Assignment 3 (Measurable Functions)

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous m-a.e. on $\mathbb{R}$, then there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ m-a.e. on $\mathbb{R}$.
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and if $g : \mathbb{R} \to \mathbb{R}$ is such that $f = g$ m-a.e. on $\mathbb{R}$, then g must be continuous m-a.e. on $\mathbb{R}$.
   (c) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous such that $f = g$ m-a.e. on $\mathbb{R}$, then it is necessary that $f(x) = g(x)$ for all $x \in \mathbb{R}$.
   (d) There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f = \chi_{[0,1]}$ m-a.e. on $\mathbb{R}$.
   (e) An almost everywhere vanishing Lebesgue measurable function need not be continuous.
   (f) Let $f : (X, S, \mu) \to \overline{\mathbb{R}}$ be a bounded function. Then $\mu\left(f^{-1}\{-\infty\}\right) > 0$.
   (g) Let $f_n : (\mathbb{R}, M, m) \to \overline{\mathbb{R}}$ be defined by $f_n = \frac{1}{n}\chi_{(0,n)}$. Then $f_n \to 0$ uniformly.
   (h) Let $|f|$ be measurable on (X, S, μ). Then f measurable.
   (i) Let $f : (X, S, \mu) \to \mathbb{R}$ be bounded a.e. Then f measurable.

2. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq 5, \\ 0 & \text{if } |f(x)| > 5. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

3. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. For each $x \in X$, let $g(x) = \begin{cases} 0 & \text{if } f(x) \in \mathbb{Q}, \\ 1 & \text{if } f(x) \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Show that $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

4. Let $f : [a, b] \to \mathbb{R}$ be Lebesgue measurable. Let $N = \{x \in [a, b] : f(x) = 0\}$. Show that $g = \chi_N + \frac{1}{f}\chi_{N^c}$ is Lebesgue measurable.

5. Show that $f_n(x) = e^{-n(1-\cos x)}$ converges to 0 a.e. m on $\mathbb{R}$. Find an interval in $\mathbb{R}$ where the sequence f_n converges to 0 uniformly.

6. Let $f_n : (\mathbb{R}, M, m) \to \overline{\mathbb{R}}$ be defined by $f_n = \chi_{(n,n+1)}$. Show that $f_n \to 0$ point wise but not uniformly.

7. Let $E \subset \mathcal{M}(\mathbb{R})$ with $m(E) < \infty$. If $\{f_n\}$ is a sequence of Lebesgue measurable functions on E such that $|f_n(x)| \leq M_x < \infty$, $\forall x \in E$ and $\forall\, n \geq 1$. Prove that for any $\epsilon > 0$, there exists a compact set $K \subset E$ such that $\{f_n\}$ is uniformly bounded on K, where $m(E \setminus K) < \epsilon$.

8. Let $f : (\mathbb{R}^2, M(\mathbb{R}^2), m) \to \mathbb{R}$ be given by $f(x, y) = \begin{cases} 1 & \text{if } \frac{x}{y} \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$
   Show that f is Lebesgue measurable.

9. Let $\mathbb{Q}$ denotes set of rationals. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = \begin{cases} 1 & \text{if } x + y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$
   Show that f is Lebesgue measurable.

10. Let $f : X \to \overline{\mathbb{R}}$ be an almost bounded measurable function on a complete measure space (X, S, μ). Then for $A_n = \{x \in X : \ |f(x)| > n\}$, show that $\lim_{n\to\infty} \mu(A_n) = 0$.

11. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$ and let $f : X \to \mathbb{R}$ be measurable. Let $A_n = \{x \in X : |f(x)| > n\}$. Show that A_n is $\mathcal{A}$-measurable and $\lim \mu(A_n) = 0$.

12. If $(X, \mathcal{A})$ is a measurable space, then show that $f : X \to [-\infty, +\infty]$ is $\mathcal{A}$-measurable iff $\{x \in X : f(x) > r\} \in \mathcal{A}$ for each $r \in \mathbb{Q}$.

13. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) = \infty$. Define a function $f : \mathbb{R} \to [0, \infty)$ by $f(x) = m\{E \cap (0, |x|)\}$. Show that f is continuous.

14. For $a_n \geq 0$, define a function f on $\mathbb{R}$ by $f = \sum_{n=0}^{\infty} a_n \chi_{(n,n+1)}$. Show that the set of discontinuities of f is Lebesgue measurable.

15. Let (X, S, μ) be a measurable space and $f : X \to [0, 1]$. Then show that f is measurable if and only if $\left\{x \in X : f(x) > \frac{k}{2^n}\right\}$ is measurable $\forall\ k = 0, 1, 2, 3, \ldots, 2^n$ and $\forall\ n \in \mathbb{N}$.

16. Let $f_n,\ f$ be real valued measurable functions on $\mathbb{R}$. Let $E = \{x \in \mathbb{R} :\ \lim f_n(x) = f(x)\}$. Show that E is Lebesgue measurable.

17. Let $f : [0,\ 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Find the measure of the set $\{x \in [0, 1] :\ f(x) \geq 0\}$.

18. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then show that $g \circ f$ is $\mathcal{A}$-measurable.

19. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. If G is an open subset of $\mathbb{R}^2$, then show that $\{x \in X : (f(x), g(x)) \in G\}$ is $\mathcal{A}$-measurable.

20. If $f : \mathbb{R} \to \mathbb{R}$ is continuous m-a.e. on $\mathbb{R}$, then show that f is Lebesgue measurable.

21. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, then show that $f' : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x,\ .)$ and $f(.,\ y)$ are measurable. Prove/disprove that f is Lebesgue measurable.

23. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x,\ .)$ is measurable and $f(.,\ y)$ is continuous. Show that f is Lebesgue measurable.

24. Let $f, g : X \to \mathbb{R}$. Define $\varphi(x) = (f(x), g(x))$. Then show that f and g are measurable if and only if φ is measurable.

25. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose for each $\epsilon > 0$ there exists an open set O such that $m(O) < \epsilon$ and f is constant on $\mathbb{R} \setminus O$. Show that f is Lebesgue measurable.

26. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Show that $\{x \in \mathbb{R} : f$ is continuous at x $\}$ is Lebesgue measurable.

27. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous one-one and onto map. Then show that f maps Borel sets onto Borel sets.

28. Let $f : \mathbb{R} \to \mathbb{R}$ be a one to one onto continuous function. Show that F is a F_σ set if and only if $f(F)$ is a F_σ set.

29. Let C be the Cantor's ternary set. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in C \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$
Show that f is Lebesgue measurable. By letting C has a non-Borel measurable subset, construct a Lebesgue measurable function which is not Borel measurable.

30. Let $f : (a, b) \to \mathbb{R}$ be a continuous bijection function. Then a Lebesgue measurable $E \subset (a, b)$ satisfies $m(E) = 0$ implies $m(f(E)) = 0$ if and only if for every Lebesgue measurable subset $A \subset (a, b)$ the set $f(A)$ is Lebesgue measurable.