Assignment 5 (L^p- spaces and Product measure)

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) $L^1(X,S,\mu)$ has an almost non-zero function for every measure space (X,S,μ).
   (b) $L^\infty(X,S,\mu)$ contain an almost non-zero function for every measure space (X,S,μ).
   (c) If $f:(X,S,\mu)\to\mathbb{R}$ is bounded almost everywhere. Then f is measurable.
   (d) If $1\le p<\infty$. Then $L^\infty(X,S,\mu)\subset L^p(X,S,\mu)$, implies μ is a finite measure.
   (e) Let $\mathcal{S}(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$ such that $|x|^\alpha f(x)$ is bounded, for any $\alpha\in\mathbb{N}$. Then $\mathcal{S}(\mathbb{R})$ is dense $L^2(\mathbb{R})$.
   (f) Let $1\le p<\infty$. If $L^\infty(X,S,\mu)\subset L^p(X,S,\mu)$. Then μ is a finite measure.
2. Let (X,S,μ) be a measure space and $0<p<1$. Then for $f,g\in L^+\cap L^p(X,S,\mu)$ show that $\|f+g\|_p\ge\|f\|_p+\|g\|_p$.
3. Let (X,S,μ) be a measure space and $1\le p<\infty$. For $f\in L^p(X,S,\mu)$ and $\alpha>0$ show that $\mu\{x\in X:|f(x)|\ge\alpha\}\le\left(\frac{\|f\|_p}{\alpha}\right)^p$.
4. Let $1\le p<\infty$ $f\in L^p(\mathbb{R},M,m)$. Then show that $\|f(x+h)-f(x)\|_p\to0$ as $|h|\to0$.
5. Let (X,S,μ) be a finite measure space and $1\le p<q\le\infty$. For $f\in L^q(X,S,\mu)$, show that $\|f\|_p\le(\mu(X))^{\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_q$. Further deduce that $L^q(X,S,\mu)$ is a proper dense subspace of $L^p(X,S,\mu)$.
6. Show that the space of all simple functions is dense in $L^\infty(X,S,\mu)$.
7. Suppose $f\in L^\infty(X,S,\mu)$ is supported on a set of finite measure. Then show that f is in $L^p(X,S,\mu)$ for all $p\ge1$ and $\lim\limits_{p\to\infty}\|f\|_p=\|f\|_\infty$.
8. Prove that $L^1(\mathbb{R},M,m)\cap L^p(\mathbb{R},M,m)$ is a proper dense subspace of $L^p(\mathbb{R},M,m)$, whenever $1<p<\infty$.
9. Let $1\le p,\,q\le\infty$ and $p^{-1}+q^{-1}=r^{-1}$. Show that for $f\in L^p(X,S,\mu)$ and $g\in L^q(X,S,\mu)$ $fg\in L^1(X,S,\mu)$ and $\|fg\|_r\le\|f\|_p\|g\|_q$. (A generalized Holder's inequality.)
10. Let $1\le p<q<r\le\infty$. Then $L^q(X,S,\mu)\subset L^p(X,S,\mu)+L^r(X,S,\mu)$.
11. Let $1\le p<q<r\le\infty$. Show that $L^p(X,S,\mu)\cap L^r(X,S,\mu)\subset L^q(X,S,\mu)$ and $\|f\|_q\le\|f\|_p^\lambda\|f\|_r^{1-\lambda}$, where $\lambda\in(0,1)$ is given by $q^{-1}=\lambda p^{-1}+(1-\lambda)r^{-1}$.
12. Let $1\le p<\infty$ and $p^{-1}+q^{-1}=1$. For $f\in L^p(X,S,\mu)$, prove that
$$\|f\|_p=\sup\left\{\left|\int_X fgd\mu\right|:g\in L^q(X,S,\mu)\text{ and }\|g\|_q=1\right\}.$$
13. Let (X,S,μ) be a σ-finite measure space. Then show that $\|f\|_\infty=\sup\limits_{\|g\|_1=1}\left|\int\limits_X fgd\mu\right|$.
14. Let $1\le p<\infty$ and $f\in L^+(X,S,\mu)\cap L^p(X,S,\mu)$. Define $f_n(x)=\min\{n,f(x)\}$. Then show that f_n increases to f point wise a.e. and $\lim\limits_{n\to\infty}\int\limits_X|f_n-f|^pd\mu=0$.

15. Let $\mathcal{B}(\mathbb{R}^2)$ be the σ-algebra generated by Borel subsets of $\mathbb{R}^2$ (i.e. σ-algebra generated by open subsets of $\mathbb{R}^2$). Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

16. Let $f : (X, S, \mu) \to \mathbb{R}$ be measurable. Show that $G_f = \{(x, y) \in X \times \mathbb{R},\ y = f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$. If $(X, S, \mu) = (\mathbb{R}, M, m)$, then show that $m \times m(G_f) = 0$.

17. Let(X, S, μ) be a σ-finite measure space. Let $f : (X, S, \mu) \to [0, \infty]$ be measurable. Show that $A_f = \{(x, y) \in X \times [0, \infty],\ y \leq f(x)\} \in S \otimes \mathcal{B}(\mathbb{R})$ and $\mu \times m(A_f) = \int_X f(x) d\mu(x)$.

18. Let $f(x, y) = e^{-xy} \sin x$ and $D = [0, \infty) \times [1, \infty)$. Show that $f\chi_D \in L^1(\mathbb{R}^2, M \otimes M, m \times m)$ and $\int_0^\infty \int_1^\infty f(x, y) dy dx = \int_1^\infty \int_0^\infty f(x, y) dx dy$.

19. Let $f(x, y) = e^{-xy} - 2e^{-2xy}$ and $D = [0, 1] \times [1, \infty)$. Show that $f\chi_D \notin L^1(\mathbb{R}^2, M \otimes M, m \times m)$.

20. Let $f \in L^1(X, S, \mu)$ and $g \in L^1(Y, T, \nu)$. Define $\varphi(x, y) = f(x)g(y)$. Show that φ is measurable and $\varphi \in L^1(X \times Y, S \otimes T, \mu \times \nu)$.

21. Let $f \in L^1(0, a)$ and define $g(x) = \int_x^a \frac{f(t)}{t} dm(t)$. Then show that $g \in L^1(0, a)$ and compute $\int_0^a g(x) dm(x)$.

22. Let $X = Y = [0, 1],\ S = T = \mathcal{B}[0, 1]$ and $\mu = \nu = m$. Define $f : [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 2y & \text{if } y \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Compute $\int_0^1 \int_0^1 f(x, y) dy dx$ and $\int_0^1 \int_0^1 f(x, y) dx dy$. Whether $f \in L^1(m \times m)$?

23. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 :\ x^2 + y^2 < 1\}$. Show that $\mathbb{D}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $m \times m(\mathbb{D})$, using product measure technique.

24. Let (X, S, μ) be a finite measure space and $f : X \to [1, \infty]$ be a measurable function. Compute $\mu \times m \{(x, y) \in X \times \mathbb{R} :\ y < f(x)\}$.

25. Let $E, F \in M(\mathbb{R})$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \chi_E(x)\chi_F(x - y)$. Then show that f is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable and $\int\limits_{\mathbb{R}^2} f\, d(m \times m) = m(E)m(F)$.

26. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 :\ y \geq x^2 \text{ and } y \leq 1\}$. Show that $\mathbb{D}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $m \times m(\mathbb{D})$, using product measure technique.

27. Let $P(x, y)$ be a polynomial on $\mathbb{R}^2$. Show that the set $G_P = \{(x, y) \in \mathbb{R}^2 : P(x, y) = 1\}$ is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $m \times m\,(G_P)$.

28. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{x^2}{1+(x-y)^2} \chi_{[-1,1]}(x)$. Then show that f is $M(\mathbb{R}) \otimes M(\mathbb{R})$ - measurable. Compute $\int\limits_{\mathbb{R}^2} f\, d(m \times m)$.