Assignment 2 (Measurable sets-II)

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) $\{x \in \mathbb{R} : x^6 - 6x^4 \text{ is irrational}\}$ is a Lebesgue measurable subset of $\mathbb{R}$.
   (b) If A is a Lebesgue measurable subset of $\mathbb{R}$ and if B is a Lebesgue non-measurable subset of $\mathbb{R}$ such that $B \subset A$, then it is necessary that $m^*(A \setminus B) > 0$.
   (c) If A and B are disjoint subsets of $\mathbb{R}$ such that A is Lebesgue measurable and B is Lebesgue non-measurable, then it is possible that $m^*(A \cup B) < m^*(A) + m^*(B)$.
   (d) If $E \subset \mathbb{R}$ is such that $m^*(E + E) < \infty$, then $m^*(E) < \infty$.
   (e) If for $E \subset [0,1]$ and $r_n \in [0,1] \cap \mathbb{Q}$, write $E_n = E + r_n$. Then $m^*\left(\bigcap_{n=1}^{\infty} E_n\right) < \lim_{n\to\infty} m^*(E_n)$.
   (f) If $O \subset \mathbb{R}$ is open and $A \subset \mathbb{R}$, then $A + O$ is Lebesgue measurable.
   (g) There exists a set $E \in \mathcal{M}(\mathbb{R}^2)$ such that $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$ is not in $\mathcal{M}(\mathbb{R})$.

2. Let E and F be two closed subsets $\mathbb{R}$. Show that $E + F$ is Lebesgue measurable. Does $E + F$ closed?

3. Let $E \subset \mathbb{R}$ be Lebesgue measurable and $m(E) < \infty$. Show that for each $\epsilon > 0$, there exists compact set K and open set O with $K \subseteq E \subseteq O$ such that $m(O \setminus K) < \epsilon$.

4. If $A \subset \mathbb{R}$, then show that there exists a Lebesgue measurable subset E of $\mathbb{R}$ such that $m^*(A) = m(E)$.

5. Let A be a bounded subset of $\mathbb{R}$. Show that $m\left(\overline{A}\right) < \infty$, where $\overline{A}$ is the closure of A.

6. Let $A \subset \mathbb{R}$ be a closed set with $m(A) = 0$. Show that A is nowhere dense in $\mathbb{R}$. Does this conclusion hold true if A is not closed?

7. Let $A \subset [0,1]$ be Lebesgue measurable with $m(A) = 1$. If $B \subset [0,1]$, then show that $m^*(A \cap B) = m^*(B)$.

8. Let $E_i \subset (0,1)$ be Lebesgue measurable such that $\sum_{i=1}^{n} m(E_i) > n - 1$. Show that $m(\bigcap_{i=1}^{n} E_i) > 0$.

9. Let $\{E_i\}$ be a decreasing sequence of Lebesgue measurable sets in $(0,1)$ such that $\sum_{i=1}^{n} m(E_i) > n - \frac{1}{n}$. Show that $m\left(\bigcap_{i=1}^{\infty} E_i\right) = 1$.

10. Let $A \subset \mathbb{R}$ be such that $m^*(A) > 0$. Show that there exist $x, y \in A$ such that $x - y \in \mathbb{R} \setminus \mathbb{Q}$.

11. Let A and B be Lebesgue measurable subsets of $(0,1)$ such that $m(A) > \frac{1}{2}$ and $m(B) > \frac{1}{2}$. Prove that there exist $a \in A$ and $b \in B$ such that $a + b = 1$.

12. Let A be an unbounded Lebesgue measurable subset of $\mathbb{R}$ such that $m(A) < \infty$. Show that for each $\varepsilon > 0$, there exists a bounded Lebesgue measurable set B in $\mathbb{R}$ such that $B \subset A$ and $m(A \setminus B) < \varepsilon$.

13. Let $A, B \subset \mathbb{R}$ such that $A \cup B$ is Lebesgue measurable and $m(A \cup B) = m^*(A) + m^*(B) < \infty$. Show that both A and B are Lebesgue measurable.

14. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{P}(\mathbb{R})$ and let $\{E_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{M}(\mathbb{R})$ such that $A_n \subset E_n$ for each $n \in \mathbb{N}$. Show that $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m^*(A_n)$.

15. Let $E \subset \mathbb{R}$ and let $\alpha \in \mathbb{R}$. If $\alpha E = \{\alpha x : x \in E\}$, then show that $m^*(\alpha E) = |\alpha| m^*(E)$. Also, show that if E is Lebesgue measurable, then αE is Lebesgue measurable.

16. If E Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < +\infty$ and $f(x) = m(E \cap (-\infty, x])$ for all $x \in \mathbb{R}$, then show that $f : \mathbb{R} \to \mathbb{R}$ is continuous.

17. Let E be a Lebesgue measurable subset of $\mathbb{R}$ that satisfies $m(E) < \infty$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = m\{E \cap (-\infty, x^2)\}$. Show that f is differentiable at 0 and $f'(0) = 0$.

18. Let $m^*(A) > 0$. Then show that there exists at least one closed set $F \subset \mathbb{R}$ with $m(F) < \infty$ such that $A \cap F \neq \emptyset$.

19. Let μ be a finite measure on $M(\mathbb{R})$. Suppose for each closed set $F \subset \mathbb{R}$ with $m(F) < \infty$, implies $\mu(F) = 0$. Then show that $\mu = 0$.

20. Let E be a measurable subset of $\mathbb{R}$ with $m(E) < \infty$ and $m\{E \cap (n, n+1)\} < \frac{1}{2^{|n|+2}} m(E)$, for all $n \in \mathbb{Z}$. Show that $m(E) = 0$.

21. Let $\{E_n\}$ be a sequence of Lebesgue measurable subset of $\mathbb{R}$ such that $\sum\limits_{n=1}^{\infty} m(E_n) < \infty$. Show that $m\left(\bigcap\limits_{n=1}^{\infty} E_n\right) = 0$.

22. Let $[-1, 1] \cap \mathbb{Q} = \{r_1, r_2, \ldots\}$. For a Lebesgue measurable set $E \subset [0, 1]$ with $m(E) > 0$, define $E_n = E + r_n;\ n \in \mathbb{N}$. Show that all of E_n's can not be pairwise disjoint.

23. Let F be a closed subset of $\mathbb{R}$ with $m(F) = 0$. Then for any $A \subset F$, show that $m^*\{x \in \mathbb{R} : d(x, A) = 0\} = 0$.

24. Let $O \subset \mathbb{R}$ be an open set (possibly disconnected) with $m(O) < \infty$. Show that for each $\epsilon > 0$ there exist non-empty disjoint open sets O_1 and O_2 such that $O = O_1 \cup O_2$ and $m(O_2) < \epsilon$.

25. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $m(E) < \infty$. Then there exists a sequence of compact set $K_n \in E$ and a set N of Lebesgue measure zero such that $E = F \cup N$, where $F = \bigcup\limits_{n=1}^{\infty} K_n$.

26. Let K be a compact subset of $\mathbb{R}$. Show that the set $E = \{x \in \mathbb{R} : d(x, K) < 1\}$ is Lebesgue measurable and $m(E) \leq \delta(K) + 2$, where $\delta(K)$ denotes diameter of K. Further, if $m(E) = 2$, then show that the set K can contains at most one point.

27. Let K be a compact subset of $\mathbb{R}$ and $O_n = \left\{x \in \mathbb{R} : d(x, K) < \frac{1}{n}\right\}$. Show that each of O_n is Lebesgue measurable and $\lim\limits_{n \to \infty} m(O_n) = m(K)$.

28. For an open subset $\mathcal{O}$ of $\mathbb{R}$, define $E_k = \left\{x \in \mathbb{R} : d(x, \mathcal{O}) \geq \dfrac{1}{k}\right\}$. Show that each E_k is Lebesgue measurable and $\lim\limits_{k \to \infty} m(E_k) = m(\mathbb{R} \setminus \overline{\mathcal{O}})$.

29. Let $E \subset \mathbb{R}$ and $m^*(E) > 0$. Then for each $0 < \alpha < 1$, there exists an open interval I such that $m^*(E \cap I) \geq \alpha m(I)$.

30. Let $E \subset \mathbb{R}$ be Lebesgue measurable and $m(E) > 0$. Show that the set $E - E = \{x - y : x, y \in E\}$ contains an open interval.