Assignment 4 (Lebesgue Integration)

1. Whether $L^1(X,S,\mu)$ has an almost non-zero function for every (X,S,μ) ?
2. Let $f:(X,S,\mu)\to[0,\infty]$ be measurable. Define a set function $\nu(E)=\int\limits_E f d\mu,\ E\in S$. Show that ν is a measure on (X,S). Does $\nu(E)=0$ imply $\mu(E)=0$?
3. Let $f\in L^1(X,S,\mu)$ and $E_n=\{x\in X:|f(x)|\geq n\}$. Show that $\lim\limits_{n\to\infty} n\ \mu(E_n)=0$.
4. Let $f:(X,S,\mu)\to\mathbb{R}$ be measurable. Define a set function $\nu:S\to\overline{\mathbb{R}}$ by $\nu(E)=\int_E f d\mu$, whenever $E\in S$. Show that $\nu(X)$ is finite if and only if $f\in L^1(X,S,\mu)$.
5. For $f\in L^+\cap L^1(\mathbb{R},M,m)$, define $g(x)=\sum\limits_{n=1}^{\infty} f(2^n x+\frac{1}{n})$. Show that $g\in L^1(\mathbb{R},M,m)$ and $\int\limits_{\mathbb{R}} g dm=\int\limits_{\mathbb{R}} f dm$.
6. Let $f:(X,S,\mu)\to[0,\infty]$ be such that $\|f\|_1=1$. Show that there exists at leat one $n\in\mathbb{N}$ such that $\mu\{x\in\mathbb{X}:|f(x)|<n\}>0$.
7. Let $f_n:X\to[0,\infty]$ be a sequence of measurable functions and $f_n\to f$ point wise. Suppose there exists $M>0$ such that $\sup\limits_{n\geq1}\int_X f_n\leq M$. Show that $f\in L^1(X,S,\mu)$.
8. Let $f\in L^1(X,S,\mu)$, then show that for each $\epsilon>0$ there exists $\delta>0$ and set $E\in S$ such that $\int\limits_E |f|\,d\mu<\epsilon$, whenever $\mu(E)<\delta$.
9. Let $\mu(\mathbb{R})<\infty$ and $f_n\in L^1(\mathbb{R},M,\mu)$ be such that $f_n\to f$ uniformly. Show that $f\in L^1(X,S,\mu)$ and $\int\limits_X f=\lim\int\limits_X f_n$.
10. Let $\mu(X)<\infty$ and $f:X\to[0,\infty]$ be a measurable function. Show that $f\in L^1(X,S,\mu)$ if and only if $\sum\limits_{n=0}^{\infty}\mu\{x\in X:f(x)\geq n\}<\infty$.
11. Let $f_n:X\to[0,\infty]$ be a decreasing sequence of measurable functions and $f_n\to f$ point wise. If $f_1\in L^1(X,S,\mu)$. Then show that $\int\limits_X f=\lim\int\limits_X f_n$.
12. Let $f\in L^1(\mathbb{R},M,m)$ be such that $\int\limits_I f=0$, for any open interval $I\subset\mathbb{R}$, then show that $f=0$ a.e. on X.
13. Let $f_n,g:X\to\overline{\mathbb{R}}$ be measurable functions such that $f_n\leq g,\ \forall\ n\in\mathbb{N}$ and $g\in L^1(X,S,\mu)$. Show that $\limsup\int\limits_X f_n\leq\int\limits_X\limsup f_n$.
14. Let $f_n:X\to[0,\infty]$ be a sequence of measurable functions and $f_n\to f$ point wise such that $\int\limits_X f=\lim\int\limits_X f_n<\infty$. Show that $\int\limits_E f=\lim\int\limits_E f_n$, for any $E\in S$.
15. Let $f_n,f,g_n,g\in L^1(X,S,\mu)$ be such that $|f_n|\leq g_n,\ f_n\to f$ and $g_n\to g$ point wise. Show that $\int\limits_X g=\lim\int\limits_X g_n$ implies $\int\limits_X f=\lim\int\limits_X f_n$.
16. Let $f_n,f\in L^1(X,S,\mu)$ be such that $f_n\to f$ point wise. Prove that $\lim\int\limits_X|f_n-f|=0$ if and only if $\int\limits_X|f|=\lim\int\limits_X|f_n|$. (Hint: Use question 12.)
17. Let μ be the counting measure on the measurable space $(\mathbb{N},\mathcal{P}(\mathbb{N}))$ and let $f:\mathbb{N}\to[0,+\infty]$. Show that $\int\limits_E f\,d\mu=\sum\limits_{n\in E} f(n)$ for every $E\subset\mathbb{N}$ and hence, in particular, $\int\limits_{\mathbb{N}} f\,d\mu=\sum\limits_{n=1}^{\infty} f(n)$.
18. Let δ_x be the Dirac measure at $x\in X$ on the measurable space $(X,\mathcal{P}(X))$. If $f:X\to[0,+\infty]$ and $E\subset X$, then show that $\int\limits_E f\,d\delta_x=\begin{cases} f(x) & \text{if } x\in E,\\ 0 & \text{if } x\notin E.\end{cases}$
(Hence, in particular, $\int\limits_X f\,d\delta_x=f(x)$.)

19. Let μ_n be a sequence of σ- finite measures on (X, S). For $E \in S$, define $\mu(E) = \sum\limits_{n=1}^{\infty} \mu_n(E)$. Show that μ is a σ- finite measure. If $f \in L^+(X, S, \mu)$, then prove that $\int\limits_X f d\mu = \sum\limits_{n=1}^{\infty} \int\limits_X f d\mu_n$. (Hint: Use simple function technique.)

20. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f = \frac{1}{\sqrt{x}}\chi_{(0,1)}$. Let $g(x) = \sum\limits_{r_n \in \mathbb{Q}} 2^{-n} f(x - r_n)$, then show that the function g belongs to $L^1(\mathbb{R}, M, m)$.

21. Let $f_n = \chi_{\left[\frac{1}{n}, \frac{1}{n+1}\right]}$. Construct an increasing sequence $\{g_n\}$ of measurable functions in terms of f_n such that $\lim\limits_{n\to\infty} \int\limits_{\mathbb{R}} g_n(x) dm(x) < \infty$.

22. For each $x \in [0, 1]$, let $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{k}{n} \text{ for some } k, n \in \mathbb{N} \text{ with g.c.d.}(k, n) = 1, \\ 0 & \text{otherwise.} \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{[0,1]} f\, dm$.

23. Let $f, g : (X, S, \mu) \to [0, +\infty]$ be measurable. If $\lambda(E) = \int\limits_E f\, d\mu$ for all $E \in \mathcal{S}$, then show that λ is a measure on $(X, \mathcal{S})$ and that $\int\limits_X g\, d\lambda = \int\limits_X gf\, d\mu$.

24. For each $x \in [0, 1]$, let $f(x) = \begin{cases} x^2 & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ x^3 & \text{if } x = \frac{1}{3^n} \text{ for some } n \in \mathbb{N}, \\ x^4 & \text{otherwise.} \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{[0,1]} f\, dm$.

25. Let $f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in [0, \frac{1}{2}] \setminus C, \\ \cos(\pi x) & \text{if } x \in (\frac{1}{2}, 1] \setminus C, \\ x^2 & \text{if } x \in C. \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{[0,1]} f\, dm$, where C denotes the Cantor ternary set in $[0, 1]$.

26. Evaluate the Lebesgue integrals: (a) $\int\limits_{[0,+\infty)} e^{-[x]}\, dx$ (b) $\int\limits_{(0,1]} \frac{1}{\sqrt[3]{x}}\, dx$

27. Let $f(x) = \begin{cases} e^{[x]} & \text{if } x \in \mathbb{Q}, \\ e^{-[x]} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{(0,+\infty)} f\, dm$.

28. Let $f(x) = \begin{cases} e^{|x|} & \text{if } x \in \mathbb{Q}, \\ e^{-|x|} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{\mathbb{R}} f\, dm$.

29. Let $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ \frac{1}{x} & \text{if } x > 1. \end{cases}$
Evaluate the Lebesgue integral $\int\limits_{(0,+\infty)} f\, dm$.

30. Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{n^2 x e^{-x^2}}{n^2 + x^2}$. Show that $f_n \in L^1([0, \infty))$, for each $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \int_{[0,\infty)} f_n dm = \frac{1}{2}$.

31. Evaluate the following: (a) $\lim\limits_{n\to\infty} \int\limits_{-2}^{2} \frac{x^{2n}}{1+x^{2n}}\, dx$ (b) $\lim\limits_{n\to\infty} \int\limits_{[0,1]} \frac{1+nx}{(1+x)^n}\, dx$ (c) $\int\limits_0^1 (\sum\limits_{n=1}^{\infty} \frac{x^n}{n})\, dx$

(d) $\lim\limits_{n\to\infty} \int\limits_1^{\infty} \frac{1}{1+x^{2n}}\, dx$ (e) $\sum\limits_{n=0}^{\infty} \int\limits_0^1 \frac{x^2}{(1+x^2)^n}\, dx$ (f) $\lim\limits_{n\to\infty} \int\limits_{[0,\infty)} \dfrac{n^2 x e^{-x^2}}{n^2 + x^2}\, dx$