G$. That is, every element of G is represented by a diagonal matrix. Hence, $L(G) \cong G$ which will imply that group G is abelian. Converse part is followed by the above corollary. □

For $f, g : G \to \mathbb{C}$, define $\langle f, g\rangle := \frac{1}{k} \sum_{t \in G} f(t)g(t^{-1})$.

Theorem 1.1.15. *Let $\pi_1, \pi_2 \in \hat{G}$ with $\dim V_i = n_i,\ i = 1, 2$. Let $\pi_1(t) = [a_{ij}(t)]$ and $\pi_2(t) = [b_{ij}(t)]$. Then,*

(a) $\langle a_{il}, b_{mj}\rangle = 0,\ \forall i, j, m, l$ *and*

(b) $\langle a_{il},\, a_{mj}\rangle = \frac{1}{n}\delta_{ij}\delta_{lm}$.

Proof. For $T : V_1 \to V_2$ to be a linear map, define an averaging linear map on V_1 by

$$T_0 = \frac{1}{k}\sum \pi_1(t)T \circ \pi_2(t^{-1}).$$

Then, $T_0 \in \mathrm{Hom}_G(\pi_1, \pi_2)$. By schur's lemma, we get $T_0 = 0$. That is,

$$\frac{1}{k}\sum_{t\in G}\sum_{l,m} a_{il}(t)x_{lm}b_{mj}(t^{-1}) = 0,$$

where $T = (x_{lm})$. Since T is arbitrary, we get

$$\frac{1}{k}\sum_{t\in G} a_{il}(t)b_{mj}(t^{-1}) = 0.$$

That is, $\langle a_{il}, b_{mj}\rangle = 0$.

Now, for a linear map $T_1 : V_1 \to V_2$, we define $T_0 = \frac{1}{k}\sum\limits_{t\in G}\pi_1(t)T\pi_1(t^{-1})$. Then, $T_0 \in \mathrm{Hom}_G(\pi_1, \pi_1)$. By Schur's lemma, $T_0 = \lambda I$, for some $\lambda \in \mathbb{C}$, where $\lambda = \frac{1}{n_1}\mathrm{tr}(T_0) = \frac{1}{n_1}$ tr (T). Thus, $\lambda = \frac{1}{n_1}\sum\limits_{l} x_{ll} = \frac{1}{n_1}\sum\limits_{lm} x_{lm}\delta_{lm}$. Observe that, the $(ij)^{th}$ entry of the matrix T_0 satisfies

$$\frac{1}{k}\sum_{t\in G} a_{il}(t)x_{lm}a_{mj}(t^{-1}) = \lambda\delta_{ij} = \frac{1}{n_1}\sum_{lm} x_{lm}\delta_{lm}\delta_{ij}.$$

Since T is arbitrary by comparing the coefficients of x_{lm}, we set

$$\frac{1}{k}\sum_{t\in G} a_{il}(t)a_{mj}(t^{-1}) = \frac{1}{n_1}\sum_{lm}\delta_{lm}\delta_{ij}.$$

That is, $\langle a_{il}, a_{mj}\rangle = \frac{1}{n_1}\sum\limits_{lm}\delta_{lm}\delta_{ij}$.

□

Chapter 2

Character theory

In this Chapter, we will construct a set of scalar valued functions from the irreducible representations of a finite group G. These functions play the role of building blocks to get an orthonormal expansion of a function on G.

2.1 Character of a representation

Let (π, V) be a representation of a group G.

Definition 2.1.1. A function $\chi_\pi : G \to \mathbb{C}$ that satisfies $\chi_\pi(gh) = \chi_\pi(g)\chi_\pi(h)$, whenever $g, h \in G$ is called the character of representation π.

Proposition 2.1.2. *For finite group G, let $\chi_\pi(g) = tr\,(\pi(g))$. Then*

(a) $\chi_\pi(1) = n$,

(b) $\chi_\pi(t^{-1}) = \overline{\chi(t)}$, *for all* $t \in G$,

(c) $\chi_\pi(tst^{-1}) = \chi_\pi(s)$, *for all* $s, t \in G$,

(d) $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$.

Proof. We have $\chi_\pi(e) = \mathrm{tr}(\pi(e)) = \mathrm{tr}(I) = n = \dim V$.

Since G is finite of order$k$, therefore, $g^m = e$, for some $m \in \mathbb{N}$ with $m \leq k$. Hence $(\pi(g))^k = I$. This in turn implies that $\lambda^k = 1$ and hence $|\lambda| = 1$. Thus

$$\mathrm{tr}(\pi(t^{-1})) = \sum_{i=1}^{k} \lambda_i^{-1} = \sum_{i=1}^{k} \overline{\lambda_i} = \overline{\mathrm{tr}(\pi(t))}.$$

We know that $\mathrm{tr}(ST) = \mathrm{tr}(TS)$, therefore, $\mathrm{tr}(STS^{-1}) = \mathrm{tr}(T)$. This implies that $\mathrm{tr}(\pi(t)\pi(s)\pi(t^{-1})) = \mathrm{tr}(\pi(s))$. That is, $\mathrm{tr}(\pi(tst^{-1})) = \mathrm{tr}(\pi(s))$. Hence $\chi_\pi(tst^{-1}) = \chi_\pi(s)$.

Finally, since we know that $(\pi_1 \oplus \pi_2)(g) = \pi_1(g) \oplus \pi_2(g)$, therefore, it is easily followed that $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$.

□

Now, for $\phi, \psi : G \to \mathbb{C}$, we define an inner product by $(\phi, \psi) = \frac{1}{k} \sum_{t \in G} \phi(t)\overline{\psi(t)}$ and $\check{\phi}(t) = \overline{\phi(t^{-1})}$. Then $\check{\chi_\pi}(t) = \overline{\chi_\pi(t^{-1})} = \chi_\pi(t)$. Thus, we can write

$$(\phi, \chi) = \frac{1}{k} \sum_{t \in G} \phi(t)\overline{\chi(t)} = \frac{1}{k} \sum_{t \in G} \phi(t)\psi(t^{-1}) = \langle \phi, \chi \rangle.$$

Theorem 2.1.3. *If χ_π is the character of representation π of group G then,*

(a) $(\chi_\pi, \chi_\pi) = 1$.

(b) $(\chi_{\pi_1}, \chi_{\pi_2}) = 0$, *whenever* $\pi_1, \pi_2 \in \hat{G}$.

Proof. In view of Theorem 1.1.15, we have

$$(\chi_\pi, \chi_\pi) = \sum_i \sum_j \langle a_{ii}, a_{jj} \rangle = \sum_i \sum_j \frac{\delta_{ij}}{n} = 1.$$

Also, we have

$$(\chi_{\pi_1}, \chi_{\pi_2}) = \sum_i \sum_j \langle a_{ij}, a_{ji} \rangle = 0.$$

□

The character corresponding to irreducible representations are called irreducible characters or simple characters and they form an orthonormal basis.

Theorem 2.1.4. *Let ϕ be the character of a representation π of a finite group G. If $\pi' \in \hat{G}$, then the multiplicity of irreducible representation that appear in the representation π is $\langle \phi, \chi_{\pi'} \rangle$.*

Proof. Since $\phi = \chi_\pi = \sum_{i=1}^{m} \chi_{\hat{\pi}_i}$, we get $\langle \phi, \chi_{\pi'} \rangle = \sum_i \langle \chi_{\hat{\pi}_i}, \chi_{\pi'} \rangle$. That is, the number of irreducible representation that appear in π. □

Corollary 2.1.5. *The number of irreducible representation of G is independent of the choice of a decomposition of π.*

Proof. Let $\{\hat{\pi}_i : i = 1, 2, \dots, m\}$ be a irreducible representation of a finite group G and $\{\chi_j : j = 1, 2, \dots, m\}$ be their irreducible characters, then π can be decomposed as $\pi = \bigoplus_{i=1}^{m} m_i \, \hat{\pi}_i$. □

Corollary 2.1.6. *Let (π, V) and (π', V') be two representation of finite group G and χ and χ' be their characters then $\pi \cong \pi'$ if and only if $\chi = \chi'$.*

Proof. If $\pi \cong \pi'$. Then there exists $T \in \mathrm{Hom}_G(V, V')$ with $\pi'(g) \circ T = T \circ \pi(g)$. That is, $\pi'(g) = T \circ \pi(g) T^{-1}$. Then by computing trace of both the sides, we get $\mathrm{tr}(\pi'(g)) = \mathrm{tr}(T \circ \pi(g) T^{-1})$. This implies $\mathrm{tr}(\pi'(g)) = \mathrm{tr}(\pi(g))$. Hence $\chi'(g) = \chi(g), \ \forall g \in G$. Conversely suppose $\chi' = \chi$, then $\pi = \sum_{i=1}^{m} m_i \hat{\pi}_i$ and $\pi' = \sum_{i=1}^{m} n_i \hat{\pi}_i$. Then by comparing characters of π and $\hat{\pi}$ we get

$$\sum_{i=1}^{m} (m_i - n_i) \hat{\chi}_i = 0.$$

Since $\{\hat{\chi}_i\}_{i=1}^{m}$ forms an orthonormal set, therefore, it follows that $m_i = n_i$. Thus we infer that $\pi \cong \pi'$. □

Theorem 2.1.7. (Irreducibility criteria) Let χ_π be the character of a representation π of finite group G. Then $\langle \chi_\pi, \chi_\pi \rangle$ is a positive integer and $\langle \chi_\pi, \chi_\pi \rangle = 1$ if and only if $\pi \in \hat{G}$.

Proof. Since, we know that $\pi = \bigoplus_{i=1}^{m} m_i \hat{\pi}_i$, where $\hat{\pi}_i \in \hat{G}$. It follows that $\chi = \sum_{i=1}^{m} m_i \chi_i$, where χ_i is the character of π_i. Hence $\langle \chi, \chi \rangle = \sum_{i=1}^{m} m_i^2 > 0$. Now, $\langle \chi, \chi \rangle = 1$ if one of the $m_i's = 1$ and rest other $m_i = 0$. That is, $\langle \chi, \chi \rangle = 1$ if and only if π is irreducible. □

2.2 Decomposition of regular representation

Let G be a finite group and $V = \mathbb{C}(G)$ be the vector space with element of G as the basis. Define $L : G \to GL(V)$ such that $L(t)(s) = ts$. If χ_L is the character of L, then $\chi_L(e) = \text{tr}(I) = k = o(G)$. If $s \neq e$, then we get $st \neq t$, $\forall t \in G$ and $[L(s)] = [a_{ij}(s)]$, where $a_{ij}(s) = \langle se_j, e_i \rangle$, $e_j, e_i \in G$. Then $a_{ij}(s) = \langle (s)e_j, e_i \rangle = \langle st, s \rangle = 0$.

Theorem 2.2.1. *If L is the left regular representation of finite group G, then*

(a) $\chi_L(e) = k$ *and*

(b) $\chi_L(s) = 0$, *if* $s \neq e$.

Corollary 2.2.2. *Every $\pi \in \hat{G}$ is contained in the regular representation L with multiplicity equals to* $\dim \pi$.

Proof. Multiplicity of $\pi = \langle \chi_L, \chi_\pi \rangle = \frac{1}{k} \sum_{t \in G} \chi_L(t^{-1}) \chi_\pi(t) = \frac{1}{k} k \chi_\pi(e) = \dim \pi$. □

If $L = \bigoplus_{i=1}^{m} \pi_i$, where $\pi_i \in \hat{G}$, $< \chi_L, \chi_i >= n_i$ and $n_i = \dim V_i$. Then in view of Theorem 2.2.1, we get $\chi_L(s) = \sum_{i=1}^{m} n_i \chi_i(s)$. Hence $\sum_{i=1}^{m} n_i^2 = \chi_L(e) = k$.

If $s \neq e$, then $\sum_{i=1}^{m} n_i \chi_i(s) = \chi_L(s) = 0$. Thus, we conclude that a necessary and sufficient condition for $L = \bigoplus_{i=1}^{m} \pi_i$ is that $\sum_{i=1}^{m} n_i^2 = k$.

Proposition 2.2.3. *Let $f : G \to \mathbb{C}$ be a class function and for $\pi \in \hat{G}$, define $\pi_f = \sum_{t \in G} f(t)\pi(t)$. Then $\pi_f = \frac{k}{n}(f, \chi^*)I = \frac{1}{n} \sum_{t \in G} f(t)\chi(t)$.*

Proof. We have $\pi(s^{-1})\pi_f\pi(s) = \sum_{t \in G} f(t)\pi(s^{-1})\pi(t)\pi(s) = \sum_{t \in G} f(t)\pi(sts^{-1})\pi_f$. By Schur's lemma, $\pi_f = \lambda I$, and hence $n\lambda = \sum_{t \in G} f(t)\chi_\pi(t)$. We conclude that $\lambda = \frac{1}{n} \sum_{t \in G} f(t)\chi_\pi(t) = (f, \chi_\pi^*)$. □

Let $\mathcal{H}$ be the space of all class function of G then, $\{\chi_{\pi_i} : \pi_i \in \hat{G}\} \subset \mathcal{H}$.

Theorem 2.2.4. *The set $\{\chi_\pi : \pi \in \hat{G}.\}$ forms an orthonormal basis for the space $\mathcal{H}$ of all the class functions on G.*

Proof. Let $f \in H$ such that $f \perp \chi_i *$. Write $\pi_f = \frac{k}{n}(f, \chi_\pi *)$, if $\pi \in \hat{G}$. Since G is a finite group, therefore, it follows that $\pi = \bigoplus_{i=1}^{m} n_i\pi_i$ and hence $\pi(f) = \sum_{i=1}^{m} \frac{k}{n_i}(f, \chi_\pi *)n_i = 0$. Let L be the left regular representation of G. Then $L = \bigoplus_{i=1}^{m} n_i\pi_i$ and $L_f = 0$. That is, $L_f(e) = \sum_{t \in G} f(t)L_t(e) = \sum_{t \in G} f(t)t = 0$. Since $V = \mathbb{C}(G)$, we infer that $f(t) = 0$, $\forall\, t \in G$. Hence, $f = 0$. This complete the proof. □

Chapter 3

Haar measure on topological group

3.1 Topological groups

In this Chapter, we study the question that how one can impose a topology structure on group which is compatible with the group law. Further, we see the existence of a measure which invariant under group action.

Definition 3.1.1. A group G is called a topological group if for $x, y \in G$, $(x, y) \to xy$ and $x \to x^{-1}$ are continuous.

Example 3.1.2. $GL(n, \mathbb{R}) \subset L(n, \mathbb{R})$ is topological group. Since the map $\det : L(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is continuous and $f^{-1}(\mathbb{R} \setminus \{0\}) = GL(n, \mathbb{R})$ is open, therefore, the topology of $GL(n, \mathbb{R})$ can be obtained from topology of $L(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$.

Definition 3.1.3. A topological group is called homogeneous if for $x, y \in G$, there exists isomorphism $f : G \to G$ such that $f(x) = y$.

For for any two sets A, $B \subseteq G$, define $AB = \{st : (s,t) \in A \times B\}$ and $A^{-1} = \{s^{-1} : s \in A\}$. A set $A \subset G$ is called symmetric if $A^{-1} = A$. Notice that if $A \cap B = \emptyset$ if and only if $e \notin AB^{-1}$ or $A^{-1}B$.

Proposition 3.1.4. *Let G be a topological group.*

(a) If O is open, then so is xO and O^{-1} for $x \in G$.

(b) For a neighborhood N of e, there exists a symmetric neighborhood V of e such that $VV \subset N$.

(c) If H is subgroup of G, then $\bar{H}$ is also a subgroup of G.

(d) Every open subgroup of G is closed.

(e) If A and B are compact sets in G, then AB is also compact.

Proof. (a) As $(x,y) \to xy$ and $x \to x^{-1}$ are homeomorphism, hence xO and x^{-1} are open and $AO = \bigcup_{a \in A} aO$ is also open.

(b) Since $\phi : (x,y) \to xy$ is continuous at e. Therefore, there exists open sets V_1 and V_2 such that $V_1V_2 \subset N$. Since $\phi^{-1}(O) = V_1 \times V_2$, then $\phi(V_1 \times V_2) \subset O$. That is, $V_1V_2 \subset O \subset N$.

(c) If $x, y \in \bar{H}$, then there exist nets $\{x_\alpha\}$ and $\{y_\beta\}$ in H such that $x_\alpha \to x$ and $y_\beta \to y$. Therefore, it follows that $x_\alpha y_\beta \to xy$ and $x_\alpha^{-1} \to x^{-1}$. Since $\bar{H}$ is closed, it implies that xy, $x^{-1} \in \bar{H}$.

(d) Let H be open and $G \backslash H = \bigcup xH$. Let $x \in G \backslash H$. If $xh \in H$, then $xhh^{-1} \in H$. Since $H \leq G$, it implies that $x \in H$, which is a contradiction. Thus, $xH \in G \backslash H$, $\forall\, x \notin H$. As $G \backslash H = \bigcup xH$ and $G \backslash H$ is open. Thus H is closed.

(e) $A \times B \mapsto AB$ under $(x, y) \to xy$ this means that AB is compact.

□

Lemma 3.1.5. *If F is closed and K is compact such that $F \cap K = \emptyset$. Then there exists an open neighborhood V of e such that $F \cap VK = \emptyset$.*

Proof. Let $x \in K$, then $x \in G \setminus F$ and $G \setminus F$ is open and thus $(G \setminus F)x^{-1}$ is an open neighborhood of e. Thus there exists an open neighborhood V_x of e such that $V_xV_x \in (G \setminus F)x^{-1}$. Now, $K \subset \bigcup\limits_{x \in K} V_x x$ implies $K \subset \bigcup\limits_{i=1}^{n} V_i x_i$. Let $V = \bigcap\limits_{i=1}^{n} V_i$. Then for $x \in K$, it follows thar $Vx \subset V_{x_i}V_{x_i}x_i \subset G \setminus F$. Hence, $F \cap Vx = \emptyset, \forall x \in K$. So, $F \cap VK = \emptyset$. □

Proposition 3.1.6. *If F is closed and K is compact in a topological group G. Then FK is closed.*

Proof. The case $FK = G$ is trivial. Now, let $y \in G \setminus FK$. Then $F \cap yK^{-1} = \emptyset$. Since $x \in F \cap yK^{-1}$ this implies $x = yK^{-1}$. So, $y = xK$. By previous lemma, there exists an open neighborhood V of e such that $F \cap VyK^{-1} = \emptyset$ that is $FK \cap Vy = \emptyset$. So, we can say that $Vy \subset G \setminus FK$. Thus $G \setminus FK$ is open and hence FK is closed. □

For a subgroup H of topological group G, we write $G \setminus H = \{xH : x \in G\}$. Then the canonical quotient map $q : G \to G \setminus H$ is continuous in the sense that $V \subset G \setminus H$ is open iff $q^{-1}(V)$ is open in G. Moreover, q sends an open set to open set. Let V be open in G, then $q^{-1}(q(V)) = VH$(Open in G). So, $q(V)$ is open in $G \setminus H$. Hence q is an open map.

Proposition 3.1.7. *Let H be a subgroup of topological group G. Then,*

(a) If H is closed, $G \setminus H$ is T_2.

(b) If G is locally compact, then $G \setminus H$ is also locally compact.

(c) If H is a normal subgroup of G, then $G \setminus H$ is a topological group.

Proof. (a) Let $\bar{x} = q(x)$ and $\bar{y} = q(y)$ are distinct in $G \setminus H$. Since H is closed xHx^{-1} is closed and $e \notin Hy^{-1}$. Therefore, there exists a symmetric neighborhood V of e such that $VV \cap xHy^{-1} = \emptyset$. Since $V = V^{-1}$ and $H = HH$, since H is a subgroup. that means $e \notin VxH(Vy)^{-1} = VxH(VyH)^{-1}$. Hence, $VxH \cap VyH = \emptyset$. Thus $q(Vx)$ and $q(Vy)$ are distinct open sets.

(b) If V is a compact neighborhood of e, then $q(Vx)$ is a compact neighborhood of $q(x)$ in $G \setminus H$.

(c) If $x, y \in G$ and V is neighborhood of $q(xy)$ in $G \setminus H$, then by continuity of $(x, y) \to xy$ there exists neighborhood V and W of x and y in G such that $VW \subset q^{-1}(V)$. Thus $q(V)$ and $q(W)$ are neighborhood of $q(x)$ and $q(y)$ such that $q(V)q(W) \subset V$. So multiplication in $G \setminus H$ is continuous. Similarly inversion is continuous.

□

Proposition 3.1.8. *Every locally compact group G has a subgroup H_0 which is open, closed and σ-complete.*

Proof. Let V be a symmetric compact neighborhood of e and let V_n be the n copies of V. Denotes $H_0 = \bigcup_{n=1}^{\infty} V_n$. Then H_0 is a subgroup of G generated by V. Now, H_0 is open, because V_{n+1} is in the neighborhood of V_n and hence it is closed too. Since each V_n is compact, H_0 is σ-compact. □

Lemma 3.1.9. *The quotient map $q : G \to G \setminus H$ is open.*

Proof. $q^{-1}(q(V)) = V(H)$ is open since $q(V)$ is open iff $q^{-1}(q(V))$ is open in G. Now, $q(V) = \{vH : v \in V\}$ and $q(V) \subset q^{-1}(q(V))$. $q^{-1}(q(V)) = \{x \in G : q(x) \in q(V)\} = \{x \in G : xH = vH$, for some $v \in H\}$. Let $y \notin q(V)$ that is $y \neq vH, \forall\, v \in V$ implies $y \notin q^{-1}(q(V))$. □

Example 3.1.10. Let $G = SO(n)$ and $H = SO(n-1)$, then $G \setminus H$ is not a group, however H is closed in G. $G \setminus H \cong S^{n-1} = \{ge_n : g \in G\}$ and $\phi : G \setminus H \to S^{n-1}$ such that $\phi(gH) = ge_n$ is topological isomorphism.

Let $f : G \to \mathbb{C}$ be a function on topological group G. The left and the right translations are defined by $L_y f(x) = f(y^{-1}x)$ and $R_y f(x) = f(xy)$. Notice that $L_{y_1} \circ L_{y_2} = L_{y_1 y_2}$ and $R_{y_1} \circ R_{y_2} = R_{y_1 y_2}$. Hence the maps $L, R : G \to U(L^2(G))$ are group homomorphisms.

Proposition 3.1.11. *If $f \in C_c(G)$, then f is left uniformly continuous.*

Proof. Let $f \in C_c(G)$ and $\epsilon > 0$, Let $K = \operatorname{supp}(f)$. Then $\forall x \in K$, there exists a neighborhood V_0 of e such that $|f(xy) - f(x)| < \frac{1}{2}\epsilon, \forall y \in V_x$ and there exists a symmetric neighborhood V_x of e such that $V_x V_x = U_x$. Now $K \subset V_x V_x, x \in K$ so their exists $x_1, x_2, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^{n} x_i V_{x_i}$. Let $V = \bigcap_{i=1}^{n} V_{x_i}$. We obtain that $||R_y f - f||_\infty < \epsilon, \forall y \in V$. If $x \in K$, then there exists j such that $x_j^{-1} x \in V x_j$, then $|f(xy) - f(x)| \leq |f(xy) - f(x_j)| + |f(x_j) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$. Similarly, if $xy \in K$, then $|f(xy) - f(x)| < \epsilon$. Now if x and $xy \notin K$, then $f(x) = f(xy) = 0$. $\square$

3.2 Radon measures

Let X be a non-empty locally compact Hausdorff space. A measure μ on a Borel σ- algebra $\mathcal{B}$ generated by the open subsets of X is called a Radon measure if

(a) $\mu(K) < \infty$, for all compact set K in X,

(b) $\mu(B) = \inf\{\mu(O) : O \supset B,\ O \text{ is open }\}$, whenever $B \in \mathcal{B}$,

(c) $\mu(B) = \sup\{\mu(K) : K \subset B,\ K \text{ is compact }\}$, whenever $B \in \mathcal{B}$.

Example 3.2.1. (a) The Borel measure on $\mathbb{R}^n$ is a Radon measure on $\mathbb{R}^n$.

(b) $\frac{d\theta}{d\pi}$ is a radon measure on S^1.

Let $\mathcal{B}(G)$ be the Borel σ- algebra generated by all open subsets of a topological group G.

Definition 3.2.2. A Left (or Right) Haar measure on a locally compact Hausdorff space topological group G is a non-zero radon measure μ on G such that $\mu(xB) = \mu(B)$ (or $\mu(Bx) = \mu(B)$) for all $E \in \mathcal{B}(G), \forall x \in G$.

Note if $\mu(G) = 1$, then μ is called the normalized Haar measure on G.

Example 3.2.3. Let $O(G) = n$, for $E \subset G$ and $\mu(E) = \frac{1}{n}\#(E)$, then μ is a normalized Haar measure on G.

Proposition 3.2.4. *Let μ be a radon measure on locally compact group G and $\tilde{\mu}(B) = \mu(B^{-1})$. Then*

(a) μ is a left haar measure if and only if $\tilde{\mu}$ is a right haar measure.

(b) μ is left haar measure if and only if $\int L_y f du = \int f du$, whenever $f \in C_c^+(G)$ and $y \in G$.

Proof. (a) It is easy to verify.

(b) Suppose $\mu(yE) = \mu(E), \forall y \in G, \forall E \in B$. Therefore $f = \chi_A$ and $\int \chi_{yE} d\mu = \int \chi_E d\mu$. For $f \in C_c^+(G)$ and $\epsilon > 0$ there exists a simple function ϕ such that $|\phi - f| < \epsilon$. $\int L_y f dy = \int f dy, \forall f \in C_c(G)$. Hence by the uniqueness in the Riesz representation theorem, it follows that μ will be equal to μ_y.

□

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