MA 101S (Mathematics I, Calculus)

Assignment 1A

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1. Let (x_n) be a convergent sequence of positive real numbers such that $\lim_{n\to\infty} x_n < 1$. Show that $\lim_{n\to\infty} x_n^n = 0$.

2. Let (x_n) be a convergent sequence in $\mathbb{R}$ with limit $\ell \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$.
   (a) If $x_n > \alpha$ for all $n \in \mathbb{N}$, then show that $\ell \geq \alpha$.
   (b) If $\ell > \alpha$, then show that there exists $n_0 \in \mathbb{N}$ such that $x_n > \alpha$ for all $n \geq n_0$.

   (Note that ℓ can be equal to α in (a).)

3. For $\alpha \in \mathbb{R}$, examine whether $\lim_{n\to\infty} \frac{1}{n^2}([\alpha] + [2\alpha] + \cdots + [n\alpha])$ exists (in $\mathbb{R}$). Also, find the value if it exists.
   (For each $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x.)

4. Let $x_1 = 6$ and $x_{n+1} = 5 - \frac{6}{x_n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find $\lim_{n\to\infty} x_n$ if (x_n) is convergent.

5. Let (x_n) be a sequence of nonzero real numbers. If (x_n) does not have any convergent subsequence, then show that $\lim_{n\to\infty} \frac{1}{x_n} = 0$.

6. Examine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ is convergent.

7. Let $x_n > 0$ for all $n \in \mathbb{N}$. Show that the series $\sum_{n=1}^{\infty} x_n$ converges iff the series $\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges.

8. Find all $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{2^n n^2}$ converges.

9. If $\alpha (\neq 0) \in \mathbb{R}$, then show that the series $\sum_{n=1}^{\infty} (-1)^n \sin(\frac{\alpha}{n})$ is conditionally convergent.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ [x] & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
    Determine all the points of $\mathbb{R}$ where f is continuous.

11. Let $f : [0, 1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Show that
    (a) there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{2}$.
    (b) there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{3}$.

    (In fact, if $n \in \mathbb{N}$, then there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{n}$. However, it is not necessary that there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{2}{5}$.)

12. Let p be an odd degree polynomial with real coefficients in one real variable. If $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

    (In particular, this shows that
    (a) every odd degree polynomial with real coefficients in one real variable has at least one real zero.
    (b) the equation $x^9 - 4x^6 + x^5 + \frac{1}{1+x^2} = \sin 3x + 17$ has at least one real root.
    (c) the range of every odd degree polynomial with real coefficients in one real variable is $\mathbb{R}$.)

13. Does there exist a continuous function from $(0, 1]$ onto $\mathbb{R}$? Justify.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $(-\delta, \delta)$ for some $\delta > 0$ and let $f''(0)$ exist (in $\mathbb{R}$). If $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$, then find $f'(0)$ and $f''(0)$.

15. For $n \in \mathbb{N}$, show that the equation $1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} = 0$ has exactly one real root if n is odd and has no real root if n is even.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$ and $f'(0) > 0$, $f'(1) > 0$. Show that there exist $c_1, c_2 \in (0, 1)$ with $c_1 \neq c_2$ such that $f'(c_1) = f'(c_2) = 0$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f''(c)$ exists (in $\mathbb{R}$), where $c \in \mathbb{R}$. Show that $\lim\limits_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$.
Give an example of an $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$ for which $f''(c)$ does not exist (in $\mathbb{R}$) but the above limit exists (in $\mathbb{R}$).

18. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

Show that f is Riemann integrable on $[-1, 1]$ and that $\int\limits_{-1}^{1} f(x)\, dx = 0$.

If $F(x) = \int\limits_{-1}^{x} f(t)\, dt$ for all $x \in [-1, 1]$, then show that $F : [-1, 1] \to \mathbb{R}$ is differentiable, and in particular, $F'(0) = f(0)$, although f is not continuous at 0.

19. Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int\limits_{a}^{b} f(x)\, dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

(The above result need not be true if f is assumed to be only Riemann integrable on $[a, b]$.)

20. If $f : [0, 1] \to \mathbb{R}$ is continuous, then show that $\int\limits_{0}^{x} (\int\limits_{0}^{u} f(t)\, dt)\, du = \int\limits_{0}^{x} (x - u) f(u)\, du$ for all $x \in [0, 1]$.

21. Examine whether the integral $\int\limits_{0}^{\infty} \sin(x^2)\, dx$ is convergent.

22. Determine all real values of p for which the integral $\int\limits_{0}^{\infty} \frac{x^{p-1}}{1+x}\, dx$ is convergent.

23. Find the area of the region that is inside the cardioid $r = a(1 + \cos\theta)$ and
(a) inside the circle $r = \frac{3}{2}a$,
(b) outside the circle $r = \frac{3}{2}a$.

24. Find the length of the curve $y = \int\limits_{0}^{x} \sqrt{\cos 2t}\, dt$, $0 \leq x \leq \frac{\pi}{4}$.