MA642: Real Analysis -1

(Assignment 2: Metric and Normed Linear Spaces)
January - April, 2023

1. State TRUE or FALSE giving proper justification for each of the following statements.
   (a) The totally boundedness property is preserved by homeomorphism.
   (b) Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n = \chi_{[0,1/n]}$ and f_n converges point-wise to f. Then the set $\{f, f_n : \ n = 1, 2, \ldots\}$ is compact in $B[0,1]$.
   (c) Let $f_n \in C^1[0,1]$. Then it implies that the set $\{f_n : n = 1, 2, \ldots\}$ is compact in $C[0,1]$.
2. If every countable closed subset of a metric space X is complete, show that X is complete.
3. Find a subset of l^∞ which is closed and bounded but not totally bounded.
4. Show that a subset A of a metric space (X, d) is totally bounded if and only if for every sequence (x_n) has a subsequence (x_{n_k}) satisfying $d(x_{n_k}, x_{n_{k+1}}) \leq 2^{-k}$.
5. Let K and F be two non-empty subsets of a metric space (X, d). If K is compact and F closed, then show that $\text{dist}(K, F) > 0$, whenever $K \cap F = \emptyset$. Does the the same conclusion holds if K is closed but not compact?
6. A function $f : (X, d) \to \mathbb{R}$ is called lower semi-continuous if for each $\alpha \in \mathbb{R}$ the set $\{x \in X : f(x) > \alpha\}$ is open in X.
   (a) Show that f is lower semi-continuous if and only if $f(x) \leq \liminf\limits_{n\to\infty} f(x_n)$, whenever $x_n \to x$.
   (b) If X is compact metric space, prove that every lower semi-continuous function is bounded below and attains its minimum.
7. If X is compact metric space, and let $f : X \to X$ satisfy $d(f(x), f(y)) = d(x, y)$ for $x, y \in X$. Show that f is an onto map. Is compactness of X is necessary?
8. If X is compact metric space, and let $f : X \to X$ satisfy $d(f(x), f(y)) \geq d(x, y)$ for $x, y \in X$. Show that f is an onto isometry.
9. Let X be a compact metric space, and let $f : X \to X$ be bijective and satisfy $d(f(x), f(y)) \leq d(x, y)$ for $x, y \in X$. Show that f is an isometry.
10. Let X be a compact metric space, and $\mathcal{F}$ is a subset of $(C(X)$.
    (a) Prove that an equicontinuous subset $\mathcal{F}$ is pointwise bounded if and only if $\mathcal{F}$ is uniformly bounded.
    (b) Prove that $\mathcal{F}$ is pointwise equicontinuous if and only if $\mathcal{F}$ uniformly equicontinuous.
11. Let X be a compact metric space, and (f_n) is a sequence in $(C(X)$.
    (a) Let (f_n) be equicontinuous and pointwise convergent. Show that $f_n)$ is uniformity convergent.
    (b) If (f_n) decreases pointwise to 0, show that (f_n) is equicontinuous.
    (c) If (f_n) is equicontinuous, show that $\{x \in X : (f_n(x)) \text{ converges}\}$ is a closed set in X.
12. For fixed $k > 0$ and $0 < \alpha \leq 1$, denote $\text{Lip}_k\alpha = \{f \in C[0,1] : |f(x) - f(y)| \leq k|x - y|^\alpha\}$. Show that $\{f \in \text{Lip}_k\alpha : f(0) = 0\}$ is compact subset of $C[0,1]$. Whether the set $\{f \in \text{Lip}_k\alpha : \int_0^1 f(t)dt = 1\}$ is compact?

13. Let $K(x,t)$ be a continuous function on the square $[0,1]\times[0,1]$. For $f \in C[0,1]$, define $Tf(x) = \int_0^1 f(t)K(x,t)dt$. Show that T maps bounded sets into equicontinuous sets.

14. If $f \in B[0,1]$, show that $B_n(f)(x) \to f(x)$ at each point of continuity of f.

15. For a given polynomial p and $\epsilon > 0$, show that there exits a polynomial q of rational coefficients such that $\|p-q\|_\infty < \epsilon$ on $[0,1]$.

16. Let (x_i) be a sequence in $(0,1)$ such that $\frac{1}{n}\sum_{i=1}^{n} x_i^k$ is convergent for each $k = 0,1,2\ldots$, then $\frac{1}{n}\sum_{i=1}^{n} f(x_i)$ is convergent for each $f \in C[0,1]$.

17. For $f \in C[0,1]$ and $\epsilon > 0$, show that there exists a polynomial p such that $\|f-p\|_\infty < \epsilon$ and $\|f'-p'\|_\infty < \epsilon$.

18. Let $f : [1,\infty) \to \mathbb{R}$ be continuous and $\lim_{x\to\infty} f(x)$ exists. For $\epsilon > 0$, show that there exists a polynomial p such that $|f(x) - p(1/x)| < \epsilon$ for all $x \geq 1$.