MA6001: Real Analysis -1

(Assignment 5: Surface Integral)

July - November, 2026

1. Let $D = \{(x, y) : a \leq x \leq b \text{ and } \varphi(x) \leq y \leq \psi(x)\}$, where φ and ψ are continuous functions on $[a, b]$. If f is a bounded continuous functions on D, then
$$\iint_D f(x, y)dxdy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x, y)dy \right) dx.$$
2. Evaluate the following integral applying Fubini's Theorem
   1. $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1 - y^2}dydx$
   2. $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}dydx$
   3. $\int_0^1 \int_y^1 x^2 e^{xy}dxdy$
3. Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. If f_x, f_y, f_{xy} and f_{yx} are continuous then, by using Fubini's theorem, show that $f_{xy} = f_{yx}$.
4. Let $f : D = [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by
$$f(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}^c \cap [0, 1]; \\ 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \text{ and } y \in \mathbb{Q}^c \cap [0, 1]; \\ 1 - \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest term and } y \in \mathbb{Q} \cap [0, 1]. \end{cases}$$
   Then f is integrable and $\iint_D f(x, y)dxdy = 1$. Does repeated integral $\int_0^1 \left(\int_0^1 f(x, y)dy \right) dx$ exist?
5. Evaluate the following iterated integrals applying Fubini's Theorem.
   (a) $\int_0^1 \int_{x=y}^1 \cos(x^2)dxdy$.
   (b) $\int_0^1 \int_{y=\sqrt{x}}^1 e^{y^3}dydx$.
   (c) $\int_0^1 \int_{y=x^2}^1 x^3 e^{y^3}dydx$.
   (d) $\int_0^1 \int_{x=y}^1 \frac{1}{1+x^4}dxdy$.
   (e) $\int_0^1 (\tan^{-1} \pi x - \tan^{-1} x)dx$.
6. Let D be the region in $\mathbb{R}^2$ bounded by the curves $y = 2x^2$ and $y = 1 + x^2$. Evaluate the double integral $\iint_D (2x^2 + y)dxdy$.
7. Using double integral, find the area enclosed by the curve $r = \sin 3\theta$ given in polar coordinates.
8. Evaluate the integral $\iint_D e^{(x-2y)}dxdy$ over the domain D bounded by the lines $x - 2y = 0$, $2x - y = 0$ and $x + y = 1$.
9. Compute $\lim_{a\to\infty} \iint_{D(a)} e^{-(x^2+y^2)}dxdy$, where

$$(a)\ \ D(a) = \{(x, y) : x^2 + y^2 \leq a^2\} \quad \text{and} \quad (b)\ \ D(a) = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq a\}$$

Hence prove that $(c)\ \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ $(d)\ \int_0^{\infty} x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{4}$

10. Evaluate $\iint\limits_R xdxdy$ where R is the region $1 \leq x(1-y) \leq 2$ and $1 \leq xy \leq 2$.
11. Using change of variables $u = x + y$ and $v = x - y$, show that

$$\int\limits_0^1 \int\limits_{y=0}^{y=x} (x-y)dydx = \int\limits_0^1 \int\limits_{u=v}^{2-v} \frac{v}{2} dudv.$$

12. Find the volume of the solid in the first octant bounded below by the surface $z = \sqrt{x^2+y^2}$ above by $x^2 + y^2 + z^2 = 8$ as well as the planes $y = 0$ and $y = x$.
13. Let D denote the solid bounded by surfaces $y = x$, $y = x^2$, $z = x$ and $z = 0$. Evaluate $\iiint\limits_D ydxdydz$.
14. Let $D = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \leq 1\}$ and $E = \{(u,v,w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \leq 1\}$. Show that $\iiint\limits_D dxdydz = \iiint\limits_E 24dudvdw$.
15. Let D denote the solid bounded above by the plane $z = 4$ and below by the cone $z = \sqrt{x^2+y^2}$. Evaluate $\iiint\limits_D \sqrt{x^2+y^2+z^2}dxdydz$.
16. Let D denote the solid enclosed by the spheres $x^2 + y^2 + (z-1)^2 = 1$ and $x^2 + y^2 + z^2 = 3$. Using the spherical coordinates, set up iterated integral that gives the volume of D.
17. Find the line integral of the vector field $F(x,y,z) = y\vec{i} - x\vec{j} + \vec{k}$ along the path $c(t) = (\cos t, \sin t, \frac{t}{2\pi})$, $0 \leq t \leq 2\pi$ joining $(1,0,0)$ to $(1,0,1)$.
18. Show that the integral $\int\limits_C yzdx + (xz+1)dy + xydz$ is independent of the path C joining $(1,0,0)$ and $(2,1,4)$.
19. Consider the curve C which is the intersection of the surfaces $x^2 + y^2 = 1$ and $z = x^2$. Assume that C is oriented counterclockwise as seen from the positive z-axis. Evaluate $\int\limits_C zdx - xydy - xdz$.
20. Let $f(x,y,z) = (x^2, xy, 1)$. Show that that there is no ϕ such that $\nabla\phi = f$.
21. Let C be a curve represented by two parametric representations such that $C = \{R_1(s) : s \in [a,b]\} = \{R_2(t) : t \in [c,d]\}$, where $R_1 : [a,b] \to \mathbb{R}^3$ and $R_2 : [c,d] \to \mathbb{R}^3$ be two distinct differentiable one-one maps.
    (a) Show that there exists a function $h : [c,d] \to [a,b]$ such that $R_2(t) = R_1(h(t))$.
    (b) If R_1 and R_2 trace out C in the same direction, then $\int\limits_C f.dR_1 = \int\limits_C f.dR_2$.
    (c) If R_1 and R_2 trace out C in the opposite direction, then $\int\limits_C f.dR_1 = -\int\limits_C f.dR_2$.
22. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2+y^2}$. Parametrize S by considering it as a graph and again by using the spherical coordinates.
23. Let S denote the part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$. Find the area of S.
    (a) By considering S as a part of the graph $z = f(x,y)$, where $f(x,y) = 10 - 2x - 5y$.
    (b) By considering S as a parametric surface $r(u,v) = (u\cos v, u\sin v, 10 - u(2\cos v + 5\sin v))$, $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$.
24. Find the area of the surface $x = uv, y = u + v, z = u - v$, where $(u,v) \in D = \{(s,t) \in \mathbb{R}^2 : s^2 + t^2 \leq 1\}$.
25. Find the area of the part of the surface $z = x^2 + y^2$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.
26. Let S be the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant. Evaluate $\iint\limits_S (z + 2xy)d\sigma$.
27. Find the surface integral $\iint\limits_S zd\sigma$, where S it the part of the paraboloid $2z = x^2 + y^2$ which lies in the cylinder $x^2 + y^2 = 1$.

28. Let S be the sphere $x^2 + y^2 + z^2 = 1$. If some $\alpha \in \mathbb{R}$ satisfies $\iint_S (zx + \alpha y^2 + xz)d\sigma = \frac{4\pi}{3}$, then find α.
29. Use Green's Theorem to compute $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where C is the boundary of the region $\{(x, y) : x, y \geq 0 \ \& \ x^2 + y^2 \leq 1\}$.
30. Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuously differentiable function. Suppose C is the boundary of the region bounded above by the graph of f, below by the x-axis and on the sides by the lines $x = a$ and $x = b$. Show that $\int_a^b f(x)dx = -\oint_C ydx$.
31. Let $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and let S be any surface that surrounds the origin. Prove that $\iint_S \vec{F} \cdot nd\sigma = 4\pi$.
32. Let D be the domain inside the cylinder $x^2 + y^2 = 1$ cut off by the planes $z = 0$ and $z = x + 2$. If $\vec{F} = (x^2 + ye^z, y^2 + ze^x, z + xe^y)$, use the divergence theorem to evaluate $\iint_{\partial D} F \cdot \hat{n} \, d\sigma$.
33. Let $\vec{F} = (xy, yz, zx)$ and S be the surface $z = 4 - x^2 - y^2$ with $2 \leq z \leq 4$. Use divergence theorem to find the surface integral $\iint_S \vec{F}.\vec{n}dS$.
34. Use Stokes' Theorem to evaluate the line integral $\int_C -y^3dx + x^3dy - z^3dz$, where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.
35. Let $F(x, y, z) = (y, -x, 2z^2 + x^2)$ and S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies below the plane $z = 4$. Evaluate $\iint_S \operatorname{curl} F \cdot \hat{n}d\sigma$, where $\hat{n}$ is the unit outward normal of S.